Lecture 6

3.5

If you are having trouble with this section, I recommend you do problem 1 to see how things work out in practice.

3.5.2: Since $\sigma^2 = \sigma\sigma$, if σ is written as a product of n transpositions, σ^2 is the product of $2n$ transpositions, whence is an even permutation.

3.5.3: Let's start with the hint: $(23)(12)(23) = (13)$ Similarly, $(i{+}1\ i{+}2)(i\ i{+}1)(i+1 i+2) = (i\ i+2)$. Assume we have shown that $(i\ i+j) \in <(i\ i+1)>$. Then $(i+j\ i+j+1)(i\ i+j)(i+j\ i+j+1) = (i\ i+j+1)$ By induction, all transpositions are in $<(i\ i+1)>$, so $S_n =< (i\ i+1)>$.

5.1

Just as we went from the plane in calculus 2 to three space in calculus 3 and $\mathbb{R}^n$ in linear algebra, this section makes formal what we've been doing with two components and expands it to as many components as we like. The idea of projection is the same as was used in calculus, just with any group now.

A p-group is a group whose order is a power of p. There are non-abelian p-groups, but we are going to study only abelian ones. The basic building block is then $\mathbb{Z}_{p^n}$, called an elementary abelian group of order p^n. Direct products, as studied in this section, are the main way to put them together.

5.1.2: Let $G_1, G_2, \ldots, G_n$ be groups and let $G = G_1 \times G_2 \times \cdots \times G_n$. Let I be a proper, nonempty subset of $\{1, 2, \ldots, n\}$, and let $J = \{1, 2, \ldots, n\} - I$, the other indices. Define $G_I = \{(g_1, g_2, \ldots, g_n) | g_j = 1_{G_j}\ \forall j \in J\}$.

5.1.2a: Define $\phi : G_I \to \Pi_{i\in I} G_i$ by $\phi((g_1, \ldots, g_n)) = \Pi_{i\in I}\pi_i(g_1, \ldots, g_n)$. Since each π_i is a homomorphism, ϕ is a homomorphism. If $\phi(g_1, \ldots, g_n) = \Pi_{i\in I} 1_{G_i}$, then $g_i = 1_{G_i}\ \forall i \in I$. Since $g_j = 1_{G_j}\ \forall j \notin I$, $(g_1, \ldots, g_n) = 1_G$. Therefore, ϕ is an injection. Since ϕ is a surjection by definition of G_I, ϕ is an isomorphism.

5.1.2b: Let $g = (g_1, \ldots, g_n) \in G_I$, and let $h = (h_1, \ldots, h_n) \in G$. Since each G_i is a group, $hgh^{-1} \in G_I$ and G_I is normal in G. Define $\rho : G \to G_J$ by $\rho((g_1, ..., g_n) = (h_1, ..., h_n)$ where $h_j = g_j \ \forall j \in J$ and $h_i = 1_{G_i} \ \forall i \in I$. By definition ρ is surjective.

$$\rho((g_1, ..., g_n)(p_1, ..., p_n)) = \rho(g_1 p_1, \ldots, g_n p_n) = \begin{cases} g_j p_j & \text{if } j \in J; \\ 1_{G_i} & \text{otherwise} \end{cases}$$

$$= (\begin{cases} g_j & \text{if } j \in J \\ 1_{G_i} & \text{otherwise.} \end{cases})(\begin{cases} p_j & \text{if } j \in J \\ 1_{G_i} & \text{otherwise} \end{cases}) = \rho(g_1, ..., g_n)\rho(p_1, ..., p_n).$$

Thus ρ is an epimorphism. $(g_1, ..., g_n) \in \ker \rho$ if and only if $\rho(g_1, ..., g_n) = \Pi_{j \in J} 1_{G_j}$ if and only if $(g_1, ..., g_n) \in G_I$. By the first isomorphism theorem $G/G_I \cong G_J$.

5.1.2c: Note that $G_I \cap G_J = 1_G$ since $I \cap J = \emptyset$. Therefore, every element of G can be written uniquely as $x_i y_j$ where $x_i \in G_I$ and $y_J \in G_J$. Thus $\sigma : G \to G_I \times G_J$ by $\sigma(x_i y_j) = (x_i, y_j)$. By part b and $G_I \cap G_J = 1_G$, $x_i y_j = y_j x_i$. Thus $\sigma((x_i y_j)(x'_i y'_j)) = \sigma(x_i x'_i y_i y'_i) = (x_i x'_i, y_i y'_i) = (x_i, y_i)(x'_i, y'_i) = \sigma((x_i y_i))\sigma((x'_i y'_i))$. Therefore σ is a homomorphism. If $(x_i y_i) \in \ker \sigma$, then $(x_i, y_i) = (1_{G_I}, 1_{G_J})$ and $x_i = 1_{G_I}, \ y_j = 1_{G_J}$, whence $x_i y_j = 1_G$ and σ is an injection. By definition of σ, σ is onto, so σ is an isomorphism as claimed.

5.1.3: Let I and K be any disjoint, non-empty subsets of $\{1, 2, ..., n\}$. Let G_I and G_K be as defined in problem 5.1.2. By 5.1.2b, G_I and G_K are both normal in G. Since $xyx^{-1}y^{-1} \in G_I \cap G_K = \{1_G\}$, $xy = yx$.

5.1.5. Let $H =< (i, \overline{1}) >= \{(i, \overline{1}), (-1, \overline{2}), (-i, \overline{3}), (1, \overline{0})$. Then
$< j, \overline{0} >< i, \overline{1} >< -j, \overline{0} >=< j, \overline{0} >< -k, \overline{1} >=< -i, \overline{1} >\notin H$. Therefore, H is not normal in $Q_8 \times \mathbb{Z}_4$.