Second-Order Invariant Domain Preserving Approximation Of The Compressible Navier–Stokes Equations

J.-L. Guermond

Department of Mathematics
Texas A&M University

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Collaborators and acknowledgments

This work done in collaboration with:
Martin Kronbichler (Technical University of Munich, Germany)
Matthias Maier (Dept. Math., TAMU, TX)
Bojan Popov (co-PI, Dept. Math., TAMU, TX)
Ignacio Tomas (Sandia National Laboratories, NM)

Support:

AFOSR AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
ARL
Lawrence Livermore National Laboratory
NSF

Outline

Background and objectives

Long term objectives of the research program

Objectives

Develop numerical techniques for solving nonlinear conservation equations (PDEs with dominant hyperbolic features) with the following guaranteed/certified properties:

- Be invariant domain preserving.
- Be asymptotic preserving (or well-balanced).
- Be (somewhat) discretization agnostic.
- Satisfy some entropy inequalities.

Key challenge: The above properties must be guaranteed/certified.

Why?

Numerical methods with certified properties

- are robust.
- can be used in confidence with very little know-how from the user.
- do not involve numerical parameter "to learn."

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Fields of applications

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- Compressible Euler equations (transonic to hypersonic)
- Euler-Poisson equations
- Compressible Navier-Stokes
- Gray radiation hydrodynamics
- Ideal magnetohydrodynamics
- Radiation transport
- Multi-material fluid flows
- Shallow water equations

Results established so far

Some results established so far

- Asymptotic and invariant domain preserving approximation of radiation transport. (First-order in streaming regime, second-order in diffusion regime). **Guermond, Popov, Ragusa (2020)**

Robustness is guaranteed for all the above methods up to second-order accuracy.

Current work

Current work

- Demonstration of extreme scalability of the proposed algorithms for the compressible Euler equations and other hyperbolic systems using the deal.ii library, **Maier, Kronbichler (2021)**
  - MPI
  - Multithreading
  - SIMD vectorisation
- Invariant domain preserving approximation of Euler equation with tabulated equation of state. **Clayton, G, Popov (2021)**.
- Topic of the today: extension to compressible Navier-Stokes using semi-implicit time stepping
  - Second-order accurate technique that is guaranteed to be invariant domain preserving technique under hyperbolic CFL. **G, Maier, Popov Tomas (2021)**
- Beyond SSP ... explicit and IMEX ... (in preparation).
- Invariant domain preserving approximation for mixed approximation.

Current work: extreme scalability

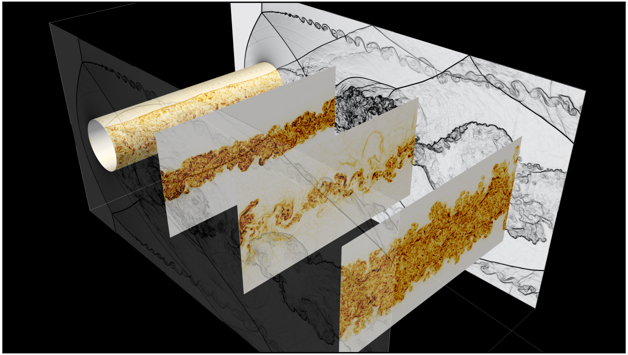

Figure: Continuous $\mathbb{Q}_1$ elements, 1.817B grid points, **Maier, Kronbichler (2021)**

Outline

Compressible Navier-Stokes

Objectives

- Conservation equation for $\mathbf{u} = (\rho, \mathbf{m}, E)$:

$$\begin{aligned} &\partial_t \rho + \nabla\cdot(\mathbf{v}\rho) = 0, \\ &\partial_t \mathbf{m} + \nabla\cdot(\mathbf{v}\otimes\mathbf{m} + p(\mathbf{u})\mathbb{I} - \mathbb{s}(\mathbf{v})) = \mathbf{f}, \\ &\partial_t E + \nabla\cdot(\mathbf{v}(E + p(\mathbf{u})) - \mathbb{s}(\mathbf{v})\mathbf{v} + \mathbf{k}(\mathbf{u})) = \mathbf{f}\cdot\mathbf{v}. \end{aligned}$$

- $+$ BC and Initial data.
- Fluid is Newtonian and heat-flux follows Fourier's law:

$$\mathbb{s}(\mathbf{v}) = 2\mu\mathbb{e}(\mathbf{v}) + (\lambda - \tfrac{2}{3}\mu)\nabla\cdot\mathbf{v}\mathbb{I}, \qquad \mathbb{e}(\mathbf{v}) = \tfrac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathsf{T}}),$$
$$\mathbf{k}(\mathbf{u}) = -c_v^{-1}\kappa\nabla e,$$

with $\mu > 0$, $\lambda \geq 0$, and $\kappa > 0$.

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Objectives

- Two invariant domains can be identified:

$$\mathcal{A} := \{\mathbf{u} \mid \rho > 0,\ e(\mathbf{u}) > 0,\ s(\mathbf{u}) > s_{\min}\}, \qquad \text{For Euler}$$
$$\mathcal{B} := \{\mathbf{u} \mid \rho > 0,\ e(\mathbf{u}) > 0\}, \qquad \text{For NS}$$

Difficulties: conflicting invariant sets and conflicting variables

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- Which invariant domain to preserve?
  - Minimum entropy principle is true for Euler.
  - Minimum entropy principle is false for NS.
- Which variable should be used?
  - "Right variable" for Euler is $\mathbf{u} = (\rho, \mathbf{m}, E)$ (conserved variables).
  - "Right variable" for NS is $(\rho, \mathbf{v}, e)$ (primitive variables).
  - Some advocate "entropy variable" and "entropy stability". Why?
- How to do the explicit-implicit time stepping?
  - Most "IMEX" methods cannot make the diference between conserved and primitive variables.
  - Very few mathematically precise/correct results on the topic: **Zhang & Shu (2017)** with $\Delta t \leq ch^2$.

Our solution

Our solution (an overview)

- Use operator splitting to separate hyperbolic part and parabolic part.
- Hyperbolic operator

$$\begin{aligned} &\partial_t \rho + \nabla\cdot(\mathbf{v}\rho) = 0, \\ &\partial_t \mathbf{m} + \nabla\cdot(\mathbf{v}\otimes\mathbf{m} + p(\mathbf{u})\mathbb{I}) = \mathbf{0}, \\ &\partial_t E + \nabla\cdot(\mathbf{v}(E + p(\mathbf{u})) = 0, \\ &\mathbf{v}\cdot\mathbf{n}_{|\partial D} = 0, \quad \text{or other admissible BC.} \end{aligned}$$

- Parabolic operator

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- But how can it be done properly?

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Our solution

Our solution (an overview)

- Combine the explicit and implicit part using Strang's splitting in some clever way.
- The devil is in the details. Just "invoking" Strang's splitting is wishful thinking.

Our solution

Our solution for the hyperbolic part (an overview)

- Use conserved variables for the hyperbolic part.
- Make the hyperbolic part explicit.
- Invoke the "invariant-domain" technology with "convex limiting" for the explicit hyperbolic part.

Our solution

Our solution for the parabolic part (an overview)

- Use primitive variables for the parabolic part.
- Make the viscous terms implicit (in some clever way).
- Make the implicit algorithm "invariant-domain" preserving up to second-order in time.

Comments about IMEX vs. Strang

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- We are not aware (yet?) of the existence of any second-order IMEX technique that is invariant domain preserving for the NS equations and that is not somewhat equivalent to Strang splitting or a variation thereof.
- There is a very fundamental difficulty here: How to go beyond second-order and guarantee some "invariant-domain" preserving properties?

Hyperbolic step

- The hyperbolic step consists of solving

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Overview of solution strategy

Three step strategy

(i) Construct low-order invariant domain preserving method (GMS-GV).

(ii) Construct a high-order scheme that may not be invariant domain preserving (entropy viscosity commutator).

(iii) Apply convex limiting with correct bounds inferred from low-order solution to get a high-order method that is invariant domain preserving.

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Brief description of the method

- Sequence of shape-regular meshes $(\mathcal{T}_h)_{h>0}$.
- Scalar-valued finite element space $P(\mathcal{T}_h)$ with basis functions $\{\varphi_i\}_{i\in\mathcal{V}}$. (Assume $P(\mathcal{T}_h) \subset C^0(\overline{D};\mathbb{R})$ for simplicity.)
- Vector-valued approximation space $\mathbf{P}(\mathcal{T}_h) := (P(\mathcal{T}_h))^{d+2}$. ($\Leftarrow$ current weakness)

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(I) Hyperbolic step; GMS-GV scheme

- Set

$$\mathbf{c}_{ij} := \int_D \varphi_i \nabla \varphi_j \,\mathrm{d}x, \quad \mathbf{n}_{ij} := \frac{\mathbf{c}_{ij}}{\|\mathbf{c}_{ij}\|_{\ell^2}},$$
$$m_i := \int_D \varphi_i \,\mathrm{d}x.$$

  (these are the only mesh-dependent coefficients of the method!)

- Let Δt be some time step.
- Let $\mathbf{u}_h(\cdot, t^n)$ approximated by $\sum_{i\in\mathcal{V}} \mathbf{U}_i^n \varphi_i$, $\mathbf{U}_i^n \in \mathbf{P}(\mathcal{T}_h) \cap \mathcal{A}$ (some current admissible state).
- Compute low-order update $\mathbf{U}_i^{\mathrm{L},n+1}$

$$\frac{m_i}{\Delta t}(\mathbf{U}_i^{\mathrm{L},n+1} - \mathbf{U}_i^n) + \sum_{j\in\mathcal{I}(i)} \mathbb{f}(\mathbf{U}_j^n)\mathbf{c}_{ij} - \sum_{j\in\mathcal{I}(i)\setminus\{i\}} d_{ij}^{\mathrm{L},n}(\mathbf{U}_j^n - \mathbf{U}_i^n) = \mathbf{0}.$$

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SKIP BORING DETAILS

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Theorem (GMS-GV, Local invariance, JLG+BP (2016-2018))

- *Let* $n \geq 0$ *and let* $i \in \mathcal{V}$.
- *Assume that* Δt *is small enough so that* $1 - 4\Delta t \frac{\sum_{j\in\mathcal{I}(i)\setminus\{i\}} d_{ij}^{\mathrm{L},n}}{m_i} \geq 0$.
- *Let* $\mathcal{B} \subset \mathcal{A}$ *be a convex invariant set*
- *Then*

$$(\mathbf{U}_j^n \in \mathcal{B},\ \forall j \in \mathcal{I}(i)) \Longrightarrow (\mathbf{U}_i^{\mathrm{L},n+1} \in \mathcal{B}).$$

- This is the generalization of the maximum principle for any discretization (any mesh), in any space dimension, for any hyperbolic system.
- GMS-GV is a bulletproof scheme. GMS-GV cannot fail.

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Key idea

Reduce the graph viscosity d_{ij}^n as much as possible to be as close as possible to the Galerkin solution (very accurate).

Be careful: do not be too greedy

- Using zero artificial viscosity, $d_{ij}^{\mathrm{H},n} = 0$ may seem to be a good idea (if your world is linear), but it is always a bad idea.
- Using linear stabilization may seem to be a good idea (if your world is linear), but it is not robust w.r.t. entropy inequalities.

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High-order viscosity: Commutator-based entropy viscosities

- Consider an entropy pair $(\eta(\mathbf{v}), \mathbf{F}(\mathbf{v}))$.
- Key idea: measure smoothness of an entropy using the chain rule.

$$\boxed{\nabla\cdot(\mathbf{F}(\mathbf{u})) = (\nabla\eta(\mathbf{u}))^{\mathsf{T}}\nabla\cdot(\mathbb{f}(\mathbf{u}))}$$

- Commutator-based entropy viscosity is defined by setting

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Strategy

- Let $\Psi : \mathcal{B} \to \mathbb{R}$ be a quasiconcave functional (ex: density, internal energy, entropy, ...).
- Assume low-order update satisfies $\Psi(\mathbf{U}_i^{\mathrm{L},n+1}) \geq 0$.
- We want to "limit" the high-order update $\mathbf{U}_i^{\mathrm{H},n+1} \to \mathbf{U}_i^{n+1}$ so that

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Overview of solution strategy

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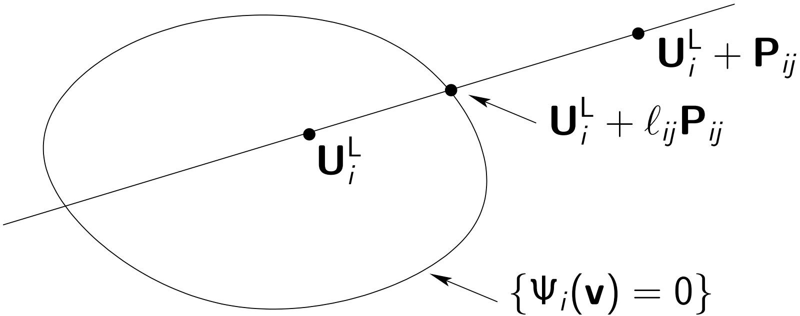
(III) Limiting strategy



Summary of the hyperbolic step

- Let $S_{1h}(t_n + \Delta t, t_n) : \mathbf{P}(\mathcal{T}_h) \to \mathbf{P}(\mathcal{T}_h)$ denote the nonlinear update for the hyperbolic problem described in Guermond, Nazarov, Popov, Tomas, (2018) (2019).

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Parabolic step

$$\begin{aligned}
&\partial_t \rho = 0,\\
&\partial_t \mathbf{m} - \nabla\cdot(\mathbb{s}(\mathbf{v})) = \mathbf{f},\\
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\end{aligned}$$

- Let Δt be some time step.
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Parabolic step: density update

- Density update

$$\varrho_i^{n+1} := \varrho_i^n, \qquad \forall i \in \mathcal{V}.$$

Parabolic step: velocity update

- Introduce bilinear form associated with viscous dissipation,

$$a(\mathbf{v},\mathbf{w}) := \int_D \mathbb{s}(\mathbf{v}){:}\mathbb{e}(\mathbf{w})\,\mathrm{d}x, \qquad \mathbf{v},\mathbf{w} \in \mathbf{H}_0^1(D) := H_0^1(D;\mathbb{R}^d).$$

- Let $\{\mathbf{e}_k\}_{k\in\{1:d\}}$ be the canonical Cartesian basis of $\mathbb{R}^d$. For any $i \in \mathcal{V}$ and $j \in \mathcal{I}(i)$ define $d\times d$ matrix $\mathbb{B}_{ij} \in \mathbb{R}^{d\times d}$ by setting

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$$\mathbf{V}_i^{n+1} := 2\mathbf{V}^{n+\frac{1}{2}} - \mathbf{V}_i^n, \qquad \mathbf{M}_i^{n+1} := \varrho_i^{n+1}\mathbf{V}_i^{n+1}, \qquad \forall i \in \mathcal{V}.$$

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Parabolic step: internal energy update

- Bilinear form associated with the thermal diffusion

$$b(e,w) := c_v^{-1}\kappa \int_D \nabla e \cdot \nabla w \,\mathrm{d}x, \qquad \forall e, w \in H^1(D).$$

- For any $i \in \mathcal{V}$ and $j \in \mathcal{I}(i)$ we set

$$\beta_{ij} := b(\varphi_j, \varphi_i).$$

- We further assume that the acute angle condition holds:

$$\beta_{ij} \leq 0, \quad \forall i \neq j \in \mathcal{V}.$$

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Parabolic step: internal energy update (high-order update)

- Introduce $\mathbf{v}_h^{n+\frac{1}{2}} := \sum_{i\in\mathcal{V}} \mathbf{V}_i^{n+\frac{1}{2}}\varphi_i$ and define

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Caution

No guarantee of positivity of the internal energy here.

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Parabolic step: internal energy update

Solution

- Use backward Euler for low-order internal energy $e_i^{L,n+1}$.
- And use FCT to limit.

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Parabolic step: internal energy update (limiting from below)

Theorem (Positivity and conservation)

Let $\mathbf{U}^n$ be an admissible state. Let $\mathbf{U}^{n+1}$ be the parabolic update. Then, $\mathbf{U}^{n+1}$ is an admissible state, i.e., $\mathbf{U}_i^{n+1} \in \mathcal{B}$ for all $i \in \mathcal{V}$ and all Δt, and the following holds for all $i \in \mathcal{V}$ and all Δt:

$$\varrho_i^{n+1} = \varrho_i^n > 0, \qquad \forall i \in \mathcal{V},$$

$$\min_{j\in\mathcal{V}} \mathrm{e}_j^{n+1} \geq \min_{j\in\mathcal{V}} \mathrm{e}_j^n > 0,$$

$$\sum_{i\in\mathcal{V}} m_i \mathrm{E}_i^{n+1} = \sum_{i\in\mathcal{V}} m_i \mathrm{E}_i^n + \sum_{i\in\mathcal{V}} \Delta t m_i \mathbf{F}_i^{n+\frac{1}{2}} \cdot \mathbf{V}_i^{n+\frac{1}{2}}.$$

Full algorithm

- Let $S_{1h}(t+\Delta t, t) : \mathbf{P}(\mathcal{T}_h) \rightarrow \mathbf{P}(\mathcal{T}_h)$ denote the nonlinear update for the hyperbolic substep from t to $t+\Delta t$.
- Let $S_{2h}(t+\Delta t, t) : \mathbf{P}(\mathcal{T}_h)\times\mathbf{P}(\mathcal{T}_h) \rightarrow \mathbf{P}(\mathcal{T}_h)$ be the nonlinear update for the parabolic substep from t to $t+\Delta t$.
- The update $\mathbf{u}_h^{n+1} \in \mathbf{P}(\mathcal{T}_h)$ is computed as follows:

$$\mathbf{u}_h^{n+1} = S_{1h}(t_n+\Delta t, t_n + \tfrac{1}{2}\Delta t) \circ S_{2h}(t_n+\Delta t, t_n) \circ (S_{1h}(t_n+\tfrac{1}{2}\Delta t, t_n)(\mathbf{u}_h^n), \mathbf{f}_h^{n+\frac{1}{2}}).$$

- Or

$$\mathbf{w}_h^1 := S_{1h}(t_n+\tfrac{1}{2}\Delta t, t_n)(\mathbf{u}_h^n),$$
$$\mathbf{w}_h^2 := S_{2h}(t_n+\Delta t, t_n)(\mathbf{w}_h^1, \mathbf{f}_h^{n+\frac{1}{2}}),$$
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- The update $\mathbf{u}_h^{n+1} \in \mathbf{P}(\mathcal{T}_h)$ is computed as follows:

$$\mathbf{u}_h^{n+1} = S_{1h}(t_n+\Delta t, t_n+\tfrac{1}{2}\Delta t) \circ S_{2h}(t_n+\Delta t, t_n) \circ (S_{1h}(t_n+\tfrac{1}{2}\Delta t, t_n)(\mathbf{u}_h^n), \mathbf{f}_h^{n+\frac{1}{2}}).$$

- Or

$$\mathbf{w}_h^1 := S_{1h}(t_n+\tfrac{1}{2}\Delta t, t_n)(\mathbf{u}_h^n),$$

$$\mathbf{w}_h^2 := S_{2h}(t_n+\Delta t, t_n)(\mathbf{w}_h^1, \mathbf{f}_h^{n+\frac{1}{2}}),$$

$$\mathbf{u}_h^{n+1} := S_{1h}(t_n+\Delta t, t_n+\tfrac{1}{2}\Delta t)(\mathbf{w}_h^2).$$

Full algorithm

- Let $S_{1h}(t+\Delta t, t) : \mathbf{P}(\mathcal{T}_h) \to \mathbf{P}(\mathcal{T}_h)$ denote the nonlinear update for the hyperbolic substep from t to $t+\Delta t$.
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- Or

$$\begin{aligned}
\mathbf{w}_h^1 &:= S_{1h}(t_n+\tfrac{1}{2}\Delta t, t_n)(\mathbf{u}_h^n),\\
\mathbf{w}_h^2 &:= S_{2h}(t_n+\Delta t, t_n)(\mathbf{w}_h^1, \mathbf{f}_h^{n+\frac{1}{2}}),\\
\mathbf{u}_h^{n+1} &:= S_{1h}(t_n+\Delta t, t_n+\tfrac{1}{2}\Delta t)(\mathbf{w}_h^2).
\end{aligned}$$

Main result

Theorem (JLG+BP+MM+IT (2020))

- *Let* $\mathbf{P}(\mathcal{T}_h)$ *be a discrete space a described in* **Guermond-Popov-Tomas (2019)**.
- *Let* $\mathbf{u}_h^n \in \mathbf{P}(\mathcal{T}_h)$ *and* $\mathbf{u}_h^n(\mathbf{x}) \in \mathcal{B}$ *for all* $\mathbf{x}$.
- *Let* $\Delta t \leq \Delta t_0(\mathbf{u}^h)$, *where* $\Delta t_0(\mathbf{u}^h)$ *is the largest hyperbolic time step that makes the algorithm in* **Guermond-Popov-Tomas (2019)** *invariant-domain preserving for the Euler problem.*
- *Let* $\mathbf{u}_h^{n+1} \in \mathbf{P}(\mathcal{T}_h)$ *be computed as above (previous slide).*
- *Then* $\mathbf{u}_h^{n+1}(\mathbf{x}) \in \mathcal{B}$ *for all* $\mathbf{x}$.
- *The algorithm is conservative (global mass and total energy conserved).*

Outline

Numerical illustration

1D convergence tests

- 1D convergence tests. Viscous shockwave. Exact solution by **Becker (1922)**.
- Truncated domain $D = (-1, 1.5)$.
- Consolidated error indicator, $q \in \{1, 2\infty\}$:

$$\delta_q(t) := \frac{\|\rho_h(t) - \rho(t)\|_{L^q(D)}}{\|\rho(t)\|_{L^q(D)}} + \frac{\|\mathbf{m}_h(t) - \mathbf{m}(t)\|_{\mathbf{L}^q(D)}}{\|\mathbf{m}(t)\|_{\mathbf{L}^q(D)}} + \frac{\|E_h(t) - E(t)\|_{L^q(D)}}{\|E(t)\|_{L^q(D)}}.$$

Table: 1D Viscous shockwave (exact solution by Becker (1922)), $\mathbb{P}_1$ meshes. Convergence tests, $t = 3$, CFL $= 0.4$.

I	$\delta_1(t)$	rate	$\delta_2(t)$	rate	$\delta_\infty(t)$	rate
50	5.85E-02	–	3.11E-01	–	8.28E-03	–
100	2.50E-02	1.23	1.91E-01	0.71	2.82E-03	1.55
200	4.83E-03	2.37	3.27E-02	2.54	5.13E-04	2.46
400	1.07E-03	2.17	9.79E-03	1.74	9.32E-05	2.46
800	2.52E-04	2.09	2.29E-03	2.10	2.02E-05	2.21
1600	6.20E-05	2.02	5.76E-04	1.99	4.89E-06	2.05
3200	1.55E-05	2.00	1.46E-04	1.98	1.23E-06	1.99

2D convergence tests

- 1D Viscous shockwave computed in 2D. Exact solution by **Becker (1922)**.
- Truncated domain: $D = (-0.5, 1)\times(0, 1)$.
- Same consolidated error indicator, $q \in \{1, 2\infty\}$ as in 1D.

Table: 2D Viscous schockwave, $\mathbb{P}_1$ nonuniform Delaunay meshes, $t = 3$, CFL $\in \{0.4, 0.9\}$.

CFL	I	$\delta_1(t)$	rate	$\delta_2(t)$	rate	$\delta_\infty(t)$	rate
	4458	8.99E-03	–	1.49E-02	–	1.20E-01	–
	17589	1.35E-03	2.76	3.04E-03	2.31	3.23E-02	1.91
0.4	34886	5.19E-04	2.80	1.47E-03	2.13	1.44E-02	2.36
	69781	2.45E-04	2.17	7.20E-04	2.05	7.93E-03	1.72
	139127	1.04E-04	2.47	3.71E-04	1.93	3.27E-03	2.56
	4458	6.99E-03	–	2.03E-02	–	1.58E-01	–
	17589	9.51E-04	2.91	3.39E-03	2.61	3.61E-02	2.15
0.9	34886	3.98E-04	2.54	1.60E-03	2.20	1.55E-02	2.47
	69781	1.79E-04	2.30	7.54E-04	2.17	8.23E-03	1.83
	139127	8.17E-05	2.28	3.67E-04	2.09	3.28E-03	2.67

2D benchmark

- Shock/viscous boundary layer interaction (**Daru&Tenaud (2000, 2009)**).

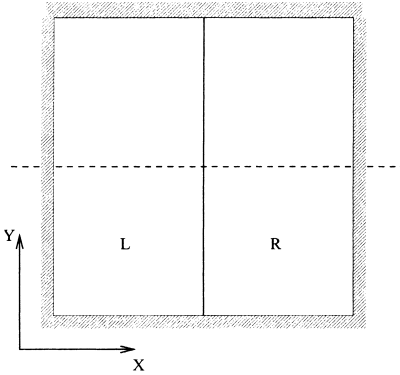


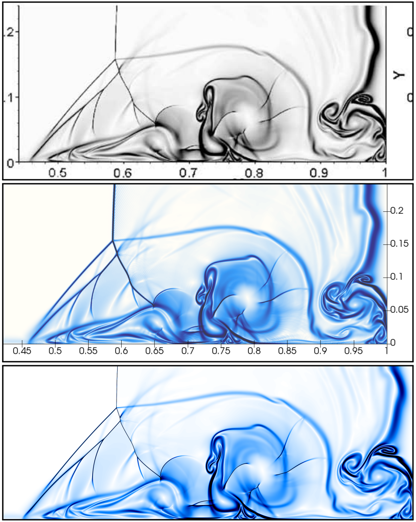
Figure: Description of the problem

- "Standard methods" are known to give "various answers" depending on the method (**Daru&Tenaud (2000)**, **Sjogreen&Yee (2003)**)

2D benchmark

SKIP BORING DETAILS

2D benchmark



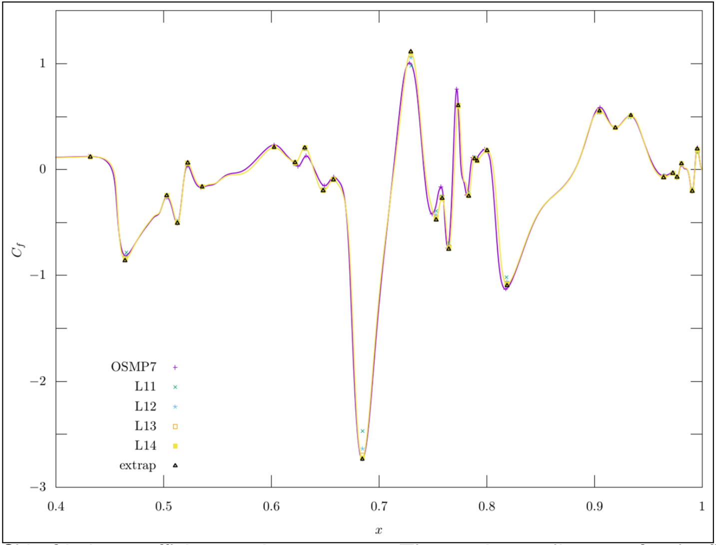
Uniform Cartesian mesh 4000×2000 (OSMP7).

Nonuniform Delaunay triangulation $\mathbb{P}_1$ FE, (0.86M grid points)

Non uniform quadrangular mesh $\mathbb{Q}_1$ FE (128M grid points)

Figure: Comparison with **Daru&Tenaud (2009)**. Density at $t = 1$ for $\mu \in \{10^{-3}\}$.

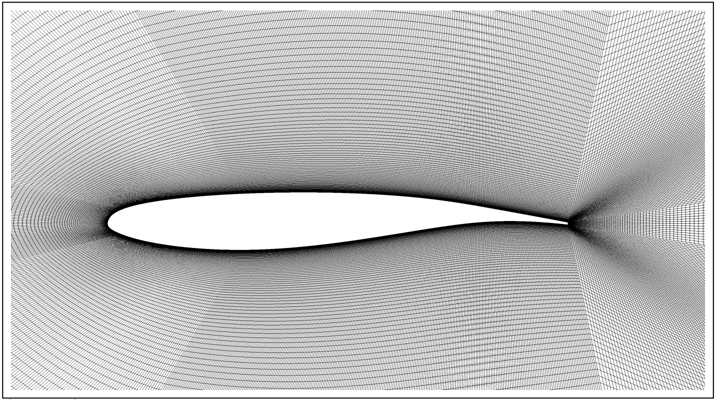
2D benchmark



Skin friction coefficient at time $t = 1.00$. The continuous lines are for the finest level and the OSMP7 scheme as reported in **Daru&Tenaud (2009)**.

AOT15a airfoil

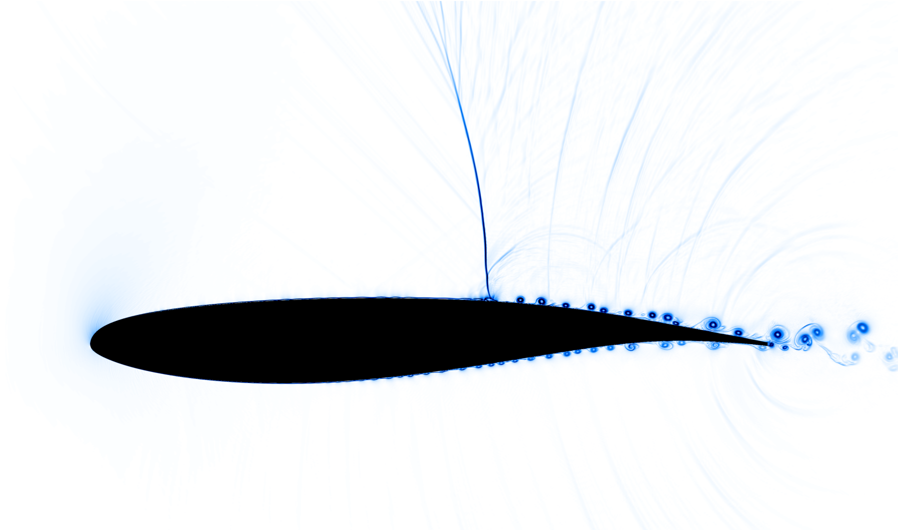
- AOTa15 airfoil at Mach 0.73, Reynolds 3×10^6, angle 3.5^o.
- Grid heavily graded with a minimal resolution in the viscous sublayer of 2.1 micrometer vertical to 60 micrometer horizontal (anisotropy 30:1).
- 274 million gridpoints.



Onera OAT15a airfoil with graded mesh used in our computations. Actual hexahedral 3D mesh is created by extruding the quadrilateral 2D mesh in the z-direction.

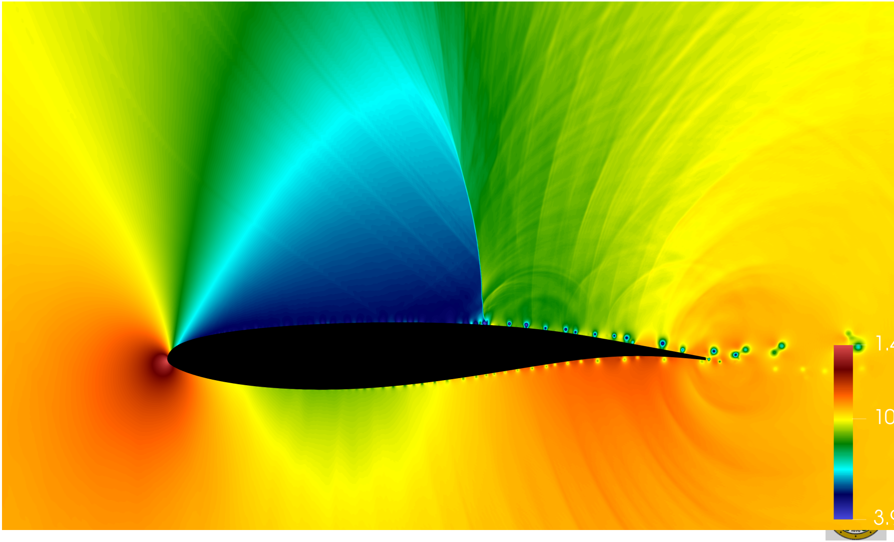
2D results

Airfoil OAT15a at Re=3 000 000, 2D, schlieren plot:



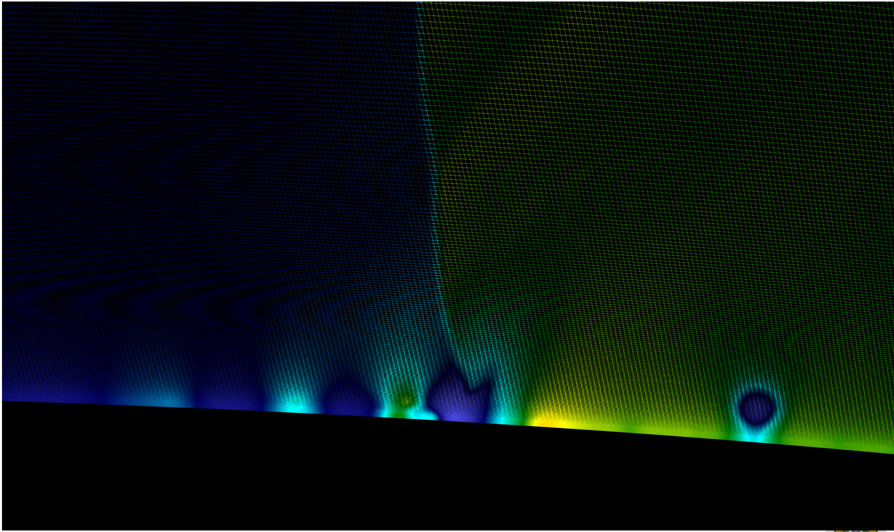
2D results

Airfoil OAT15a at Re=3 000 000, 2D, pressure:



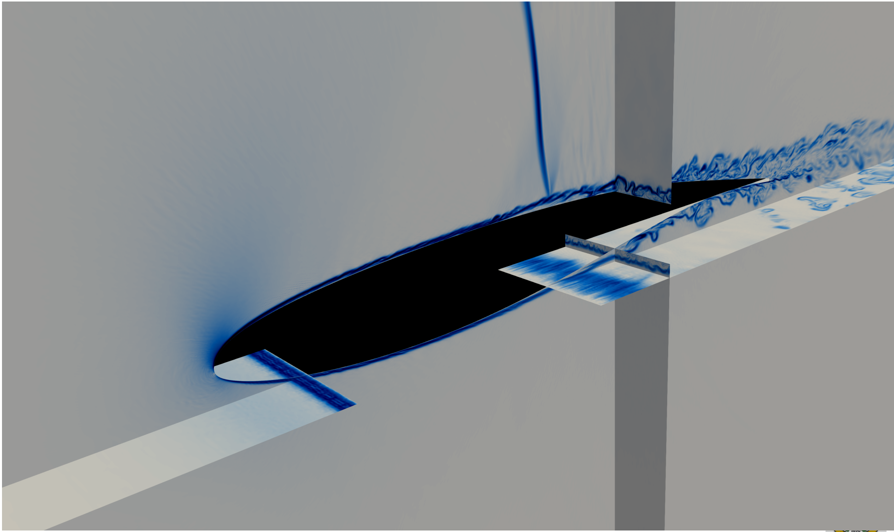
2D results

Airfoil OAT15a at Re=3 000 000, 2D, mesh resolution:



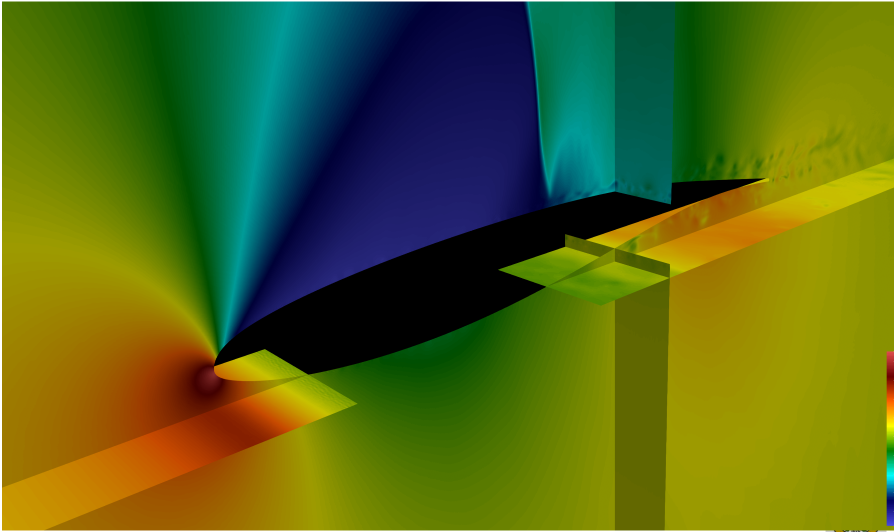
3D results

Airfoil OAT15a at Re=3 000 000, 3D, schlieren:



3D results

Airfoil OAT15a at Re=3 000 000, 3D, pressure



3D results

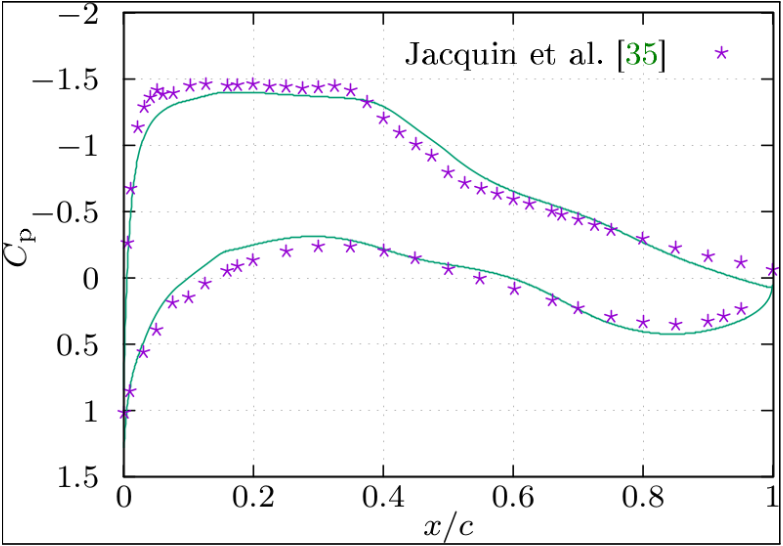


Figure: time- and z-averaged pressure coefficient. Comparison with experiments.

Strong scaling tests

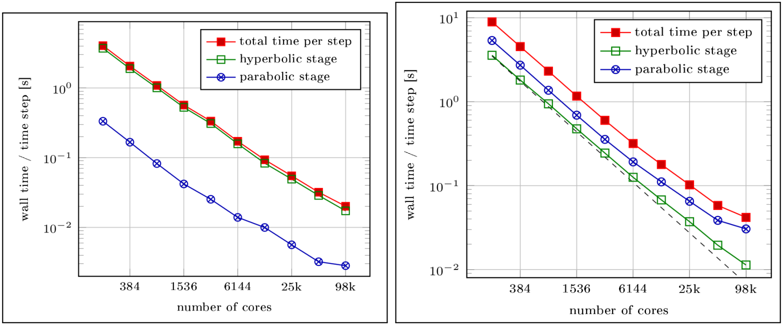


Figure: Scaling analysis broken down to show the contributions of the hyperbolic and parabolic parts. Left: 3D Onera OAT15a airfoil, with 34.5 million grid points. Right: 2D shocktube with 134 million grid points.

Current work

Collaborative team: J.-L. Guermond, M. Kronbichler, M. Maier, M. Nazarov, B. Popov, L. Saavedra, M. Sheridan, I. Tomas, E. Tovar.

- Implementation in Deal.II (Ryujin) of our shallow water code.
- Extension beyond second-order. Current " one-size fits all IMEX" technology is inadequate.
- Gray radiation hydrodynamics.
- Euler-Poisson.
- Third- and fourth-order in space with guaranteed properties and reasonable low-order stencil.