Vectors Cast New Darkness, and New Light, on a Classic Related Rates Problem

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Once upon a time, a calculus class was studying this problem [**2**, Sec. 2.8, Exercise 11]:

> Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

Many classes had studied similar problems before. One can imagine Cadet Julius Caesar reading, "Marcus's chariot is moving north at 3 leagues per day. Lucius's chariot is moving east . . . " (Seriously, problems involving time-dependent right triangles were among those condemned by Luise Lange in an article [**1**] in the Monthly in 1950. Evidently they were already trite at that time.)

But this particular class was part of an experimental program [**3**] in engineering education, in which the students had been introduced to vectors already in the first semester. As one team of students was working out the problem by the standard textbook method (a special case of (6) below), another student — let's call him "Red" — interrupted, "Why are we doing it this complicated way? Isn't the answer obvious?" In effect, Red's argument was this: The velocity of the first car is $\mathbf{v}_1 = \langle 0, -60 \rangle$ (see Figure 1). The velocity of the second car is $\mathbf{v}_2 = \langle -25, 0 \rangle$. So the relative velocity is

$$\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = \langle -25, 60 \rangle,$$

and its magnitude is

$$|\mathbf{v}| = \sqrt{25^2 + 60^2} = 65 \text{ mi/h}.$$

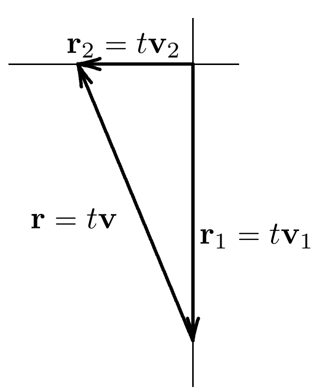

FIGURE 1

The professor stammered a bit, did a quick private calculation on the blackboard (something like (6) and (7) below), and said, "I guess you're right. I never thought of looking at the problem that way." (Not while teaching that section of the book, at least.)

Fast-forward to the second semester. The class was given a review quiz containing this question [**2**, Sec. 2.8, Example 4]:

> A car is traveling west at 50 miles per hour and a truck is traveling north at 60 miles per hour. At what rate are the vehicles approaching each other when the car is 0.3 miles and the truck 0.4 miles from the intersection of the roads? (Both are moving toward the intersection, and the roads are exactly perpendicular.)

When the papers were graded, it turned out that half the class had calculated a relative speed of 78.1025 miles per hour, and the other half had calculated 78.00, exactly! Closer examination revealed that the students who got the (correct) answer 78 had used the textbook method, whereas the students who got 78.1025 had used Red's method:

$$\sqrt{50^2 + 60^2} = 78.1025.$$

Apparently, knowing about vectors can be hazardous to your grade.

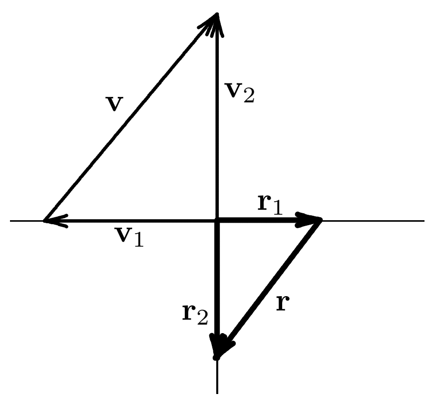

FIGURE 2

Mystery 1: Why did Red's method give the wrong value in the second problem? Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of the two vehicles (see Figure 2), and let

$$\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1 \tag{1}$$

be the displacement of one vehicle from the other. Then $|\mathbf{r}|$ is the distance between them, and the problem is asking for

$$\frac{d}{dt}|\mathbf{r}|. \tag{2}$$

On the other hand, the relative velocity is

$$\mathbf{v} \equiv \mathbf{v}_2 - \mathbf{v}_1 \equiv \frac{d\mathbf{r}_2}{dt} - \frac{d\mathbf{r}_1}{dt} = \frac{d\mathbf{r}}{dt}\,, \tag{3}$$

and Red's prescription is to calculate

$$|\mathbf{v}| = \left|\frac{d\mathbf{r}}{dt}\right|. \tag{4}$$

Quantities (2) and (4) are not the same. Red's formula is the answer to a legitimate question ("What is the magnitude of the relative velocity?"), but not the question that was asked.

To see the distinction starkly, consider the extreme case where one car is at rest and the other one moves around it in a circle. The relative velocity is the velocity of the second car, which is not zero; but the distance between the cars is not changing at all.

Mystery 2: Why did Red's method give the right value in the first problem? Let's look carefully at the correct solution method [**2**, p. 159], which is to differentiate the Pythagorean theorem applied to the right triangle with the cars at the vertices. In vector notation this amounts to

$$\begin{aligned}\frac{d}{dt}|\mathbf{r}| &= \frac{1}{2|\mathbf{r}|}\,\frac{d}{dt}|\mathbf{r}|^2 \\ &= \frac{1}{|\mathbf{r}|}\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right) \\ &= \left(\frac{\mathbf{r}}{|\mathbf{r}|}\right)\cdot\mathbf{v}, \end{aligned} \tag{5}$$

which is the component of $\mathbf{v}$ parallel to $\mathbf{r}$ (also known as the scalar projection of $\mathbf{v}$ onto $\mathbf{r}$). It will be equal to $|\mathbf{v}|$ (up to sign) if and only if $\mathbf{v}$ has no component perpendicular to $\mathbf{r}$ (the opposite extreme from the circular motion mentioned earlier). This condition will hold if $\mathbf{r}$ always points in the same direction — in other words, if the hypotenuse of the triangle always has the same slope. In the standard textbook problems with uniform vehicle speeds, it is easy to see that this is true if and only if *the two cars depart from (or arrive at) the right-angled corner at the same time*. (In this case one has *similar triangles* at all times; in other cases the triangle collapses to a line segment whenever one car is at the corner.) This was so in the first problem, but not in the second one.

(In the more traditional elementary notation, where x and y are the lengths of the sides of the right triangle, $\frac{d}{dt}|\mathbf{r}|$, quantity (2) or (5), is

$$\frac{1}{\sqrt{x^2+y^2}}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right), \tag{6}$$

while $|\mathbf{v}|$, quantity (4), is

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \tag{7}$$

These are equal precisely when the vectors $\langle x, \pm y\rangle$ and $\langle \frac{dx}{dt}, \pm\frac{dy}{dt}\rangle$ are proportional; and these vectors can be identified, up to sign, with $\mathbf{r}$ and $\mathbf{v}$. The annoying signs depend on the orientation of the triangle in the plane (which is different in our two example problems).)

Mystery 3: Now that we understand why the two numbers, 78.0 and 78.1025, are different, why are they *so very close*? The angle θ between the two crucial vectors, $-\mathbf{r} = \langle 0.3, 0.4\rangle$ and $\mathbf{v} = \langle 50, 60\rangle$, is rather small (in this particular problem, at least). The parallel component of $\mathbf{v}$, quantity (5), is $-|\mathbf{v}|\cos\theta$. The difference between $\cos\theta$ and 1 is of *second order* in the small quantity θ: $\cos\theta \approx 1 - \frac{\theta^2}{2}$. That's why it doesn't show up until the third decimal place.

Finally, it should be noted that two students solved the quiz problem by a correct vectorial method (not the standard related-rates method). They calculated the two acute angles in Figure 2 at the instant in question to be 53.13° and 36.87°. Hence they were able to find the projection of each velocity vector onto the hypotenuse direction, and add them:

$$50\cos 53.13 + 60\cos 36.87 = 78.0 \text{ mi/h}.$$

Acknowledgments. The students who made interesting observations are Jonathan Blue Bradley, Joshua Merriam, and Kurt Wenger. The Foundation Coalition and the writing of the paper are supported by NSF Grant No. CFDA 47.041 to Texas A&M University.

References

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