

Quiz 2 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded.**

Question 1: Let ϕ be a non-zero solution to the eigenvalue problem $\partial_x((3+x^2)\partial_x\phi(x)) = \lambda\phi(x)$, $x \in (0, \pi)$, $\phi(\pi) = 0$, $-\partial_x\phi(0) + \phi(0) = 0$. Determine the sign of λ using the energy method.

Multiply the equation by ϕ , integrate over $(0, \pi)$, and apply the fundamental theorem of calculus (i.e. integrate by parts):

$$\begin{aligned} \lambda \int_0^\pi (\phi(x))^2 dx &= \int_0^\pi \phi(x) \partial_x((3+x^2)\partial_x\phi(x)) dx = \int_0^\pi (\partial_x(\phi(x)(3+x^2)\partial_x\phi(x)) - (1+x^2)(\partial_x\phi(x))^2) dx \\ &= (3+\pi^2)\phi(\pi)\partial_x\phi(\pi) - 3\phi(0)\partial_x\phi(0) - \int_0^\pi (3+x^2)(\partial_x\phi(x))^2 dx \\ &= -3(\phi(0))^2 - \int_0^\pi (3+x^2)(\partial_x\phi(x))^2 dx. \end{aligned}$$

In conclusion

$$-3(\phi(0))^2 - \int_0^\pi (3+x^2)(\partial_x\phi(x))^2 dx = \lambda \int_0^\pi (\phi(x))^2 dx.$$

Assuming that ϕ is nonzero, we obtain that $\lambda = -(3\phi^2(0) + \int_0^\pi (3+x^2)(\partial_x\phi(x))^2 dx) / \int_0^\pi (\phi(x))^2 dx \leq 0$, i.e., λ is non-positive. If $\lambda = 0$ then $\phi(0) = 0$ and $\partial_x\phi = 0$, which implies that ϕ is constant. The other condition $\phi(0) = 0$ implies that $\phi = 0$ which contradicts our assumption that ϕ is non-zero. In conclusion λ is negative.

Question 2: Let $k, f : [-1, +1] \rightarrow \mathbb{R}$ be such that $k(x) = 5 + 2x$, $f(x) = \sin(x)$ if $x \in [-1, 0]$ and $k(x) = 3 + 3x$, $f(x) = \cos(x)$ if $x \in (0, 1]$. Consider the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = f(x)$ with $T(-1) = 5$ and $T(1) = -1$. (a) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1, 0]$ and T^+ the solution on $[0, +1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 5$ and $k^+(0) = 3$, i.e., $5\partial_x T^-(0) = 3\partial_x T^+(0)$.

Question 3: Consider the heat equation $\partial_t u(x, t) - 2\partial_{xx} u(x, t) = 0$, $\partial_x u(0, t) = 0$, $\partial_x u(1, t) = 0$, $u(x, 0) = u_0(x)$, $t > 0$, $x \in (0, 1)$. The general solution is $u(x, t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-2n^2\pi^2 t}$. Compute the solution corresponding to the initial data $u_0(x) = 3 \cos(2\pi x) - 5 \cos(3\pi x)$.

The solution contains two terms only, corresponding to $n = 2$ and $n = 3$,

$$u(x, t) = 3 \sin(2\pi x) e^{-16\pi^2 t} - 5 \sin(3\pi x) e^{-18\pi^2 t}.$$

Question 4: Assume that the following equation has a smooth solution: $-\partial_x((3+x^4)\partial_x T(x)) - 7\partial_x T(x) + (1+b-x)T(x) = \cos(x)$, $T(a) = 1$, $T(b) = \pi$, $x \in [a, b]$, $t > 0$, where $k > 0$. Prove that this solution is unique by using the energy method. (Hint: Do not try to simplify $-\partial_x((1+x^2)\partial_x T)$.)

Assume that there are two solutions T_1 and T_2 . Let $\phi = T_2 - T_1$. Then

$$-\partial_x((3+x^4)\partial_x \phi(x)) - 7\partial_x \phi(x) + (1+b-x)\phi(x) = 0, \quad \phi(a) = 0, \quad \phi(b) = 0$$

Multiply the PDE by ϕ , integrate over (a, b) , and integrate by parts (i.e. apply the fundamental theorem of calculus):

$$\begin{aligned} 0 &= \int_a^b (-\partial_x((3+x^4)\partial_x \phi(x))\phi(x) - 7(\partial_x \phi(x))\phi(x) + (1+b-x)(\phi(x))^2) dx \\ &= \int_a^b (-\partial_x(\phi(x)(3+x^4)\partial_x \phi(x)) + (3+x^4)(\partial_x \phi(x))^2 - 7\partial_x\left(\frac{1}{2}\phi(x)^2\right) + (1+b-x)(\phi(x))^2) dx \\ &= \int_a^b ((3+x^4)(\partial_x \phi(x))^2 + (1+b-x)(\phi(x))^2) dx \geq \int_a^b \phi^2(x) dx, \end{aligned}$$

since $1+b-x \geq 1$ for all $x \in [a, b]$. This implies $\int_a^b (\phi(x))^2 dx = 0$, i.e., $\phi = 0$, meaning that $T_2 = T_1$.