Quiz 5 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**. Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}\mathrm{d}x, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{1}$$

$$\mathcal{F}(S_\lambda(x)) = \frac{1}{\pi}\frac{\sin(\lambda\omega)}{\omega}, \quad \text{where} \quad S_\lambda(x) = \begin{cases} 1 & \text{if } |x| \le \lambda \\ 0 & \text{otherwise.} \end{cases} \qquad \sqrt{\frac{\pi}{\alpha}}\mathcal{F}(e^{-\frac{x^2}{4\alpha}}) = e^{-\alpha\omega^2}. \tag{2}$$

Question 1: Solve the following integral equation (Hint: $x^2 - 3xa + 2a^2 = (x-a)(x-2a)$):

$$\int_{-\infty}^{+\infty} f(y)f(x-y)\mathrm{d}y - 3\sqrt{2}\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2\pi}} f(x-y)\mathrm{d}y = -4\pi e^{-\frac{x^2}{4\pi}}. \qquad \forall x \in \mathbb{R}.$$

This equation can be re-written using the convolution operator:

$$f * f - 3\sqrt{2}e^{-\frac{x^2}{2\pi}} * f = -4\pi e^{-\frac{x^2}{4\pi}}.$$

We take the Fourier transform and use (2) to obtain

$$2\pi\mathcal{F}(f)^2 - 2\pi 3\sqrt{2}\mathcal{F}(f)\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{2\pi}}} = -4\pi\frac{1}{\sqrt{4\pi\frac{1}{4\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{4\pi}}}$$

$$\mathcal{F}(f)^2 - 3\mathcal{F}(f)e^{-\omega^2\frac{\pi}{2}} + 2e^{-\omega^2\pi} = 0$$

$$(\mathcal{F}(f) - e^{-\omega^2\frac{\pi}{2}})(\mathcal{F}(f) - 2e^{-\omega^2\frac{\pi}{2}}) = 0.$$

This implies

$$\text{either} \quad \mathcal{F}(f) = e^{-\omega^2\frac{\pi}{2}}, \quad \text{or} \quad \mathcal{F}(f) = 2e^{-\omega^2\frac{\pi}{2}}.$$

Taking the inverse Fourier transform, we obtain

$$\text{either} \quad f(x) = \sqrt{2}e^{-\frac{x^2}{2\pi}}, \quad \text{or} \quad f(x) = 2\sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

Question 2: (i) Let f be an integrable function on $(-\infty, +\infty)$. Prove that for all $a, b \in \mathbb{R}$, and for all $\xi \in \mathbb{R}$, $\mathcal{F}([e^{ibx}f(ax)])(\xi) = \frac{1}{a}\mathcal{F}(f)(\frac{\xi+b}{a})$.

The definition of the Fourier transform together with the change of variable $ax \longmapsto x'$ implies

$$\begin{aligned}
\mathcal{F}[e^{ibx}f(ax)])(\xi) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(ax)e^{ibx}e^{i\xi x}dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(ax)e^{i(b+\xi)x}dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{a}f(x')e^{i\frac{(\xi+b)}{a}x'}dx' \\
&= \tfrac{1}{a}\mathcal{F}(f)(\tfrac{\xi+b}{a}).
\end{aligned}$$

Question 3: Consider the telegraph equation $\partial_{tt}u+2\alpha\partial_t u+\alpha^2 u-c^2\partial_{xx}u=0$ with $\alpha\geq 0$, $u(x,0)=0$, $\partial_t u(x,0)=g(x)$, $x\in\mathbb{R}$, $t>0$ and boundary condition at infinity $u(\pm\infty,t)=0$. Solve the equation by the Fourier transform technique. (Hint: the solution to the ODE $\phi''(t)+2\alpha\phi'(t)+(\alpha^2+\lambda^2)\phi(t)=0$ is $\phi(t)=\mathrm{e}^{-\alpha t}(a\cos(\lambda t)+b\sin(\lambda t))$)

Applying the Fourier transform with respect to x to the equation, we infer that

$$\begin{aligned}0 &= \partial_{tt}\mathcal{F}(u)(\omega,t)+2\alpha\partial_t\mathcal{F}(u)(\omega,t)+\alpha^2\mathcal{F}(u)(\omega,t)-c^2(-i\omega)^2\mathcal{F}(u)(\omega,t)\\ &= \partial_{tt}\mathcal{F}(u)(\omega,t)+2\alpha\partial_t\mathcal{F}(u)(\omega,t)+(\alpha^2+c^2\omega^2)\mathcal{F}(u)(\omega,t)\end{aligned}$$

Using the hint, we deduce that

$$\mathcal{F}(u)(\omega,t)=\mathsf{e}^{-\alpha t}(a(\omega)\cos(\omega ct)+b(\omega)\sin(\omega ct)).$$

The initial condition implies that $a(\omega)=0$ and $\mathcal{F}(g)(\omega)=\omega c b(\omega)$; as a result, $b(\omega)=\mathcal{F}(g)(\omega)/(\omega c)$ and

$$\mathcal{F}(u)(\omega,t)=\mathsf{e}^{-\alpha t}\mathcal{F}(g)\frac{\sin(\omega ct)}{\omega c}.$$

Then using (2), we have

$$\mathcal{F}(u)(\omega,t)=\frac{\pi}{c}\mathsf{e}^{-\alpha t}\mathcal{F}(g)\mathcal{F}(S_{ct}).$$

The convolution theorem implies that

$$u(x,t)=\mathsf{e}^{-\alpha t}\frac{1}{2c}g*S_{ct}=\mathsf{e}^{-\alpha t}\frac{1}{2c}\int_{-\infty}^{\infty}g(y)S_{ct}(x-y)\mathsf{d}y.$$

Finally the definition of S_{ct} implies that $S_{ct}(x-y)$ is equal to 1 if $-ct<x-y<ct$ and is equal zero otherwise, which finally means that

$$u(x,t)=\mathsf{e}^{-\alpha t}\frac{1}{2c}\int_{x-ct}^{x+ct}g(y)\mathsf{d}y.$$

Question 4: Find the inverse Fourier transform of $\frac{1}{\pi}\frac{\sin(\lambda\omega}{\omega}$.

Since $\mathcal{F}(S_\lambda(x))=\frac{1}{\pi}\frac{\sin(\lambda\omega)}{\omega}$, the inverse Forier transform theorem implies that

$$\mathcal{F}^{-1}\left(\frac{1}{\pi}\frac{\sin(\lambda\omega)}{\omega}\right)(x)=\begin{cases}1 & \text{if } |x|<\lambda\\ \frac{1}{2} & \text{if } |x|=\lambda\\ 0 & \text{otherwise.}\end{cases}$$
