WHAT WE SHOULD HAVE LEARNED FROM G. H. HARDY ABOUT QUANTUM FIELD THEORY UNDER EXTERNAL CONDITIONS

S. A. FULLING

Mathematics Department, Texas A&M University,
College Station, TX 77843, USA
E-mail: fulling@math.tamu.edu

1 Reminder of the Phenomena

To establish the setting of this presentation, consider as an example the classic problem of a scalar field in a box. More precisely, consider a "Casimir slab" of width L in 4-dimensional space-time, with the energy-momentum tensor associated with minimal gravitational coupling. With the spatial operator in the field equation, $H = -\nabla^2$, several quantities are associated:

1.1 Vacuum energy

In the 1970s considerable attention was directed to defining and calculating the local density of energy in such situations. [2] It was found that there are two contributions. First, there is a constant energy density throughout the space, proportional to L^{-4}. It is variously described as being related to the finite size of the box, to the discreteness of the spectrum of normal modes, and to the existence of a closed geodesic (or classical path) of length $2L$. This is the scalar analog of the classic Casimir effect. It persists in a closed universe without boundary. Second, there is a divergent distribution of energy clinging to the walls: $T_{00}(x) \propto x^{-4}$, when x is the distance to the nearest wall. It persists in an infinite space with only one wall. We may say that this effect is caused by the existence of the boundary, the spatial inhomogeneity near the boundary of the set of mode functions, and the length $(2x)$ of a path that reflects from the boundary and returns to the observation point. Therefore, with some exaggeration in the eyes of an experimentalist, we can say: By observing vacuum energy *locally*, we can tell how big the world is (L) and how far we are from its edge (x).

1.2 The heat kernel expansion

Expanding heat kernels has long been a favorite industry of many, myself included. For any positive operator H, the heat kernel $K(t,x,y) \equiv e^{-tH}(x,y)$ is the Green function such that $u(x) = \int K(t,x,y)f(y)\,dy$ solves the initial-value problem $-\frac{\partial u}{\partial t} = Hu$, $u(0,x) = f(x)$. It is well known that K has an asymptotic expansion

$$K(t,x,x) \sim (4\pi t)^{-m/2}\left[1 + \sum_{n=1}^{\infty} a_n(x)t^n\right], \tag{1}$$

(m = spatial dimension) where $a_n(x)$ is a *local* functional of the curvature, etc. (covariant functionals of the coefficient functions in the seond-order differential operator H) at x. In particular, a_n is identically zero inside our box (where $H = -\nabla^2$). By studying the heat kernel expansion, you will never discover the Casimir effect!

1.3 Schrödinger and Schwinger–DeWitt kernels

The function $U(t,x,y) \equiv e^{-itH}(x,y)$ is obtained formally by replacing t in the heat kernel by it. When H is an elliptic operator, a quantum-mechanical Hamiltonian, U is the propagator that solves the time-dependent Schrödinger equation. When H is a hyperbolic operator and t a fictitious proper time, U is the Schwinger–DeWitt kernel used in renormalization of quantum field theories. The rotation of the t coordinate in Eq. (1) is algebraically trivial, but the resulting expansion is, in general, invalid if taken literally! This is most easily seen by letting our space be the half-space $\mathbf{R}_+$, for which the problem can be solved exactly by the method of images:

$$U(t,x,y) \sim (4\pi it)^{-1/2}\left[e^{i|x-y|^2/4t} - e^{i|x+y|^2/4t}\right]. \tag{2}$$

Passing to the diagonal, we have

$$U(t,x,x) \sim (4\pi it)^{-1/2}\left[1 - e^{ix^2/t}\right], \tag{3}$$

and we see that the reflection term is exactly as large as the "main" term, in blatant contradiction to the alleged asymptotic expansion (1) (which in this case consists just of the main term and an implied error term vanishing faster than any power of t). Does this mean that the Schwinger–DeWitt series, to which so many graduate students have devoted their thesis years, is nothing but a snare and a delusion? Heaven forbid! The information in that series is meaningful when used correctly (for example, in renormalization theory). In fact, the second term in Eq. (3), although large, is rapidly oscillatory, and consequently the series is indeed valid in various distributional senses, as I shall partially explain below.

1.4 The cylinder kernel

The Green function $T(t,x,y) \equiv e^{-t\sqrt{H}}(x,y)$ solves the (elliptic) boundary-value problem $\frac{\partial^2 u}{\partial t^2} = Hu$, $u(0,x) = f(x)$, $u(t,x) \to 0$ as $t \to +\infty$ (i.e., in a semi-infinite cylinder with our spatial manifold as its base) by $u(x) = \int T(t,x,y) f(y)\, dy$. The cylinder kernel T shares many of the properties of vacuum energy ($\langle T_{00}\rangle$) and the latter's progenitor, the Wightman two-point function ($W(t,x,y) \equiv \langle \phi(t,x)\phi(0,y)\rangle$), a Green function for the wave equation $\frac{\partial^2 u}{\partial t^2} = -Hu$; however, T is technically simpler in several ways. Its study therefore deserves our attention, even though it has no direct physical interpretation with t as a time coordinate. For an example we once again consider one space dimension, where $H = -\frac{\partial^2}{\partial x^2}$ (so we're dealing with very classical Green functions for the two-dimensional Laplace equation). If the spatial manifold is the entire real line, the kernel is

$$T_0(t,x,y) = \frac{t}{\pi} \frac{1}{(x-y)^2 + t^2}\,, \tag{4}$$

whereas if the space is $\mathbf{R}_+$, the problem can again be solved by images:

$$T_+(t,x,y) = \frac{t}{\pi} \left[\frac{1}{(x-y)^2 + t^2} - \frac{1}{(x+y)^2 + t^2} \right]. \tag{5}$$

Passing to the diagonal and making a Taylor expansion, we get

$$T_+(t,x,x) = \frac{1}{\pi t} \left[1 - \frac{t^2}{(2x)^2} + \cdots \right]. \tag{6}$$

Thus we see that the asymptotic expansion of the cylinder kernel does probe x (and also L, in a finite universe), as the vacuum energy does. On the other hand, $\langle T_{00}\rangle$ and W share some of the delicate analytical complications of U, whereas T is about as well-behaved as K.

1.5 Summary and synopsis

The four Green functions T, U, K, W demonstrate two distinctions (Table 1): that between local and global dependence on the geometry, and that between pointwise and distributional validity of their asymptotic expansions (and, as we'll see, of their eigenfunction expansions also).

Table 1: Asymptotic properties of Green functions.

	Pointwise	Distributional
Local	K	U
Global	T	W

The main points I wish to make are:

- As already remarked, the small-t expansion of T contains nonlocal information not present in the corresponding expansion of K.
- Nevertheless, both these expansions are determined by the high-energy asymptotic behavior of the density of states, or, more generally, the spectral measures, of the operator H.
- The detailed relationships among all these asymptotic developments are, at root, not a matter of quantum field theory, nor even of partial differential equations or operators in Hilbert space. They are instances of some classical theory on the summability of infinite series and integrals, developed circa 1915. [9,10]

2 Spectral Densities

The Green functions have spectral expansions in terms of the eigenfunctions of H. If the manifold is $\mathbf{R}$ and $H = -\frac{\partial^2}{\partial x^2}$, the heat, cylinder, and Wightman kernels are

$$K_0(t,x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-tk^2}\, e^{ikx} e^{-iky}, \tag{7}$$

$$T_0(t,x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-t|k|}\, e^{ikx} e^{-iky}, \tag{8}$$

$$W_0(t,x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dk\, \frac{e^{-it|k|}}{|k|}\, e^{ikx} e^{-iky}. \tag{9}$$

On the other hand, we can vary the space, or the operator; for instance, the heat kernel for $\mathbf{R}_+$ is

$$K_+(t,x,y) = \frac{2}{\pi}\int_0^{\infty} d\omega\, e^{-t\omega^2}\, \sin\omega x \sin\omega y, \tag{10}$$

while that for a box of length L is

$$\begin{aligned} K_L(t,x,y) &= \frac{2}{L}\sum_{n=1}^{\infty} e^{-t(n\pi/L)^2}\,\sin\frac{n\pi x}{L}\sin\frac{n\pi y}{L} \\ &= \frac{2}{L}\int_0^\infty d\omega\, e^{-t\omega^2}\sum_{n=1}^{\infty}\delta\left(\omega-\frac{n\pi}{L}\right)\sin\omega x\sin\omega y. \end{aligned} \tag{11}$$

In general, each object can be written in the schematic form

$$\int_0^\infty g(t\omega)\,dE(\omega,x,y), \tag{12}$$

where dE is integration with respect to the spectral measure of a given H (on a given manifold) and g is the kernel of a certain integral transform (Laplace, Fourier, etc.) defining the Green function in question. (To fit some of the kernels into the mold (12), it is necessary to redefine some variables, for example replacing t by $\sqrt{t}$.)

Spectral measures or densities are in a sense more fundamental and more directly relevant than Green functions, since *all* functions of H can be expressed immediately in terms of them. However, the kernels (especially K and T) are more accessible to calculation and analytical investigation.

3 Riesz–Cesàro Means

Eq. (12) is an instance of the general structure

$$f(\lambda) \equiv \int_0^\lambda a(\sigma)\,d\mu(\sigma), \tag{13}$$

where μ is some measure, such as $E(\sigma,x,y)$. (For a totally continuous spectrum, $d\mu(\sigma)$ equals $\mu'(\sigma)\,d\sigma$ where μ' is a function, which we call the spectral density. For a totally discrete spectrum, $d\mu(\sigma)$ is of the form $\sum_n c_n\delta(\sigma-\sigma_n)\,d\sigma$. If $E(\sigma,x,x)$ is integrated over a compact manifold, then the corresponding $\mu'(\sigma)$ becomes the density of states, and $\mu(\lambda)$ becomes the counting function — the number of eigenvalues less than λ.) If $a=1$, then f is μ itself. If $a=g(t\sigma)$ and $\lambda\to\infty$, then f is one of the kernels previously discussed.

Derivatives of negative order of f are defined as iterated indefinite integrals, which can be represented as single integrals:

$$\begin{aligned} \partial_\lambda^{-\alpha} f(\lambda) &\equiv \int_0^\lambda d\sigma_1\cdots\int_0^{\sigma_{\alpha-1}} d\sigma_\alpha\, f(\sigma_\alpha) \\ &= \frac{1}{\alpha!}\int_0^\lambda (\lambda-\sigma)^\alpha\,df(\sigma). \end{aligned} \tag{14}$$

The *Riesz–Cesàro means* of f are defined from these by a change of normalization:

$$R^\alpha_\lambda f(\lambda) \equiv \alpha!\,\lambda^{-\alpha}\partial_\lambda^{-\alpha} f(\lambda) = \int_0^\lambda \left(1-\frac{\sigma}{\lambda}\right)^\alpha df(\sigma). \tag{15}$$

If the limit of $f(\lambda)$ as λ approaches ∞ exists, then $R^\alpha_\lambda f(\lambda)$ also approaches that limit (though perhaps more slowly). On the other hand, $R^\alpha_\lambda f(\lambda)$ may converge at infinity when $f(\lambda)$ does not. In that case, $R^\alpha_\lambda f(\infty)$ serves to *define* $f(\infty)$. This extended notion of summability of an infinite integral generalizes the summation of classically divergent Fourier series by Cesàro means (the averages of the first N partial sums).

4 Riesz Means with Respect to Different Variables

To this point our notation has been rather loose, so as to discuss all the various kernels in a unified, schematic framework without propounding cumbersome definitions. Now it is necessary to tighten up. Henceforth λ will denote the eigenvalue parameter of our second-order elliptic differential operator H, and ω will denote its square root, the frequency parameter. (On the real line, where the prototype eigenfunction is $(2\pi)^{-1/2}e^{ikx}$, we have $\lambda = k^2$ and $\omega = |k|$.) Thus the basic relations are

$$\lambda = \omega^2, \quad d\lambda = 2\omega\, d\omega. \tag{16}$$

This seeming triviality of calculus has surprising impact. For $\alpha > 0$ the Riesz means $R^\alpha_\lambda\mu$ and $R^\alpha_\omega\mu$ are not the same things. In fact, one can calculate $R^\alpha_\lambda\mu$ in terms of the $R^\beta_\omega\mu$ with $\beta \le \alpha$, and vice versa.[9,11] The striking result is that *the $\lambda \to \infty$ behavior of the $R_\lambda(\mu)$ is completely determined by that of the $R_\omega(\mu)$, but the converse is false; the asymptotics of the $R_\omega(\mu)$ depend on integrals of the $R_\lambda(\mu)$ over all λ, not just on their asymptotic values.* That is, in the passage from λ to ω, new terms in the asymptotic development of $R^\alpha_\omega\mu$ arise as undeterminable constants of integration; in going from ω to λ, there are "magical cancellations" that cause these terms to disappear from the asymptotics of $R^\alpha_\lambda\mu$.

In more detail: The rigorous asymptotic approximation

$$R^\alpha_\lambda\mu(\lambda) = \sum_{s=0}^{\alpha} a_{\alpha s}\lambda^{(m-s)/2} + O(\lambda^{(m-\alpha-1)/2}) \tag{17}$$

can be shown, as can the corresponding asymptotic approximation

$$R^\alpha_\omega\mu(\omega) = \sum_{s=0}^{\alpha} c_{\alpha s}\omega^{m-s} + \sum_{\substack{s=m+1\\ s-m\ \mathrm{odd}}}^{\alpha} d_{\alpha s}\omega^{m-s}\ln\omega + O(\omega^{m-\alpha-1}\ln\omega), \tag{18}$$

where

- $c_{\alpha s} =$ constant $\times\, a_{\alpha s}$ if $s \le m$ or $s - m$ is even;
- $c_{\alpha s}$ is undetermined by $a_{\alpha s}$ if $s > m$ and $s - m$ is odd;
- $d_{\alpha s} =$ constant $\times\, a_{\alpha s}$ if $s > m$ and $s - m$ is odd.

The constants are combinatorial structures (involving mostly gamma functions) whose detailed form is not important now.

5 Some Conclusions

We can now relate these facts to the spectral theory and physics discussed earlier.

1. From the foregoing it is clear that the ω expansion coefficients of μ (Eq. (18)) contain more information than the λ expansion coefficients (Eq. (17)). In the application to quantum field theory it turns out that the new data are "global" and the old ones are "local". Indeed, the λ coefficients are in one-to-one correspondence with the terms in the heat kernel expansion, while the whole list of ω coefficients correspond to the terms in the expansion of the cylinder kernel (see below).

2. The Riesz means of $\mu(\lambda) \equiv E(\lambda, x, y)$ give rigorous meaning to the formal high-frequency expansions obtained by formally inverting the asymptotics of K and T (i.e., applying the inverse of the appropriate integral transform (12) to the asymptotic expansion term-by-term). The existence of discrete spectra (which makes the exact μ a step function) shows that those high-frequency expansions cannot be literally asymptotic. That they nevertheless can be given a precise meaning in terms of some averaging procedure was pointed out by Brownell [1] in the '50s; the connection with Riesz means was drawn by Hörmander. [11]

3. Similarly, the concepts of Riesz–Cesàro summation tighten up the convergence of the spectral expansions of U and W, and also the $t \to 0$ asymptotics of those distributions.

6 Kernel Expansions from Riesz Means

When f and μ are related by Eq. (13), a Riesz mean of f can be expressed in terms of the corresponding Riesz mean of μ:

$$R_\lambda^\alpha f(\lambda) = a(\lambda) R_\lambda^\alpha \mu(\lambda) + \lambda^{-\alpha} \sum_{j=1}^{\alpha+1} \frac{(-1)^j}{(j-1)!} \binom{\alpha+1}{j} \int_0^\lambda d\sigma\, (\lambda-\sigma)^{j-1} \sigma^\alpha \partial_\sigma^j a(\sigma) R_\sigma^\alpha \mu(\sigma). \quad (19)$$

In the case of Eq. (12), Eq. (19) relates the asymptotics of the spectrum to the asymptotics of the kernels. Eq. (17) corresponds to

$$K(t) \equiv \int_0^\infty e^{-\lambda t}\, d\mu(\lambda) \sim \sum_{s=0}^\infty b_s t^{(-m+s)/2}, \quad (20)$$

where

$$b_s = \frac{\Gamma((m+s)/2+1)}{\Gamma(s+1)}\, a_{ss}\,. \quad (21)$$

Eq. (18) corresponds to

$$T(t) \equiv \int_0^\infty e^{-\omega t}\, d\mu \sim \sum_{s=0}^\infty e_s t^{-m+s} + \sum_{\substack{s=m+1 \\ s-m \text{ odd}}}^\infty f_s t^{-m+s} \ln t, \quad (22)$$

where e_s and f_s are related to c_{ss} and d_{ss} in much the same way as b_s is related to a_{ss}.[7] (One can concentrate on the spectral coefficients with $\alpha = s$, because those with $\alpha \neq s$ contain no additional information.)

One has to wonder whether there is something more profound and general in the Riesz–Hardy theory of spectral asymptotics with respect to different variables, waiting to be discovered and applied. We have concentrated on the counterpoint of $\lambda^{1/2}$ versus λ, where the extra data associated with the former is geometrically global in a spectral problem, that associated with the latter strictly local. This geometrical significance, of course, we could not have learned from Hardy and Riesz; it has to be observed in the application. The entire development (with different exponents in the series) could be repeated for λ versus λ^2 and applied to the spectral asymptotics of H versus H^2 (a fourth-order differential operator). Examination of the formulas of Gilkey[8] for the heat kernel expansion of a fourth-order operator verify that, indeed, there are terms in the asymptotics of H that disappear from the asymptotics of H^2 (because the combinatorial coefficients multiplying them vanish). In this case, however, both series are totally local, so there does not appear to be a qualitative difference between the two sets of spectral invariants. Is the effect, therefore, a mere curiosity in that case, or does it have a deeper significance? In the other direction, one can expect that the Riesz means with respect to $\lambda^{1/3}$ (and hence, the Green function for the operator $e^{-tH^{1/3}}$) contain new information different from that contained in T; and that $\lambda^{1/6}$ subsumes them both; and so on. Should we care about this new spectral data? Finally, as a wild speculation, might the investigation of these Riesz means with respect to arbitrarily extreme fractional powers of the eigenparameter provide an easy way to get access to at least a part of the detailed spectral information that is encoded in the lengths of the closed geodesics (or periodic classical orbits) of H?

7 Distributional/Cesàro Theory of Spectral Expansions

Finally, I shall summarize some more technical mathematics surrounding the Green function integrals (12):[4]

- The locality of the asymptotics of G hinges on the regularity of g at 0. For example, the different behavior of K and T arises from the distinction between e^{-x} (which has a Taylor series at 0) and $e^{-\sqrt{x}}$ (which does not).
- The pointwise versus distributional/Cesáro validity of the asymptotics of G hinges on the behavior of g at infinity. Thus, the difference between K and U arises from the contrast between e^{-x} and e^{-ix}.
- The large-λ asymptotics of E is valid (and local) in a distributional/Cesà-ro sense (but usually not pointwise).

It remains to explain the term "distributional/Cesàro". The point is that Riesz–Cesàro limits (defined after Eq. (15)) are equivalent to distributional limits in t, or in λ or ω, under scaling.[3] *If f belongs to the distribution space $\mathcal{D}'(\mathbf{R}_+)$, then for α in the interval $(-k-2,-k-1)$, $f(t) = O(t^\alpha)$ in the Cesàro sense if and only if there exist "moments" $\mu_0,\ldots,\mu_k$ such that*

$$f(\sigma t) = \sum_{j=0}^{k} \frac{(-1)^j \mu_j \delta^{(j)}(t)}{j!\,\sigma^{j+1}} + O(\sigma^\alpha) \tag{23}$$

in the topology of $\mathcal{D}'$, as $\sigma \to +\infty$ (with a similar but more complicated statement when α is an integer). This theorem has applications to both $G(t)$ and $E(\lambda)$ in the role of f.[4,5]

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References

1. F. H. Brownell, Extended asymptotic eigenvalue distributions for bounded domains in n-space, *J. Math. Mech.* **6**, 119 (1957).
2. B. S. DeWitt, Quantum field theory in curved spacetime, *Phys. Reports* **19**, 295 (1975).
3. R. Estrada, The Cesàro behavior of distributions, *Proc. Roy. Soc. London A*, to appear.
4. R. Estrada and S. A. Fulling, Distributional asymptotic expansions of spectral functions and of the associated Green kernels, submitted to *Electronic Journal of Differential Equations* (`funct-an/9710003`).
5. R. Estrada, J. M. Gracia-Bondía, and J. C. Várilly, On summability of distributions and spectral geometry, *Commun. Math. Phys.* **191**, 219 (1998)
6. R. Estrada and R. P. Kanwal, *Asymptotic Analysis: A Distributional Approach* (Birkhäuser, Boston, 1994).
7. S. A. Fulling, with appendix by R. A. Gustafson, Some properties of Riesz means and spectral expansions, submitted to *Electronic Journal of Differential Equations* (`physics/9710006`).
8. P. B. Gilkey, The spectral geometry of the higher order Laplacian, *Duke Math. J.* **47**, 511 (1980).
9. G. H. Hardy, The second theorem of consistency for summable series, *Proc. London Math. Soc.* **15**, 72 (1916).
10. G. H. Hardy and M. Riesz, *The General Theory of Dirichlet's Series* (Cambridge U. Press, Cambridge, 1915).
11. L. Hörmander, The spectral function of an elliptic operator, *Acta Math.* **121**, 193 (1968).