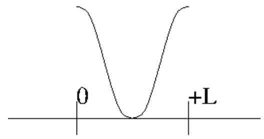
Quiz 4 (Notes, books, and calculators are not authorized)

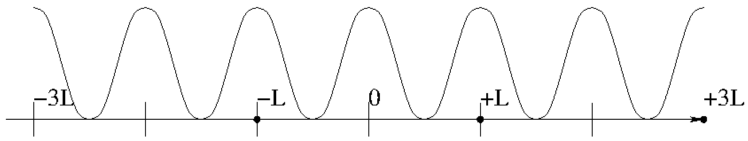
Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ be defined by $f(x) = 1 + \cos(x)$. (a) Draw the graph of f, the graph of the cosine series of f, and the graph of the sine series of f.

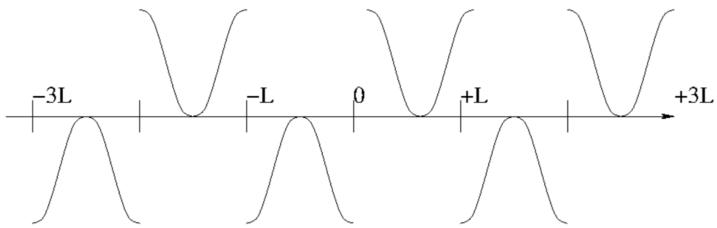
Here is the graph of f with $L = 2\pi$:



the graph of the cosine series of f:

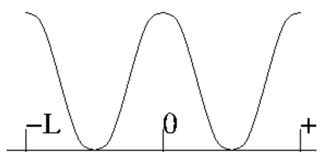


and the graph of the sine series of f (with $L = 2\pi$):

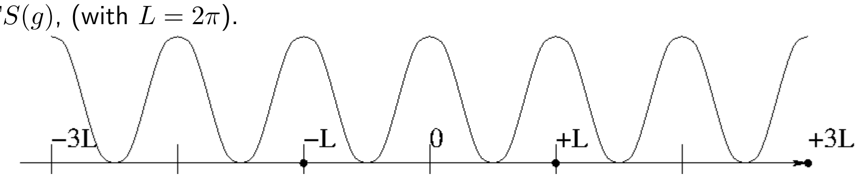


(b) Let $g : [-2\pi, 2\pi] \longrightarrow \mathbb{R}$ be defined by $g(x) = 1 + \cos(x)$. What is the restriction of the Fourier series of g over $[-2\pi, 2\pi]$? (Use the notation $FS(g)$.) Give all the details (a correct picture is enough if you run out of time).

(1) Here is the graphs of g, (with $L = 2\pi$).



Here is the graphs of $FS(g)$, (with $L = 2\pi$).



g is smooth and $g(+2\pi) = g(-2\pi)$, hence $FS(g)$ coincide with g over $[-2\pi, 2\pi]$.

Notice in passing that we observe that $FS(g)(s) = 1 + \cos(x)$.

(2) Another (rigorous) way to answer consists of observing that

$$a_0 = \frac{1}{4\pi}\int_{-2\pi}^{2\pi}(1+\cos(x))\mathrm{d}x = 1; \quad a_1 = \frac{1}{2\pi}\int_{-2\pi}^{2\pi}(1+\cos(x))\cos(2\pi\frac{x}{2\pi})\mathrm{d}x = 1; \quad a_n = 0,\ \forall n \geq 2,$$
$$b_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi}(1+\cos(x))\sin(2\pi\frac{x}{2\pi})\mathrm{d}x = 0.$$

Hence, recalling that by definition

$$FS(g) = \sum_{0}^{\infty} a_n \cos(2\pi n\frac{x}{2\pi}) + \sum_{1}^{\infty} b_n \sin(2\pi n\frac{x}{2\pi}),$$

we have $FS(g) = a_0 + a_1\cos(2\pi\frac{x}{2\pi}) + 0 = 1 + \cos(x)$. Hence $FS(g)$ and g coincide over $[-2\pi, 2\pi]$.

Question 2: Let $u \in C^2(\mathbb{R}^2)$ be a harmonic function. State the mean value theorem for u.

For any $\mathbf{x} \in \mathbb{R}$ and any circle $\mathcal{C}(\mathbf{x}, r)$ centered at $\mathbf{x}$ and of radius r, we have

$$u(\mathbf{x}) = \frac{1}{2\pi r} \int_{\mathcal{C}(\mathbf{x},r)} u(\mathbf{y}) \mathrm{d}l.$$

Question 3: Consider the square $D = (-1, +1) \times (-1, +1)$. Let $f(x, y) = x^2 - y^4 - 3$. Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ solve $-\Delta u = 0$ in D and $u|_{\partial D} = f$. Compute $\min_{(x,y)\in\overline{D}} u(x, y)$ and $\max_{(x,y)\in\overline{D}} u(x, y)$. (*Hint:* A point $\mathbf{x} = (x, y)$ belongs to ∂D if and only if $x^2 = 1$ and $y \in [-1, 1]$ or $y^2 = 1$ and $x \in [-1, 1]$.)

We use the maximum principle (u is harmonic and has the required regularity). Then

$$\min_{(x,y)\in\overline{D}} u(x, y) = \min_{(x,y)\in\partial D} f(x, y), \quad \text{and} \quad \max_{(x,y)\in\overline{D}} u(x, y) = \max_{(x,y)\in\partial D} f(x, y).$$

A point (x, y) is at the boundary of D if and only if $x^2 = 1$ and $y \in [-1, 1]$ or $y^2 = 1$ and $x \in [-1, 1]$.

(i) In the first case, $x^2 = 1$ and $y \in (-1, 1)$, we have

$$f(x, y) = 1 - y^4 - 3, \qquad y \in (-1, 1).$$

The maximum is -2 and the minimum is -3.

(ii) In the second case, $y^2 = 1$ and $x \in (-1, 1)$, we have

$$f(x, y) = x^2 - 1 - 3, \qquad x \in (-1, 1).$$

The maximum is -3 and the minimum is -4. We finally can conclude

$$\min_{(x,y)\in\partial D} f(x, y) = \min_{-1\leq x\leq 1} x^2 - 4, = -4, \qquad \max_{(x,y)\in\partial D} f(x, y) = \max_{-1\leq y\leq 1} -2 - y^2 = -2.$$

(iii) In conclusion

$$\min_{(x,y)\in\overline{D}} u(x, y) = -4, \qquad \max_{(x,y)\in\overline{D}} u(x, y) = -2$$