Quiz 6 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Consider the wave equation $\partial_{tt}w - \partial_{xx}w = 0$, $x \in (0,4)$, $t > 0$, with

$$w(x,0) = f(x), \quad x \in (0,4), \quad \partial_t w(x,0) = 0, \quad x \in (0,4), \quad \text{and} \quad w(0,t) = 0, \quad w(4,t) = 0, \quad t > 0.$$

where $f(x) = 1$, if $x \in [1,2]$ and $f(x) = 0$ otherwise. (i) Give a simple expression of the solution in terms of an extension of f.

We know from class that with Dirichlet boundary conditions, the solution to this problem is given by the D'Alembert formula where f must be replaced by the periodic extension (of period 8) of its odd extension, say $f_{\mathsf{o,p}}$, where

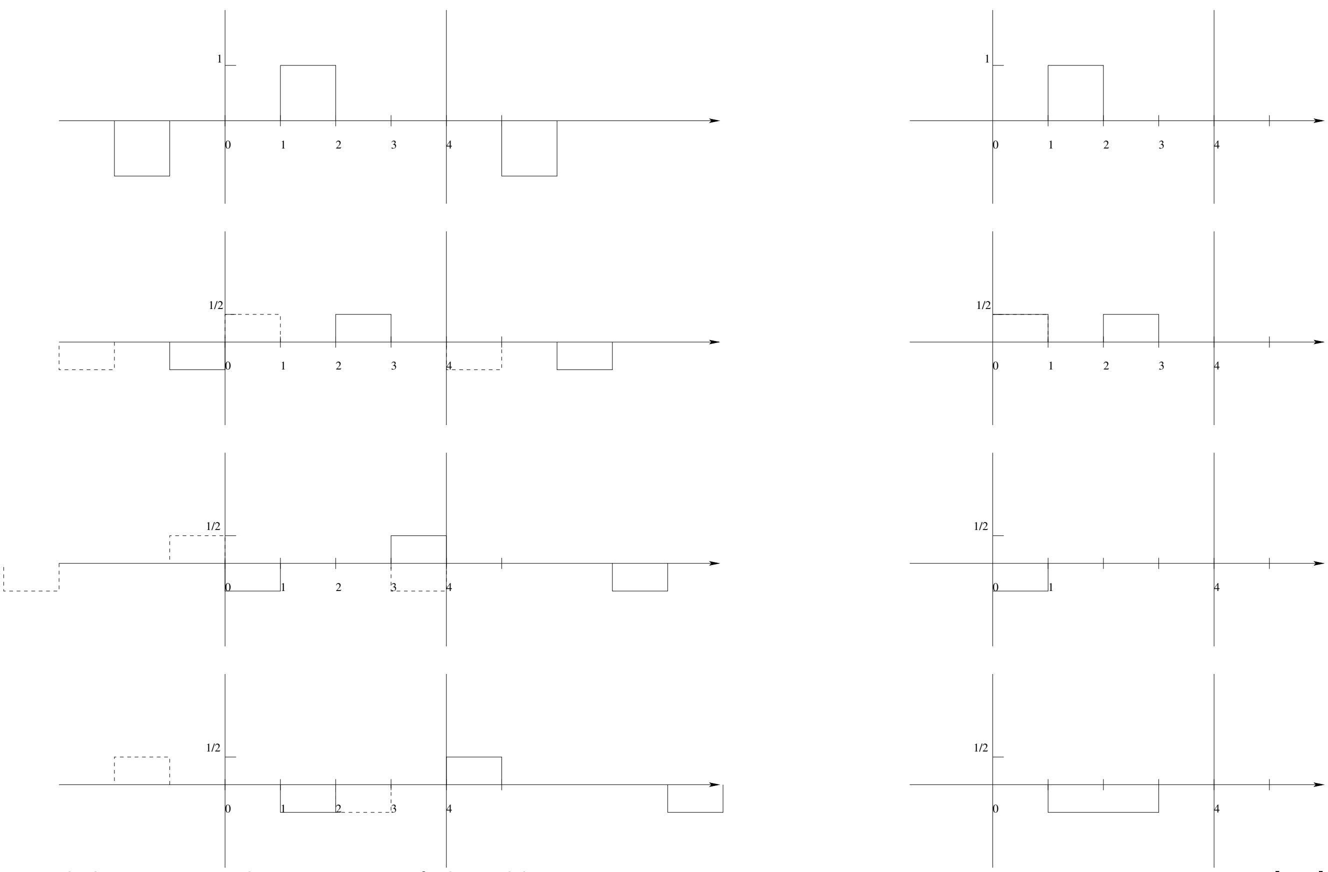
$$f_{\mathsf{o,p}}(x+8) = f_{\mathsf{o,p}}(x), \qquad \forall x \in \mathbb{R}$$
$$f_{\mathsf{o,p}}(x) = \begin{cases} f(x) & \text{if } x \in [0,4] \\ -f(-x) & \text{if } x \in [-4,0) \end{cases}$$

The solution is

$$u(x,t) = \frac{1}{2}(f_{\mathsf{o,p}}(x-t) + f_{\mathsf{o,p}}(x+t)).$$

(ii) Give a graphical solution to the problem at $t = 0$, $t = 1$, $t = 2$, and $t = 3$ (draw four different graphs and explain).

I draw on the left of the figure the graph of $f_{\mathsf{o,p}}$. Half of the graph moves to the right with speed 1, the other half moves to the left with speed 1.



Initial data + periodic extension of the odd extension at $t = 0, 1, 2, 3$.

Solution in domain $[0,4]$ at $t = 0, 1, 2, 3$

Question 2: Let $H : \mathbb{R} \longrightarrow \mathbb{R}$ be the Heaviside function. (a) Show that the derivative of $H(x)\sin(x)$ in the distribution sense is equal to $H(x)\cos(x)$. (Hint: Compute $-\int_{-\infty}^{\infty} H(x)\sin(x)\partial_x\psi(x)\mathrm{d}x$ for any $\psi \in \mathcal{C}_c^1(\mathbb{R})$, integrate by parts ...).

Recall that by definition $\partial_x(H\sin)$ is the distribution that is such that

$$\langle \partial_x(H\sin), \psi\rangle = \int \partial_x(H\sin)\psi := -\int_{-\infty}^{\infty} H(x)\sin(x)\partial_x\psi(x)\mathsf{d}x$$

for all $\psi \in \mathcal{C}_c^1(\mathbb{R})$. We then follow the hint and integrate by parts:

$$\begin{aligned}\langle \partial_x(H\cos), \psi\rangle &= -\int_{-\infty}^{\infty} H(x)\sin(x)\partial_x\psi(x)\mathsf{d}x\\ &= -\int_{0}^{\infty} \sin(x)\partial_x\psi(x)\mathsf{d}x = \int_{0}^{\infty}\cos(x)\psi(x)\mathsf{d}x\\ &= \int_{-\infty}^{\infty} H(x)\cos(x)\psi(x)\mathsf{d}x.\end{aligned}$$

This means that $\partial_x(H(x)\cos(x)) = H(x)\cos(x)$.

(b) Show that the derivative of $H(x)\cos(x)$ in the distribution sense is equal to $-H(x)\sin(x) + \delta_0$ where δ_0 is the Dirac measure at 0. (Hint: Compute $-\int_{-\infty}^{\infty} H(x)\cos(x)\partial_x\psi(x)\mathrm{d}x$ for any $\psi \in \mathcal{C}_c^1(\mathbb{R})$, integrate by parts ...).

Recall that by definition $\partial_x(H\cos)$ is the distribution that is suchy that

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for all $\psi \in \mathcal{C}_c^1(\mathbb{R})$. We then follow the hint and integrate by parts:

$$\begin{aligned}\langle \partial_x(H\cos), \psi\rangle &= -\int_{-\infty}^{\infty} H(x)\cos(x)\partial_x\psi(x)\mathsf{d}x\\ &= -\int_{0}^{\infty} \cos(x)\partial_x\psi(x)\mathsf{d}x = -\int_{0}^{\infty}\sin(x)\psi(x)\mathsf{d}x + \psi(0)\\ &= \langle \delta_0, \psi\rangle - \int_{-\infty}^{\infty} H(x)\sin(x)\psi(x)\mathsf{d}x\\ &= \langle \delta_0 - H\sin, \psi\rangle.\end{aligned}$$

This means that $\partial_x(H(x)\cos(x)) = \delta_0 - H(x)\sin(x)$.