Quiz 4 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let N be a positive integer and let $\mathbb{P}_N$ be the set of trigonometric polynomials of degree at most N; that is, $\mathbb{P}_N = \text{span}\{1, \cos(x), \sin(x), \ldots, \cos(Nx), \sin(Nx)\}$. Consider the function $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ defined by $f(x) = \sum_{n=0}^{9} \frac{9}{n^3+9} \sin(9n)\cos(4nx)$. (a) Compute the Fourier series of f.

The Fourier series of f is of the following form

$$FS(f)(x) = \sum_{m=0}^{\infty} a_n \cos(m\pi \frac{x}{\pi}) + \sum_{m=1}^{\infty} b_m \sin(m\pi \frac{x}{\pi}) = \sum_{n=0}^{\infty} a_m \cos(mx) + \sum_{n=1}^{\infty} b_m \sin(mx).$$

with

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x, \quad a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos(m\pi\frac{x}{L})\mathrm{d}x, \quad b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin(m\pi\frac{x}{L})\mathrm{d}x.$$

The orthogonality properties of the cosine and sine families implies that

$$\begin{aligned} a_{4n} &= \frac{9}{n^3+9}\sin(9n), \quad \text{and} \quad a_{4n+1} = a_{4n+2} = a_{4n+3} = 0, \quad 0 \le n \le 9. \\ a_n &= 0, \quad \forall n \ge 37, \\ b_n &= 0, \quad \forall n \ge 1. \end{aligned}$$

In conclusion

$$FS(f)(x) = \sum_{n=0}^{9} \frac{9}{n^3+9}\sin(9n)\cos(4nx).$$

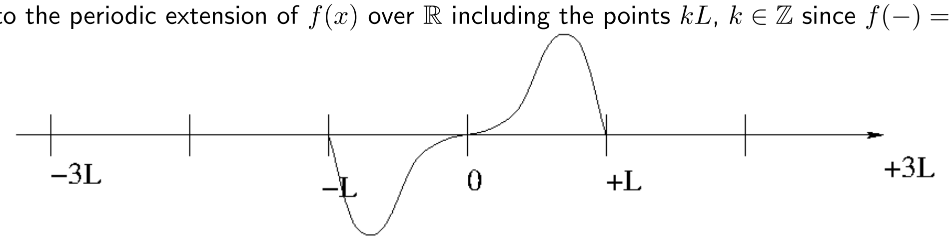
Compute the best L^2-approximation of f in $\mathbb{P}_{33}$ over $(-\pi, \pi)$.

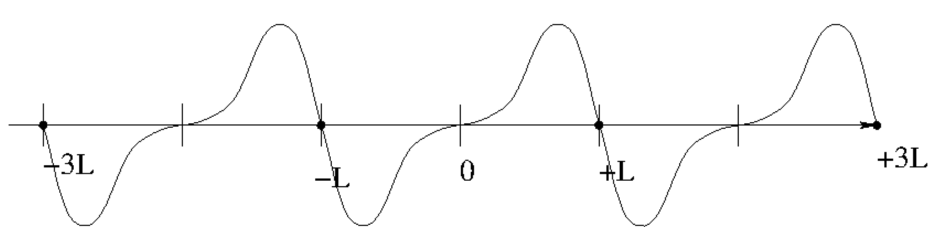
We know from class that the best L^2-approximation of f in $\mathbb{P}_{33}$ over $(-\pi, \pi)$ is the truncated Fourier series $FS_{33}(f)$:

$$FS_{33}(f) = \sum_{n=0}^{8} \frac{9}{n^3+9}\sin(9n)\cos(4nx).$$

Question 2: Consider $f : [-L, L] \longrightarrow \mathbb{R}$, $f(x) = |x|\sin(\pi\frac{x}{L})$. (a) Sketch the graph of f and the graph of the Fourier series of f.

$FS(f)$ is equal to the periodic extension of $f(x)$ over $\mathbb{R}$ including the points kL, $k \in \mathbb{Z}$ since $f(-) = f(+L)$.





(b) For which values of $x \in \mathbb{R}$ is $FS(f)$ equal to $|x|\sin(\pi\frac{x}{L})$? (Explain)

The periodic extension of $f(x) = |x|\sin(\pi\frac{x}{L})$ over $\mathbb{R}$ is smooth over each interval $[(2k-1)L,(2k+1)L]$, $k \in \mathbb{Z}$ and is continuous at all the points $(2k+1)L$, $k \in \mathbb{Z}$. This means that the Fourier series is equal to the periodic extension of f over the entire real line and equal to $f(x) = |x|\sin(\pi\frac{x}{L})$ over $[-L,+L]$. Note that f and the periodic extension of f are two different objects. f is defined over $[-L,+L]$ whereas the periodic extension of f is defined over $\mathbb{R}$.

(c) What is the derivative of $f(x) = |x|\sin(\pi\frac{x}{L})$?

If $x \geq 0$ then

$$f'(x) = \sin(\pi\frac{x}{L}) + x\frac{\pi}{L}\cos(\pi\frac{x}{L}) = \sin(\pi\frac{|x|}{L}) + |x|\frac{\pi}{L}\cos(\pi\frac{x}{L}).$$

If $x \leq 0$ then

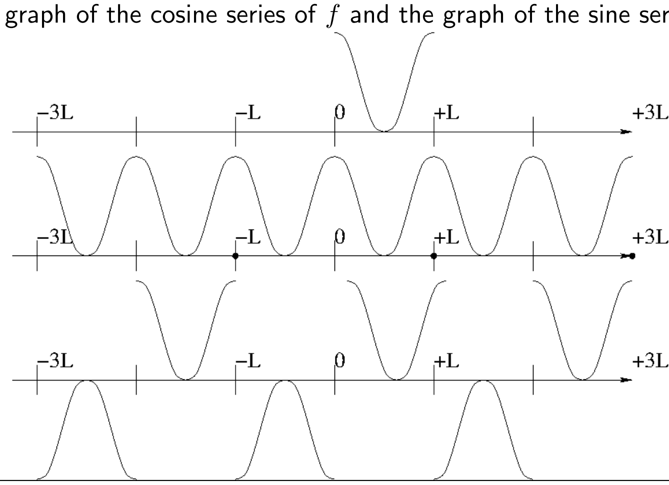
$$f'(x) = -\sin(\pi\frac{x}{L}) - x\frac{\pi}{L}\cos(\pi\frac{x}{L}) = \sin(\pi\frac{|x|}{L}) + |x|\frac{\pi}{L}\cos(\pi\frac{x}{L}).$$

(c) Is it possible to obtain $FS(f')(x)$ by differentiating $FS(f)(x)$ term by term? (Explain)

Yes, it is possible for every $x \in \mathbb{R}$ since the Fourier series of f and f' are continuous.

Question 3: Let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ be defined by $f(x) = 1 + \cos(x)$. (a)Draw the graph of f, the graph of the cosine series of f and the graph of the sine series of f.

Here are the graph of f, the graph of the cosine series of f and the graph of the sine series of f (with $L = 2\pi$).



(b) Let $g : [-2\pi, 2\pi] \longrightarrow \mathbb{R}$ be defined by $g(x) = 1 + \cos(x)$. Is the Fourier series of g equal to the sum of the cosine and sine series of f. Give all the details (a correct picture would be enough).

No. The sine series of f over the interval $[-2\pi, 0]$ is equal to the odd extension of $f(x) = 1 + \cos(x)$. The odd extension in question is $f_{\text{odd}}(x) = -1 - \cos(x)$ when $x \in [-2\pi, 0]$. The cosine series of f over the interval $[-2\pi, 0]$ is equal to the even extension of $f(x) = 1 + \cos(x)$. The even extension in question is $f_{\text{even}}(x) = 1 + \cos(x)$ when $x \in [-2\pi, 0]$. In conclusion the sum of the cosine and sine series of f is equal to 0 over the interval $[-2\pi, 0]$, which is obviously different from $g(x) = FS(g)(x) = 1 + \cos(x)$.
Here are the graphs of g, the graphs of $FS(g)$ and the graph of the sum of the cosine and sine series of f (with $L = 2\pi$).