Quizz 2 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let ϕ be a non-zero solution to the eigenvalue problem $-\partial_x\big((1+x^2)\partial_x\phi(x)\big) = \lambda\phi(x)$, $x \in (0,\pi)$, $\phi(\pi) = 0$, $-\partial_x\phi(0) + \phi(0) = 0$. Determine the sign of λ using the energy method.

Multiply the equation by ϕ, integrate over $(0,\pi)$, and apply the fundamental theorem of calculus (i.e. integrate by parts):

$$\begin{aligned}\lambda\int_0^\pi (\phi(x))^2 \mathsf{d}x &= -\int_0^\pi \phi(x)\partial_x\big((1+x^2)\partial_x\phi(x)\big)\mathsf{d}x = -\int_0^\pi (\partial_x(\phi(x)(1+x^2)\partial_x\phi(x)) - (1+x^2)(\partial_x\phi(x))^2)\mathsf{d}x \\ &= -(1+\pi^2)\phi(\pi)\partial_x\phi(\pi) + \phi(0)\partial_x\phi(0) + \int_0^\pi (1+x^2)(\partial_x\phi(x))^2\mathsf{d}x \\ &= (\phi(0))^2 + \int_0^\pi (1+x^2)(\partial_x\phi(x))^2\mathsf{d}x.\end{aligned}$$

In conclusion

$$(\phi(0))^2 + \int_0^\pi (1+x^2)(\partial_x\phi(x))^2\mathsf{d}x = \lambda\int_0^\pi (\phi(x))^2\mathsf{d}x.$$

Assuming that ϕ is nonzero, we obtain that $\lambda = (\phi^2(0) + \int_0^\pi (1+x^2)(\partial_x\phi(x))^2\mathsf{d}x)/\int_0^\pi(\phi(x))^2\mathsf{d}x \geq 0$, i.e. λ is non-negative. If $\lambda = 0$ then $\phi(0) = 0$ and $\partial_x\phi = 0$, which implies that ϕ is constant. The other condition $\phi(0) = 0$ implies that $\phi = 0$ which contradicts our assumption that ϕ is non-zero. In conclusion λ is positive.

Question 2: Let $k, f : [-1,+1] \longrightarrow \mathbb{R}$ be such that $k(x) = 2+x$, $f(x) = 0$ if $x \in [-1,0]$ and $k(x) = 1+2x$, $f(x) = 2$ if $x \in (0,1]$. Consider the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = f(x)$ with $T(-1) = 5$ and $T(1) = -1$.
(a) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1,0]$ and T^+ the solution on $[0,+1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 2$ and $k^+(0) = 1$, i.e., $2\partial_x T^-(0) = \partial_x T^+(0)$.

Question 3: Consider the heat equation $\partial_t u(x,t) - 2\partial_{xx}u(x,t) = 0$, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = u_0(x)$, $t > 0$, $x \in (0,1)$. The general solution is $u(x,t) = \sum_{n=0}^{\infty} A_n \sin(n\pi x)\mathsf{e}^{-2n^2\pi^2 t}$. Compute the solution corresponding to the initial data $u_0(x) = 3\sin(4\pi x) - 5\sin(\pi x)$.

The solution contains two terms only, corresponding to $n = 1$ and $n = 4$,

$$u(x,t) = -5\sin(\pi x)\mathsf{e}^{-2\pi^2 t} + 3\sin(4\pi x)\mathsf{e}^{-32\pi^2 t}.$$

Question 4: Assume that the following equation has a smooth solution: $-\partial_x((1+x^2)\partial_x T(x)) - 5\partial_x T(x) + (1+b-x)T(x) = \cos(x)$, $T(a) = 1$, $T(b) = \pi$, $x \in [a,b]$, $t > 0$, where $k > 0$. Prove that this solution is unique by using the energy method. (Hint: Do not try to simplify $-\partial_x((1+x^2)\partial_x T)$.

Assume that there are two solutions T_1 and T_2. Let $\phi = T_2 - T_1$. Then

$$-\partial_x((1+x^2)\partial_x\phi(x)) - 5\partial_x\phi(x) + (1+b-x)\phi(x) = 0, \quad \phi(a) = 0, \quad \phi(b) = 0$$

Multiply the PDE by ϕ, integrate over (a,b), and integrate by parts (i.e. apply the fundamental theorem of calculus):

$$\begin{aligned}
0 &= \int_a^b \big(-\partial_x((1+x^2)\partial_x\phi(x))\phi(x) - 5(\partial_x\phi(x))\phi(x) + (1+b-x)(\phi(x))^2\big)\mathsf{d}x \\
&= \int_a^b \big(-\partial_x(\phi(x)(1+x^2)\partial_x\phi(x)) + (1+x^2)(\partial_x\phi(x))^2 - 5\partial_x\Big(\frac{1}{2}\phi(x)^2\Big) + (1+b-x)(\phi(x))^2\big)\mathsf{d}x \\
&= \int_a^b \big((1+x^2)(\partial_x\phi(x))^2 + (1+b-x)(\phi(x))^2\big)\mathsf{d}x \geq \int_a^b \phi^2(x)\mathsf{d}x,
\end{aligned}$$

since $1+b-x \geq 1$ for all $x \in [a,b]$. This implies $\int_a^b (\phi(x))^2\mathsf{d}x = 0$, i.e., $\phi = 0$, meaning that $T_2 = T_1$.