

Quiz 6 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**. Here is a formula you may want to use,

$$\int_{\Omega} \psi \Delta \phi dx = \int_{\Omega} \phi \Delta \psi dx + \int_{\Gamma} \psi \partial_n \phi d\sigma - \int_{\Gamma} \phi \partial_n \psi d\sigma$$

**Question 1:** Let  $\Omega$  be a five-dimensional domain with boundary  $\Gamma$ , and consider the PDE

$$u - \Delta u = f(x), \quad x \in \Omega, \quad \text{with} \quad \partial_n u(x) = h(x) \quad \text{on the boundary } \Gamma.$$

Let  $G(x, x_0)$  be the Green's function of this problem (the exact expression of  $G(x, x_0)$  does not matter; just assume that  $G(x, x_0)$  is known). (a) what is the PDE solved by  $G(x, x_0)$ ?

Either you remember from class that the operator  $u \mapsto u - \Delta u$  with zero Neumann boundary condition is self-adjoint, or you redo the computation. Multiply the PDE by  $G(x, x_0)$ , integrate over  $\Omega$ , and integrate by parts (using the hint):

$$\begin{aligned} \int_{\Omega} G(x, x_0) f(x) dx &= \int_{\Omega} G(x, x_0) (u(x) - \Delta u(x)) dx \\ &= \int_{\Omega} G(x, x_0) u(x) dx - \int_{\Omega} u(x) \Delta G(x, x_0) dx - \int_{\Gamma} G(x, x_0) \partial_n u(x) d\sigma + \int_{\Gamma} u(x) \partial_n G(x, x_0) d\sigma \\ &= \int_{\Omega} (G(x, x_0) - \Delta G(x, x_0)) u(x) dx - \int_{\Gamma} G(x, x_0) h(x) d\sigma + \int_{\Gamma} u(x) \partial_n G(x, x_0) d\sigma \end{aligned}$$

We then define  $G(x, x_0)$  so that

$$G(x, x_0) - \Delta G(x, x_0) = \delta_{x-x_0}, \quad \partial_n G(x, x_0)|_{\Gamma} = 0,$$

where  $\delta_{x-x_0}$  is the Dirac measure:  $\int \delta_{x-x_0} \varphi = \varphi(x_0)$  for all  $\varphi \in C^0(\mathbb{R}^5)$ . This means that

$$\int_{\Omega} G(x, x_0) f(x) dx = u(x_0) - \int_{\Gamma} G(x, x_0) h(x) d\sigma.$$

(b) Give a representation of  $u(x)$  in terms of  $G$ ,  $f$  and  $h$ .

The above computation shows that

$$u(x_0) = \int_{\Omega} f(x) G(x, x_0) dx + \int_{\Gamma} h(x) G(x, x_0) dx.$$

**Question 2:** Consider the equation  $\partial_{xx}u(x) = f(x)$ ,  $x \in (0, L)$ , with  $u(0) = a$  and  $\partial_x u(L) = b$ .

(a) Compute the Green's function of the problem.

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Let  $x_0$  be a point in  $(0, L)$ . The Green's function of the problem is such that

$$\partial_{xx}G(x, x_0) = \delta_{x_0}, \quad G(0, x_0) = 0, \quad \partial_x G(L, x_0) = 0.$$

The following holds for all  $x \in (0, x_0)$ :

$$\partial_{xx}G(x, x_0) = 0.$$

This implies that  $G(x, x_0) = ax + b$  in  $(0, x_0)$ . The boundary condition  $G(0, x_0) = 0$  gives  $b = 0$ . Likewise, the following holds for all  $x \in (x_0, L)$ :

$$\partial_{xx}G(x, x_0) = 0.$$

This implies that  $G(x, x_0) = cx + d$  in  $(x_0, L)$ . The boundary condition  $\partial_x G(L, x_0) = 0$  gives  $c = 0$ . The continuity of  $G(x, x_0)$  at  $x_0$  implies that  $ax_0 = d$ . The condition

$$\int_{-\epsilon}^{\epsilon} \partial_{xx}G(x, x_0) dx = 1, \quad \forall \epsilon > 0,$$

gives the so-called jump condition:  $\partial_x G(x_0^+, x_0) - \partial_x G(x_0^-, x_0) = 1$ . This means that  $0 - a = 1$ , i.e.,  $a = -1$  and  $d = -x_0$ . In conclusion

$$G(x, x_0) = \begin{cases} -x & \text{if } 0 \leq x \leq x_0, \\ -x_0 & \text{otherwise.} \end{cases}$$


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(b) Give the integral representation of  $u$  using the Green's function.

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Let  $x_0$  be a point in  $(0, L)$ . The definition of the Dirac measure at  $x_0$  is such that

$$\begin{aligned} u(x_0) &= \langle \delta_{x_0}, u \rangle = \langle \partial_{xx}G(\cdot, x_0), u \rangle \\ &= - \int_0^L \partial_x G(x, x_0) \partial_x u(x) dx + [\partial_x G(x, x_0) u(x)]_0^L \\ &= \int_0^L G(x, x_0) \partial_{xx} u(x) dx - [G(x, x_0) \partial_x u(x)]_0^L + [\partial_x G(x, x_0) u(x)]_0^L \\ &= \int_0^L G(x, x_0) f(x) dx - G(L, x_0) \partial_x u(L) + G(0, x_0) \partial_x u(0) + \partial_x G(L, x_0) u(L) - \partial_x G(0, x_0) u(0). \end{aligned}$$

This finally gives the following representation of the solution:

$$u(x_0) = \int_0^L G(x, x_0) f(x) dx - G(L, x_0) b - \partial_x G(0, x_0) a$$


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