Quiz 1 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $\nabla\times$ be the curl operator acting on vector fields: i.e., let $\mathbf{A} = (A_1, A_2, A_3)^\mathsf{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a three-dimensional vector field over $\mathbb{R}^3$, then $\nabla\times\mathbf{A} = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1)$. Accept as a fact that $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$ for all smooth vector fields $\mathbf{A}$ and $\mathbf{B}$. Let Ω be a subset of $\mathbb{R}^3$ with a smooth boundary ∂D. Find an integration by parts formula for $\int_\Omega \mathbf{B}\cdot\nabla\times\mathbf{A}\mathrm{d}x$.

Using the divergence Theorem we infer that

$$\int_\Omega (\mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B})\mathsf{d}x = \int_\Omega \nabla\cdot(\mathbf{A}\times\mathbf{B}) = \int_{\partial D} (\mathbf{A}\times\mathbf{B})\cdot\mathbf{n}\mathsf{d}s.$$

which implies that

$$\int_\Omega \mathbf{B}\cdot\nabla\times\mathbf{A}\mathsf{d}x = \int_\Omega \mathbf{A}\cdot\nabla\times\mathbf{B}\mathsf{d}x + \int_{\partial D} (\mathbf{A}\times\mathbf{B})\cdot\mathbf{n}\mathsf{d}s.$$

Question 2: Let $u, f : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions of class C^1. (a) Compute $\partial_x f(u(x))$.

Using the chain rule we obtain

$$\partial_x f(u(x)) = f'(u(x))\partial_x u.$$

where f' denotes the derivative of f, (we could also use the notation $\partial_u f$ or $d_u f$, etc.).

(b) Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be functions of class C^1. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $F(v) = \int_0^v f'(t)\psi'(t)\mathrm{d}t$. Use (a) to compute $\partial_x(F(u(x)) - \partial_x(f(u(x)))\psi'(u(x))$.

Using the chain rule we obtain

$$\partial_x(F(u(x)) = F'(u(x))\partial_x u(x) = f'(u(x))\psi'(u(x))\partial_x u(x) = \partial_x(f(u(x)))\psi'(u(x)).$$

This means that $\partial_x(F(u(x)) - \partial_x(f(u(x)))\psi'(u(x)) = 0$.

(c) Using the notation of (a) and (b), assume that $u(\pm\infty) = 0$ and compute $\int_{-\infty}^{+\infty} \partial_x(f(u(x)))\psi'(u(x))\mathrm{d}x$.

Using (b) and $u(\pm\infty) = 0$ we have

$$\int_{-\infty}^{+\infty} \partial_x(f(u(x)))\psi'(u(x))\mathsf{d}x = \int_{-\infty}^{+\infty} \partial_x(F(u(x)))\mathsf{d}x = F(u(x))|_{-\infty}^{+\infty} = F(0) - F(0) = 0.$$

Question 3: Consider $\partial_t c(x,t) + \partial_x\big(\sin(\pi\frac{x}{L})c(x,t)\big) - \partial_x\big((1+|x|)\partial_x c(x,t)\big) = 6x/L^2 + \sin(t)/L$, where $x \in [0,L]$, $t > 0$, with $c(x,0) = f(x)$, $-\partial_n c(0,t) = 2$, $-\partial_n c(L,t) = \frac{1}{1+L}$, ($\partial_n$ is the normal derivative). Compute $E(t) := \int_0^L c(\xi,t)\mathrm{d}\xi$.

We integrate the equation with respect to x over $[0,L]$

$$\int_0^L \partial_t c(\xi,t)\mathsf{d}\xi + \int_0^L \partial_x\big(\sin(\pi\frac{\xi}{L})c(\xi,t)\big)\mathsf{d}\xi - \int_0^L \partial_\xi\big((1+|\xi|)\partial_\xi c(\xi,t)\big)\mathsf{d}\xi = \int_0^L (\frac{6}{L^2}\xi + \frac{\sin(t)}{L})\mathsf{d}\xi.$$

Using that $\int_0^L \partial_t c(\xi,t)\mathsf{d}\xi = \mathsf{d}_t \int_0^T c(\xi,t)\mathsf{d}\xi$ together with the fundamental theorem of calculus, we infer that

$$\mathsf{d}_t E(t) - (1+L)\partial_x c(L,t) + \partial_x c(0,t) = 3 + \sin(t).$$

The boundary conditions $\partial_x c(0,t) = -\partial_n c(0,t) = 2$, $-\partial_x c(L,t) = -\partial_n c(L,t) = \frac{1}{1+L}$ give

$$\mathsf{d}_t E(t) + 1 + 2 = 3 + \sin(t).$$

We now apply the fundamental theorem of calculus with respect to t

$$E(t) - E(0) = \int_0^t \mathsf{d}_\tau E(\tau)\mathsf{d}\tau = \int_0^t \sin(\tau)\mathsf{d}\tau = -\cos(t) + 1.$$

In conclusion

$$E(t) = 1 - \cos(t) + \int_0^L f(\xi)\mathsf{d}\xi, \qquad \forall t \geq 0.$$

Question 4: Let $\phi(\mathbf{x}) = \log(1 + x_1^2 + x_2^2)$ and $k(\mathbf{x}) = 1 + x_1^2 + x_2^2$, where $\mathbf{x} = (x_1, x_2)^\mathsf{T}$ (a) Compute $\nabla\cdot(k(\mathbf{x})\nabla\phi(\mathbf{x}))$.

We compute first $\nabla\phi$,

$$\nabla\phi(\mathbf{x}) = \frac{1}{1+x_1^2+x_2^2}\begin{pmatrix}2x_1\\2x_2\end{pmatrix}$$

Then

$$k(\mathbf{x})\nabla\phi(\mathbf{x}) = \begin{pmatrix}2x_1\\2x_2\end{pmatrix}$$

We conclude that

$$\nabla\cdot(k(\mathbf{x})\nabla\phi(\mathbf{x})) = 2 + 2 = 4.$$

(b)Let Ω be the disk of radius 1 centered at $(0,0)$ and let Γ be the boundary of Ω. Compute $\int_\Gamma k(\mathbf{x})\partial_n\phi(\mathbf{x})\mathrm{d}\Gamma$.

The fundamental theorem of calculus (also known as the divergence theorem) implies that

$$\int_\Gamma k(\mathbf{x})\partial_n\phi(\mathbf{x})\mathsf{d}\Gamma = \int_\Gamma n\cdot(k(\mathbf{x})\nabla\phi(\mathbf{x}))\mathsf{d}\Gamma = \int_\Omega \mathsf{div}(k(\mathbf{x})\nabla\phi(\mathbf{x}))\mathsf{d}\Omega = 4\int_\Omega \mathsf{d}\Omega = 4\pi,$$

because the surface of Ω, $\int_\Omega \mathsf{d}\Omega$, is equal to 4π.