IMMANANTS, TENSOR NETWORK STATES AND THE GEOMETRIC COMPLEXITY THEORY PROGRAM

A Dissertation

by

KE YE

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2012

Major Subject: Mathematics

IMMANANTS, TENSOR NETWORK STATES AND THE GEOMETRIC COMPLEXITY THEORY PROGRAM

A Dissertation

by

KE YE

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	Joseph Landsberg
Committee Members,	Paulo Lima-Filho
	Frank Sottile
	Colleen Robles
	Katrin Becker
Head of Department,	Emil Straube

Aug 2012

Major Subject: Mathematics

ABSTRACT

IMMANANTS, TENSOR NETWORK STATES AND THE GEOMETRIC COMPLEXITY THEORY PROGRAM. (Aug 2012)

Ke Ye, B.S., Sichuan University

Chair of Advisory Committee: J.M. Landsberg

We study the geometry of immanants, which are polynomials on n^2 variables that are defined by irreducible representations of the symmetric group $\mathfrak{S}_n$. We compute stabilizers of immanants in most cases by computing Lie algebras of stabilizers of immanants. We also study tensor network states, which are special tensors defined by contractions. We answer a question about tensor network states asked by Grasedyck. Both immanants and tensor network states are related to the Geometric Complexity Theory program, in which one attempts to use representation theory and algebraic geometry to solve an algebraic analogue of the P versus NP problem.

We introduce the Geometric Complexity Theory (GCT) program in Chapter one and we introduce the background for the study of immanants and tensor network states. We also explain the relation between the study of immanants and tensor network states and the GCT program.

Mathematical preliminaries for this dissertation are in Chapter two, including multilinear algebra, representation theory and complex algebraic geometry.

In Chapter three, we first give a description of immanants as trivial $(SL(E) \times SL(F)) \rtimes \Delta(\mathfrak{S}_n)$-modules contained in the space $S^n(E \otimes F)$ of polynomials of degree n on the vector space $E \otimes F$, where E and F are n dimensional complex vector spaces equipped with fixed bases and the action of $\mathfrak{S}_n$ on E (resp. F) is induced by permuting elements in the basis of E (resp. F). Then we prove that the stabilizer of an immanant for any non-symmetric partition is $T(GL(E) \times GL(F)) \rtimes \Delta(\mathfrak{S}_n) \rtimes \mathbb{Z}_2$, where $T(GL(E) \times GL(F))$

is the group of pairs of $n \times n$ diagonal matrices with the product of determinants equal to 1, $\Delta(\mathfrak{S}_n)$ is the diagonal subgroup of $\mathfrak{S}_n \times \mathfrak{S}_n$. We also prove that the identity component of the stabilizer of any immanant is $T(GL(E) \times GL(F))$.

In Chapter four, we prove that the set of tensor network states associated to a triangle is not Zariski closed and we give two reductions of tensor network states from complicated cases to simple cases.

In Chapter five, we calculate the dimension of the tangent space and weight zero subspace of the second osculating space of $\overline{GL_{n^2}.[perm_n]}$ at the point $[perm_n]$ and determine the $\mathfrak{S}_n \times \mathfrak{S}_n$-module structure of this space. We also determine some lines on the hyper-surface determined by the permanent polynomial.

In Chapter six, we give a summary of this dissertation.

To my wife, mother and father

ACKNOWLEDGMENTS

I would like to thank my committee chair, Dr. Joseph Landsberg, and my committee members, Dr. Paulo Lima-Filho, Dr. Frank Sottile, Dr. Colleen Robles and Dr. Katrin Becker, for the guidance and support throughout the course of this research.

I also would like to thank my friends and colleagues and the department faculty and staff for making my time at Texas A&M University a great experience.

Finally, I would like to thank my mother and father for encouragement and my wife for her patience and love.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION AND BACKGROUND

A. The history and background of the P versus NP problem

This section is based on [39], [23] and Wikipedia.

The P versus NP problem is a major unsolved problem in computer science and mathematics. The idea of using brute force search to solve certain problems is very old and natural. Theoretically, many problems can be solved in this way, though it becomes impractical if the search space is large. In the 1950's, researchers in the Soviet Union were aware of the issue of brute force search. Yablonski described the issue for general problems and focused on the specific problem of constructing the smallest circuit for a given function. In 1956, Gödel described in his letter to Von Neumann the issue in a remarkably modern way, formulating it in terms of the time required on a Turing machine to test whether a formula in the predicate calculus has a proof of length n. In 1965, Edmonds gave the first lucid account of the issue of brute force search appearing in western literature. The study of the issue of brute force search led to the definition of the classes P and NP. It was introduced by Stephen Cook in his seminal paper [12]. Informally, the P versus NP problem asks whether every problem whose solution can be quickly verified by a computer can also be quickly solved by a computer. For example, given a set of integers $\{-2, -3, 15, 14, 7, -10\}$, the statement "$\{2, -3, -10, 15\}$ adds up to zero" can be quickly verified with three additions. However, there is no known algorithm to find a subset of $\{-2, -3, 15, 14, 7, -10\}$ adding up to zero in polynomial time. To solve the P versus NP problem, computer scientists and mathematicians proposed many variants of this problem. The determinant versus the permanent is a famous algebraic variant of the P versus NP problem. The determinant can be computed in polynomial time by Gaussian elimination. There is no such a fast algorithm found for the permanent though it is known that the permanent polynomial can be realized as a linear projection of the determinant of a matrix of sufficiently large size. (Sufficiently large means exponential.) The intuition is that the permanent is much harder than the determinant. Many computer scientists and mathematicians are working to verify this intuition.

Leslie Valiant first defined in [41] in 1979 an algebraic analogue of the P versus NP problem, which is now called the VP versus VNP problem. Readers are referred to the appendix of this dissertation or [8] for the definition of these complexity classes. In 1979, Valiant gave the following conjecture in his paper [40]:

Conjecture A.1. *The permanent is not in VP.*

K. Mulmuley and M.Sohoni outlined an algebraic approach towards the P versus NP problem in a series of papers [28]-[34]. This approach is called the Geometric Complexity Theory (GCT) program. The GCT program studies the orbit closures of the determinant and the permanent

$$\widehat{\mathcal{D}et_n} := \overline{GL_{n^2} \cdot det_n} \subset S^n\mathbb{C}^{n^2},$$

$$\widehat{\mathcal{P}erm_n^m} := \overline{GL_{n^2} \cdot l^{n-m} perm_m} \subset S^n\mathbb{C}^{n^2}.$$

Here det_n is the determinant polynomial of $n \times n$ matrices and $perm_m$ is the permanent polynomial of $m \times m$ matrices. l is a linear coordinate on $\mathbb{C}$ and one takes any linear inclusion $\mathbb{C} \oplus \mathbb{C}^{m^2} \subset \mathbb{C}^{n^2}$ to have $l^{n-m}perm_m$ be a homogeneous degree n polynomial on $\mathbb{C}^{n^2}$. GL_{n^2} denotes the group of $n^2 \times n^2$ invertible matrices and $S^n\mathbb{C}^{n^2}$ denotes the space of homogeneous polynomials of degree n in n^2 variables. The overline denotes closure.

The following is an algebraic analogue of the P versus NP problem conjectured by Mulmuley and Sohoni:

Conjecture A.2. *There does not exist a constant $c \geq 1$ such that for all sufficiently large m, $\widehat{\mathcal{P}erm_{m^c}^m} \subset \widehat{\mathcal{D}et_{m^c}}$.*

One can use representation theory, algebraic geometry and local differential geometry to study $\widehat{\mathcal{P}erm_n^m}$ and $\widehat{\mathcal{D}et_n}$. Representation theory deals with modules and homomorphisms of modules. Roughly speaking, if we can decompose a module into the direct sum of some irreducible sub-modules then to compare two modules, it is sufficient to compare the multiplicity of each irreducible component. In our case of the GCT program, for example, to compare varieties $\widehat{\mathcal{P}erm_n^m}$ and $\widehat{\mathcal{D}et_n}$, one can study the coordinate rings of these two varieties and by construction these rings are all GL_{n^2}-modules. But the difficulty is that although the coordinate ring of the orbit is theoretically understood (although efficient computation of it is not known), it is very hard to determine the coordinate ring of the orbit closure, even theoretically. One can use algebraic geometry and local differential geometry to study the geometry of these varieties. For example, by computing the differential invariants one obtains a lower bound for a permanent polynomial being a projection of a determinant polynomial, see [27] for more details.

The study of immanants and tensor network states in this dissertation is related to the GCT program and other theoretical problems about efficient and feasible computations. The rest of this chapter explains the importance of these two objects and their relations to the GCT program.

B. Introduction to immanants

In Chapter three, we study stabilizers of immanants. Before we state main results of Chapter three, we first introduce immanants. D.E. Littlewood defined immanants in [25] as polynomials of degree n in n^2 variables associated to irreducible representations of the symmetric group $\mathfrak{S}_n$. These polynomials generalize the notion of the determinant and the permanent.

Definition B.1. For any partition $\pi \vdash n$, i.e., π is a non-increasing sequence of integers $(\pi_1, \ldots, \pi_k)$ such that $\sum_{i=1}^k \pi_i = n$, we define a polynomial of degree n in matrix variables

$(x_{ij})_{n\times n}$ associated to π as follows:

$$im_\pi := \sum_{\sigma\in\mathfrak{S}_n} \chi_\pi(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)},$$

where χ_π is the character of the representation of $\mathfrak{S}_n$ associated to π. This polynomial is called the immanant associated to π.

Example B.2. If $\pi = (1,1,...,1)$ then im_π is exactly the determinant of the matrix $(x_{ij})_{n\times n}$.

Example B.3. If $\pi = (n)$ then im_π is the permanent $\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n} x_{i\sigma(i)}$.

In Chapter three, we determine the identity component of the stabilizer of the immanant associated to any partition.

Proposition B.4. *Let π be a partition of $n \geq 6$ such that $\pi \neq (1,...,1)$, $(4,1,1,1)$ or (n), then the identity component of the stabilizer of the immanant im_π in GL_{n^2} is $T(GL_n \times GL_n)/N$. Here $T(GL_n \times GL_n)$ is the group consisting of pairs of $n\times n$ diagonal matrices with the product of determinants equal to 1, acting by left and right matrix multiplication. N is the subgroup of $T(GL_n \times GL_n)$ consisting of pairs of diagonal matrices of the form $(aId_n, a^{-1}Id_n)$, where a is a nonzero complex number.*

Also we determine the stabilizer of the immanant associated to any non-symmetric partition:

Theorem B.5. *Let $n \geq 5$ and let π be a partition of n which is not symmetric, that is, π is not equal to its conjugate, and $\pi \neq (1,...,1)$ or (n), then the stabilizer of the immanant im_π in GL_{n^2} is $T(GL_n \times GL_n) \rtimes \Delta(\mathfrak{S}_n) \rtimes \mathbb{Z}_2/N$. Here $\Delta(\mathfrak{S}_n)$ is the diagonal of $\mathfrak{S}_n \times \mathfrak{S}_n$ acting by sending x_{ij} to $x_{\sigma(i)\sigma(j)}$ for any $\sigma \in \mathfrak{S}_n$ and $\mathbb{Z}_2$ is the finite group $\mathbb{Z}/2\mathbb{Z}$ acting by sending x_{ij} to x_{ji}. N is the subgroup of $T(GL_n \times GL_n)$ defined in Proposition B.4.*

In the GCT program, we study orbit closures of the determinant and the permanent. Since immanants are generalizations of the determinant and the permanent, it is natural to study the same objects for immanants. The study of the complexity theory of immanants indicates that immanants hopefully interpolate the complexity from the determinant to the permanent. Although most immanants have the same statbilizer, immanants belong to different complexity classes.

Proposition B.6. *([7])* $IM_{(2,1^{n-2})} = (im_{(2,1^{n-2})})_{n\in\mathbb{N}}$ *is contained in* VP*.*

Theorem B.7. *([6]) For each fixed* k*, the sequence* $IM_{(n-k,1^k)} = (im_{n-k,1^k})_{n\in\mathbb{N}}$ *is* VNP*-complete.*

Therefore, to have a richer source of examples of the GCT program, we can also study orbit closures of immanants associated to $(2, 1^{n-2})$ and $(n-1, 1)$.

C. Introduction to tensor network states

In the GCT program, we want to compare orbit closures of the determinant and the permanent. An idea to solve this problem is to understand the difference between an orbit and its closure. Although the study of tensor network states is motivated by a question in quantum information theory, it turns out that tensor network states are closely related to the GCT program. For example, tensor network states provide some examples of the GCT type varieties, i.e., the set of tensor network states associated to a triangle is related to the orbit of the matrix multiplication tensor and this orbit and its closure are studied in the complexity theory, because it is important in computer science to know a bound of the number of operations needed to compute the product of matrices.

Tensor network states are also related to making computations that would be impossible if done in the naive space feasible by finding a good smaller space to work in. For example, if we want to study some tensors in a tensor product of n vector spaces of dimension 2, then we need to work in an ambient vector space of dimension 2^n, which is impossible to study via a computer when n is large. But if these tensors are special, i.e., these tensors are defined by the contraction of tensors, then we can just study a smaller ambient space consisting of tensor network states.

Tensor network states are special tensors associated to a graph with vector spaces attached to its vertices and edges. We use $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$ to denote the set of tensor network states associated to the graph Γ. (See Chapter IV for the formal definition of this set.). The main result of Chapter four is:

Theorem C.1. *$TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$ is not Zariski closed for any Γ containing a cycle whose vertices have sufficiently large dimensions.*

D. Notations

1. $\mathfrak{S}_n$ is the symmetric group on n elements. Given $\sigma \in \mathfrak{S}_n$, we can express σ as disjoint product of cycles, and we can denote the conjugacy class of σ by $(1^{i_1}2^{i_2}...n^{i_n})$, meaning that σ is a disjoint product of i_1 1-cycles, i_2 2-cycles, ..., i_n n-cycles. Sometimes we might use $(k_1, k_2, ..., k_p)$ (where $n \geq k_1 \geq k_2 \geq ... \geq k_p \geq 1$ and $\sum_{i=1}^{p} k_i = n$) to indicate the cycle type of σ. This notation means that σ contains a k_1-cycle, a k_2-cycle, ..., and a k_p-cycle.

2. $\pi \vdash n$ denotes the partition π of n.

3. $\pi^{'}$ is the conjugate partition of π.

4. $[\pi]$ is the irreducible representation of the symmetric group $\mathfrak{S}_n$ corresponding to the partition π of n.

5. Let λ be a partition of at most n parts, i.e., $\lambda = (\lambda_1, ..., \lambda_n)$, where $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, then we use $S_\lambda E$ to denote the irreducible representation of $GL(E)$ associated to λ.

6. χ_π is the character of $[\pi]$.

7. For a polynomial P in variables $(x_{ij})_{n\times n}$, denote the stabilizer of P in GL_{n^2} by $G(P)$.

8. Let V be a complex vector space. We use $GL(V)$ to denote the group of invertible transformations of V, and $End(V)$ denotes the vector space of endomorphisms of V.

9. Let E be a complex vector space of dimension n. Then $SL(E)$ is the group of linear transformations such that the induced linear transformations on $\wedge^n E$ is the identity.

10. Let V be a representation of $SL(E)$, then the weight-zero-subspace of V is denoted as V_0.

CHAPTER II

PRELIMINARIES

A. Multilinear algebra

1. Tensor product

Definition A.1. Let U and V be complex vector spaces and let U^* and V^* be dual vector spaces of U and V. We define $U^* \otimes V^*$ to be the set of all bilinear functions $f : U \times V \mapsto \mathbb{C}$.

Remark A.2. For vector spaces $V_1, ..., V_n$, one can either define $V_1^* \otimes \cdots \otimes V_n^*$ inductively or define it directly as the set of all multi-linear maps $f : V_1 \times \cdots \times V_n \mapsto \mathbb{C}$.

2. Symmetric tensors

Definition A.3. Let V be a finite dimensional complex vector space. Define the d-th symmetric power S^dV of V to be the linear space spanned by elements of the form

$$v_1 \circ \cdots \circ v_n := \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

We call $v_1 \circ \cdots \circ v_n$ the symmetric product of $v_1, ..., v_n$.

Remark A.4. Let $n = dim(V)$, then S^dV is a complex vector space of dimension $\binom{n+d-1}{d}$.

We define S^dV^* as the space of symmetric d-linear forms on V. We can also identify S^dV^* as the space of homogeneous polynomials in degree d on V, since we have the polarization of any homogeneous polynomial. Let Q be a homogeneous polynomial in degree d on V, then the polarization $\overline{Q}$ of Q is defined as a d-linear form:

$$\overline{Q}(x_1, ..., x_d) = \frac{1}{d!} \sum_{I \subset [d], I \neq \emptyset} (-1)^{d-|I|} Q(\sum_{i \in I} x_i),$$

where $[d] = \{1, ..., d\}$ and $x_1, ..., x_d$ are elements in V.

B. Representation theory

1. Representations of finite groups

Definition B.1. A representation of a finite group G on a finite dimensional complex vector space V is a group homomorphism $\rho : G \mapsto GL(V)$ from G to $GL(V)$. We say that ρ gives V a structure of a G-module.

Example B.2. Let $G = \mathfrak{S}_d$ be the symmetric group on d elements. Let $V = \mathbb{C}^d$, and let $\{\epsilon_1, ..., \epsilon_d\} \subset V$ be the standard basis of V. Let $\sigma \in G$ be any permutation, and define the action of σ on V by $\sigma \cdot \epsilon_i = \epsilon_{\sigma(i)}$. Then this action gives a representation of G on V.

Definition B.3. A sub-representation of a representation V is a linear subspace W of V which is invariant under the action of G. A representation V is irreducible if V contains no proper non-zero sub-representation. A representation V is called indecomposable if V cannot be expressed as the direct sum of some proper sub-representations.

Example B.4. Let V be the representation of $\mathfrak{S}_d$ described in Example B.2. V is not irreducible because it contains sub-representations U spanned by the vector $\epsilon_1 + \cdots + \epsilon_d$ and W spanned by vectors $\{\epsilon_1 - \epsilon_j \mid j = 2, ..., d\}$. U is called the trivial representation of $\mathfrak{S}_d$ and W is called the standard representation of $\mathfrak{S}_d$.

Proposition B.5. *Let G be a finite group and let V be any representation of G, then V is a direct sum of irreducible representations.*

Proof. See [16, Corollary 1.6]. $\square$

Example B.6. In Example B.4 we see that V contains two sub-representations U and W. Indeed V is the direct sum of U and W.

Definition B.7. Define the regular representation R_G of G to be the vector space with basis $\{e_x \mid x \in G\}$ and let G act on R_G by

$$g \cdot \sum_{x \in G} a_x e_x = \sum_{x \in G} a_x e_{gx}.$$

Definition B.8. Let G be a finite group, we define the group algebra $\mathbb{C}G$ associated to G to be the underlying vector space of the regular representation equipped with the algebra structure given by

$$e_g \cdot e_h = e_{gh}.$$

Schur's lemma is very useful in representation theory.

Lemma B.9. *If V and W are irreducible representations of G and $\phi : V \mapsto W$ is a G-module homomorphism, then*

1. *Either ϕ is an isomorphism or $\phi = 0$*
2. *If $V = W$ then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$ and I the identity map.*

Proof. Both claims follow from the fact that the kernel and the image of any G-module homomorphism is again a G-module and that both V and W are irreducible representations of G. □

Definition B.10. If V is a representation of G via $\rho : G \mapsto GL(V)$, its character χ_V is the complex-valued function on the group defined by

$$\chi_V(g) = tr(\rho(g)),$$

the trace of g on V.

Remark B.11. The character χ_V of a representation V is invariant under conjugation, i.e., $\chi_V(h^{-1}gh) = \chi_V(g)$, such a function is called a class function. Note that $\chi_V(1) = dim(V)$.

Example B.12. Let $\sigma \in \mathfrak{S}_d$ be a permutation of d elements and let W be the standard representation of $\mathfrak{S}_d$. Then $\chi_W(\sigma)$ is the number of elements fixed by σ minus one.

Proposition B.13. *Any representation of a finite group is uniquely determined by its character.*

Proof. See [16, Corollary 2.14]. □

Let G be a finite group. We can define a Hermitian inner product on the space $\mathbb{C}_{class}(G)$ of all class functions on G to be

$$(\alpha, \beta) := \frac{1}{|G|} \sum_{g \in G} \overline{(\alpha(g))} \beta(g).$$

Then we have the following results (See [16, Chap. 2])

1. A representation V is irreducible if and only if $(\chi_V, \chi_V) = 1$.
2. Let V_i and V be two representations of G, then the multiplicity of V_i in V is the inner product of χ_{V_i} and χ_V.
3. Any irreducible representation V of G appears in the regular representation $dim(V)$ times.
4. The characters of the irreducible representations of G are orthonormal.
5. The number of irreducible representations of G is equal to the number of conjugacy classes of G.

2. Representations of $\mathfrak{S}_d$

A partition λ of d is a non-increasing sequence of natural numbers $(\lambda_1, ..., \lambda_k)$ such that $\lambda_1 + \cdots + \lambda_k = d$. We can associate a Young diagram to each partition λ of d with λ_i boxes in the i-th row, the rows of boxes lined up on the left and we use λ to denote the Young diagram associated to the partition λ. A Young tableau is a Young diagram with boxes filled with numbers from 1 to d, such that numbers are non-decreasing along each row and strictly increasing down each column. We use $\overline{\lambda}$ to denote a Young tableau associated to a Young diagram λ.

Let $\overline{\lambda}$ be a Young tableau, and let

$$P = P_\lambda = \{g \in \mathfrak{S}_d \mid g \text{ preserves each row of } \overline{\lambda}\},$$

and

$$Q = Q_\lambda = \{g \in \mathfrak{S}_d \mid g \text{ preserves each column of } \overline{\lambda}\}.$$

Let $a_\lambda = \sum_{g\in P} e_g$ and $b_\lambda = \sum_{g\in Q} sgn(g)e_g$. We define $c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}\mathfrak{S}_d$. c_λ is called the Young symmetrizer associated to $\overline{\lambda}$.

Theorem B.14. *Some scalar multiple of c_λ is idempotent, i.e., $c_\lambda^2 = n_\lambda c_\lambda$ for some $n_\lambda \in \mathbb{C}$. The image of c_λ (by right multiplication on $\mathbb{C}\mathfrak{S}_d$) is an irreducible representation $[\lambda]$ of $\mathfrak{S}_d$. Every irreducible representation of $\mathfrak{S}_d$ can be obtained in this way for a unique partition.*

Proof. See [16, Thereom 4.3]. □

Example B.15. There are 5 irreducible representations of $\mathfrak{S}_4$: $[(4)]$, $[(3,1)]$, $[(2,2)] = [(2^2)]$, $[(2,1,1)] = [(2,1^2)]$, $[(1,1,1,1)] = [(1^4)]$.

Let $C = (1^{i_1}, ..., d^{i_d})$ be a partition of d, i.e., C contains i_k k's. Let $x_1, ..., x_d$ be independent variables. Define the power sums $P_j(x)$, $1 \le j \le d$ and the discriminant $\Delta(x)$ by

$$P_j(x) = x_1^j + ... + x_d^j,$$

$$\Delta(x) = \prod_{i<j}(x_i - x_j).$$

If $f(x) = f(x_1, ..., x_d)$ is a formal power series, and $(l_1, ..., l_d)$ is a d-tuple of non-negative integers, let

$$[f(x)]_{(l_1,\ldots,l_d)} = \text{ coefficient of } x_1^{l_1}\cdots x_d^{l_d} \text{ in } f.$$

Let $\lambda = (\lambda_1, ..., \lambda_k)$ be a partition of d, and set $l_j = \lambda_j + d - j$ for each $j = 1, ..., k$. The character of $[\lambda]$ evaluated on a permutation g of cycle type C is given by

Theorem B.16 (Frobenius formula [16])**.** $\chi_\lambda(g) = [\Delta(x)\cdot\prod_{j=1}^{d} P_j(x)^{i_j}]_{(l_1,\ldots,l_d)}$.

From the Frobenius formula, we have the following formula to compute the dimension of $[\lambda]$([16]):

$$dim([\lambda]) = \frac{d!}{l_1!\cdots l_k!}\prod_{i<j}(l_i - l_j).$$

If we define the hook length of a box in a Young diagram to be the number of boxes directly below or directly to the right of the box, including the box once, then we have another expression of the dimension formula above. It is called the hook length formula.

$$dim([\lambda]) = \frac{d!}{\prod(\text{hook lengths})}.$$

Example B.17. $dim([(d-1,1)]) = \frac{d!}{d(d-2)(d-3)\cdots 1} = d-1$.

Definition B.18. A skew hook for a Young diagram λ is a connected region of boundary boxes for λ such that removing them leaves a smaller Young diagram.

We have an efficient inductive method for computing character values. This method is called the Murnaghan-Nakayama rule.

Theorem B.19 ([37])**.** *Let λ be a partition of d and let $g \in \mathfrak{S}_d$ be a permutation written as a product of an m-cycle and a disjoint permutation $h \in \mathfrak{S}_{d-m}$, then*

$$\chi_\lambda(g) = \sum(-1)^{r(\mu)}\chi_\mu(h),$$

where the sum is over all partitions μ of $d-m$ that are obtained from λ by removing a skew hook of length m, and $r(\mu)$ is the number of rows in the skew hook minus 1.

3. Lie groups

Definition B.20. Let G be a smooth manifold. G is called a Lie group if there exist smooth maps $m : G \times G \mapsto G$ and $\iota : G \mapsto G$ such that m and ι give a group structure on G, in which m is the multiplication and ι is the inversion.

Example B.21. Let V be a complex vector space, then $G = GL(V)$ is a Lie group where the map m is the usual matrix multiplication and the map ι is the inverse of a matrix.

Theorem B.22. *Let $dim(V) = n$, then irreducible representations of $GL(V)$ are indexed by non-increasing sequence $\lambda = (\lambda_1, ..., \lambda_n)$ such that $\lambda_j \geq 0$. These modules are denoted by $S_\lambda V = c_\lambda \cdot V^{\otimes d}$, where c_λ is the Young symmetrizer associated to a Young tableau $\overline{\lambda}$ of shape λ and the action of c_λ on $V^{\otimes d}$ is induced by the action of $\mathfrak{S}_d$ on $V^{\otimes d}$ given by*

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_d) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}, \text{ for any } \sigma \in \mathfrak{S}_d.$$

Proof. See [16, Theorem 6.3]. □

4. Pieri's fomula

Let V be a complex vector space and let π and μ be two partitions. Then the tensor product of $S_\pi V$ and $S_\mu V$ has a decomposition as $GL(V)$-modules:

$$S_\pi V \otimes S_\mu V = \sum_\nu (S_\nu V)^{\oplus c^\nu_{\pi\mu}}.$$

$c^\nu_{\pi\mu}$ is called the Littlewood-Richardson coefficient.

In particular, we can consider the case where $\mu = (d)$, then we have the following Pieri's formula.

Theorem B.23. *[16, Chap.6] If ν is obtained from λ by adding d boxes to the rows of λ with no two in the same column, then $c^\nu_{\lambda,(d)} = 1$, and $c^\nu_{\lambda,(d)} = 0$ otherwise.*

Example B.24. $S^n V \otimes V = S_{(n)} V \otimes V = S_{(n+1)} V \oplus S_{(n,1)} V.$

C. Complex algebraic geometry

1. Affine and projective varieties

Definition C.1. Let X be a topological set. We say that X is reducible if there exists two proper closed subsets X_1 and X_2 such that $X = X_1 \cup X_2$. Otherwise we say that X is irreducible.

Definition C.2. Let $V = \mathbb{C}^n$ be a complex n dimensional vector space. An algebraic set in V is the set of common zeroes of a set of polynomials on V. Let X be an algebraic subset of V, we define the ideal of X to be the ideal $I(X)$ generated by polynomials on V vanishing on X. An affine variety is an irreducible algebraic subset of V.

Example C.3. Any linear subspace of V is an affine variety.

Definition C.4. For an affine variety $X \subset V$ we define the coordinate ring of X to be $\mathbb{C}[X] := \mathbb{C}[x_1, ..., x_n]/I(X)$.

Proposition C.5. *An affine variety is uniquely determined by its coordinate ring.*

Proof. See [20, Chap. 1, Corollary 3.7]. □

Definition C.6. Let $V = \mathbb{C}^n$ be a complex vector space of dimension n, and let $\mathbb{C}^*$ be the multiplication group of nonzero complex numbers. We define the projective space $\mathbb{P}V$ to be $V - \{\text{origin}\}/\mathbb{C}^*$, and we define an algebraic subset of $\mathbb{P}V$ to be the common zero set of a set of homogeneous polynomials on V. We use $I(X)$ to denote the ideal of all homogeneous polynomials vanishing on X. And we define a projective variety to be an irreducible algebraic subset X of $\mathbb{P}V$.

Remark C.7. We can define the homogeneous coordinate ring of a projective variety to be $\mathbb{C}[x_1, ..., x_n]/I(X)$, but a projective variety is not uniquely determined by its homogeneous coordinate ring. For example, the projective line $\mathbb{P}^1$ is isomorphic to the curve defined by $xz - y^2$ in $\mathbb{P}^2$, but their homogeneous coordinate rings are not isomorphic.

Definition C.8. Let X be an affine variety, then $dim(X)$ is defined to be the Krull dimension of the coordinate ring of X.

2. Singular locus of a variety

Definition C.9. Let $I(X)$ be the ideal of a variety X of dimension r and let $\{f_1, ..., f_s\}$ be a set of generators of $I(X)$. Then the common zero set of $(n-r) \times (n-r)$ minors of the Jacobian matrix $(\dfrac{\partial f_i}{\partial x_j})$ is called the singular locus of X. Any point in X that is not in the singular locus is called a non-singular point or a smooth point.

Remark C.10. The definition of the singular locus depends on the choices of generators of the ideal, but it turns out that different choices of generators give the same singular locus, see [20, Chap. 1].

Definition C.11. Inductively, one can define the k-th singular locus of a variety X as follows. Assume the $(k-1)$-th singular locus $X^{(k-1)}$ is defined, then the k-th singular locus of X is defined to be the singular locus of $X^{(k-1)}$ and is denoted by $X^{(k)}$.

D. Abstract algebra

Definition D.1. Let N and H be two groups, and let $\pi : H \mapsto Aut(N)$ be a group homomorphism. There exists a group $N \rtimes_\pi H$ called the semidirect product of N and H, with respect to π, defined as follows:

- As a set $N \rtimes_\pi H$ is the Cartesian product $N \times H$.
- Let n_1, n_2 be elements of N and let h_1, h_2 be elements of H. Define the multiplication on $N \rtimes_\pi H$ by

  $$(n_1, h_1) \cdot (n_2, h_2) = (n_1(\pi(h_1)(n_2)), h_1h_2).$$

  The identity element of $N \rtimes_\pi H$ is (e_N, e_H) where e_N and e_H are identity elements of N and H, respectively. The inverse of (n_1, h_1) is $(\pi(h_1^{-1})(n_1^{-1})^{-1}, h_1^{-1})$.

Remark D.2. We use $N \rtimes H$ to denote $N \rtimes_\pi H$ if π is obvious.

CHAPTER III

STABILIZERS OF IMMANANTS

This chapter is based on [44].

A. Introduction

Let E and F be $\mathbb{C}^n$ equipped with bases $\{e_1, ..., e_n\}$ and $\{f_1, ..., f_n\}$. For simplicity, sometimes we use x_{ij} to denote $e_i \otimes f_j$. Since immanants are homogeneous polynomials of degree n in n^2 variables, we can identify them as elements in $S^n(E \otimes F)$. (Identify the space $E \otimes F$ with the space of $n \times n$ matrices.) The space $S^n(E \otimes F)$ is a representation of $GL(E \otimes F)$, in particular, it is a representation of $GL(E) \times GL(F) \subset GL(E \otimes F)$. So we can use the representation theory of $GL(E) \times GL(F)$ to study immanants. The explicit expression of an immanant im_π in $S^n(E \otimes F)$ is:

$$\sum_{\sigma \in \mathfrak{S}_n} \chi_\pi(\sigma) \prod_{i=1}^{n} e_i \otimes f_{\sigma(i)},$$

where $\prod$ is interpreted as the symmetric tensor product.

In section B we prove that immanants can be defined as trivial $\mathfrak{S}_n$ modules (Proposition B.3). Duffner, M. Antónia found the system of equations determining the stabilizer of immanants (except the determinant and the permanent) for $n \geq 4$ in [13] in 1994. Two years later, Coelho, M. Purificação proved in [10] that if the system of equations in [13] has a solution, then using the notation of [10], permutations τ_1 and τ_2 in the system must be the same. Building on works of Duffner and Coelho, we prove the main results Proposition A.1 and Theorem A.3 of this Chapter in section C.

Proposition A.1. *Let π be a partition of $n \geq 6$ such that $\pi \neq (1, ..., 1)$, $(4, 1, 1, 1)$ or (n), then the identity component of the stabilizer of the immanant im_π in $GL(E \otimes F)$ is $T(GL(E) \times GL(F))/N$. Here $T(GL(E) \times GL(F))$ is the group consisting of pairs of*

$n \times n$ diagonal matrices with the product of determinants equal to 1, acting on $E \otimes F$ in the obvious way. N is the subgroup of $T(GL(E) \times GL(F))$ consisting of pairs of diagonal matrices of the form $(aI_E, a^{-1}I_F)$, where a is a nonzero complex number.

Remark A.2. It is well-known that Proposition A.1 is true for the permanent as well, but our proof does not recover this case.

Theorem A.3. *Let $n \geq 5$ and let π be a partition of n which is not symmetric, that is, π is not equal to its conjugate, and $\pi \neq (1, ..., 1)$ or (n), then the stabilizer of the immanant im_π is $T(GL(E) \times GL(F)) \rtimes \Delta(\mathfrak{S}_n) \rtimes \mathbb{Z}_2/N$. Here N is the subgroup of $T(GL(E) \times GL(F))$ defined in PropositionA.1 and the action of $\Delta(\mathfrak{S}_n)$ on $E \otimes F$ is induced by*

$$(\sigma, \sigma) \cdot x_{ij} = x_{\sigma(i)\sigma(j)},$$

and the action of $\mathbb{Z}_2$ on $E \otimes F$ is induced by

$$\epsilon \cdot x_{ij} = x_{ji},\ \epsilon \text{ is the generator of } \mathbb{Z}_2.$$

Remark A.4. We use the semidirect product without specifying the action of $\Delta(\mathfrak{S}_n)$ on $T(GL(E) \times GL(F))$ and the action of $\mathbb{Z}_2$ on $T(GL(E) \times GL(F)) \rtimes \Delta(\mathfrak{S}_n)$ because these actions are induced by actions of $T(GL(E) \times GL(F))$, $\Delta(\mathfrak{S}_n)$ and $\mathbb{Z}_2$ on $E \otimes F$.

Remark A.5. One can compute directly from the system of equations determined by Duffner, M. Antónia for the case $n = 4$ and $\pi = (2, 2)$ and see that in this case, Theorem A.3 fails, since there will be many additional components. For example,

$$C = \begin{pmatrix} e & -e & -e & e \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \frac{1}{e} & -\frac{1}{e} & \frac{1}{e} & -\frac{1}{e} \end{pmatrix}$$

stabilizes the immanant $im_{(2,2)}$, but it is not in the identity component.

B. The description of immanants as modules

Consider the action of $T(E) \times T(F)$ on immanants, where $T(E)$, $T(F)$ are maximal tori (diagonal matrices) of $SL(E)$, $SL(F)$, respectively. For any $(A,B) \in T(E) \times T(F)$,

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}, \mathbf{B} = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{pmatrix}$$

For the immanant $im_\pi = \sum_{\sigma \in \mathfrak{S}_n} \chi_\pi(\sigma) \prod_{i=1}^n x_{i\sigma(i)}$, the action of (A,B) on im_π is given by

$$(A,B).im_\pi := \sum_{\sigma \in \mathfrak{S}_n} \chi_\pi(\sigma) \prod_{i=1}^n a_i b_{\sigma(i)} x_{i\sigma(i)} \tag{3.1}$$

$$= im_\pi. \tag{3.2}$$

So $T(E) \times T(F)$ acts on im_π trivially. That is, immanants are in the $SL(E) \times SL(F)$ weight-zero-subspace of $S^n(E \otimes F)$. On the other hand, we have the decomposition of $S^n(E \otimes F)$ as $GL(E) \times GL(F)$-modules:

$$S^n(E \otimes F) = \sum_{\lambda \vdash n} S_\lambda E \otimes S_\lambda F.$$

Accordingly, we have a decomposition of the weight-zero-subspace:

$$(S^n(E \otimes F))_0 = \sum_{\lambda \vdash n} (S_\lambda E)_0 \otimes (S_\lambda F)_0.$$

Proposition B.1. *For $\lambda \vdash n$, $(S_\lambda E)_0 \cong [\lambda]$ as $\mathfrak{S}_n$-modules.*

Proof. See [36, p.272]. □

Thus we can identify $(S_\lambda E)_0 \otimes (S_\lambda F)_0$ with $[\lambda] \otimes [\lambda]$ as an $\mathfrak{S}_n \times \mathfrak{S}_n$ module. Also, the diagonal $\Delta(\mathfrak{S}_n)$ of $\mathfrak{S}_n \times \mathfrak{S}_n$ is isomorphic to $\mathfrak{S}_n$, so $[\lambda] \otimes [\lambda]$ is a $\mathfrak{S}_n$-module. $[\lambda] \otimes [\lambda]$

is an irreducible $\mathfrak{S}_n \times \mathfrak{S}_n$ module, but it is reducible as an $\mathfrak{S}_n$-module, so that we can decompose it. Consider the action of $\mathfrak{S}_n$ on $S^n(E \otimes F)$, let $\sigma \in \mathfrak{S}_n$, then:

$$\sigma.x_{ij} = x_{\sigma(i)\sigma(j)}.$$

So immanants are invariant under the action of $\mathfrak{S}_n$, hence are contained in the isotypic component of the trivial $\mathfrak{S}_n$ representation of $\bigoplus_{\lambda \vdash n}(S_\lambda E)_0 \otimes (S_\lambda F)_0 = \bigoplus_{\lambda \vdash n}[\lambda] \otimes [\lambda]$.

Proposition B.2. *As an $\mathfrak{S}_n$ module, $[\lambda] \otimes [\lambda]$ contains only one copy of trivial representation.*

Proof. Denote the character of $\sigma \in \mathfrak{S}_n$ on $[\lambda] \otimes [\lambda]$ by $\chi(\sigma)$, and let $\chi_{trivial}$ be the character of the trivial representation. From the general theory of characters, it suffices to show that the inner product $(\chi, \chi_{trivial}) = 1$. First, the character $\chi(\sigma, \tau)$ of $(\sigma, \tau) \in \mathfrak{S}_n \times \mathfrak{S}_n$ on the module $[\lambda] \otimes [\lambda]$ is $\chi_\lambda(\sigma)\chi_\lambda(\tau)$. So in particular, the character χ of $[\lambda] \otimes [\lambda]$ on σ is $(\chi_\lambda(\sigma))^2$. Next,

$$\begin{aligned}
(\chi, \chi_{trivial}) =& \frac{1}{n!}\Big(\sum_{\sigma \in \mathfrak{S}_n} \chi(\sigma)\overline{\chi_{trivial}(\sigma)}\Big) \\
=& \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} (\chi_\lambda(\sigma))^2 \\
=& (\chi_\lambda, \chi_\lambda) \\
=& 1
\end{aligned}$$

since $\chi_{trivial}(\sigma) = 1, \forall\, \sigma \in \mathfrak{S}_n$. $\square$

By the above proposition, $[\lambda] \otimes [\lambda] = \mathbb{C}_\lambda \oplus \cdots$, where $\mathbb{C}_\lambda$ means the unique copy of trivial representation in $[\lambda] \otimes [\lambda]$, and dots means other components in this module. Hence

$$im_\pi \in \bigoplus_{\lambda \vdash n} \mathbb{C}_\lambda.$$

We can further locate immanants:

Proposition B.3. *Let im_π be the immanant associated to the partition $\pi \vdash n$. Assume*

$\mathbb{C}_\pi$ is the unique copy of the trivial $\mathfrak{S}_n$-representation contained in $(S_\pi E)_0 \otimes (S_\pi F)_0$. Then: $im_\pi \in \mathbb{C}_\pi$.

Before proving this proposition, we remark that it gives an equivalent definition of the immanant: im_π is the element of the trivial representation $\mathbb{C}_\pi$ of $[\pi] \otimes [\pi]$ such that $im_\pi(I) = dim([\pi])$. For more information about this definition, see for example, [5].

Example B.4. If $\pi = (1, ..., 1)$, then $S_\pi E = \bigwedge^n E$, which is already a 1 dimensional vector space. If $\pi = (n)$, then $S_\pi E = S^n E$, in which there is only one (up to scale) $SL(E)$ weight zero vector $e_1 \circ \cdots \circ e_n$.

Proof. Fix a partition $\pi \vdash n$, we want to show that the immanant im_π is in $\mathbb{C}_\pi$, but we know that im_π is in the weight-zero-subspace $(S^n(E \otimes F))_0 = \oplus_{\lambda \vdash n} (S_\lambda(E))_0 \otimes (S_\lambda(F))_0$. Since $(S_\lambda(E))_0 \otimes (S_\lambda(F))_0 \subset S_\lambda(E) \otimes S_\lambda(F)$, it suffices to show that $im_\pi \in S_\pi E \otimes S_\pi F$. Then it suffices to show that for any young symmetrizer c_λ not of the shape π, $c_\lambda \otimes c_\lambda(im_\pi) = 0$. It suffices to check that $1 \otimes c_\lambda(im_\pi) = 0$, since $c_\lambda \otimes c_\lambda = (c_\lambda \otimes 1) \circ (1 \otimes c_\lambda)$. Express im_π as an element in $S^n(E \otimes F)$:

$$im_\pi = \sum_{\sigma \in \mathfrak{S}_n} \sum_{\tau \in \mathfrak{S}_n} \chi_\pi(\sigma) (\otimes_{i=1}^n e_{\tau(i)}) \otimes (\otimes_{i=1}^n f_{\sigma \circ \tau(i)})$$

The young symmetrizer $c_\lambda = \sum_{p \in P_\lambda,\, q \in Q_\lambda} sgn(q)\, p\, q$. So

$$1 \otimes c_\lambda(im_\pi) = \sum_{\tau \in \mathfrak{S}_n} \otimes_{i=1}^n e_{\tau(i)} \otimes (\sum_{\substack{p \in P_\lambda q \in Q_\lambda \\ \sigma \in \mathfrak{S}_n}} \chi_\pi(\sigma) sgn(q) \otimes_{i=1}^n f_{\sigma \cdot \tau \cdot q \cdot p(i)})$$

In the above expression, c_λ acts on f_i's. Now it suffices to show that:

$$\sum_{\substack{p \in P_\lambda \\ q \in Q_\lambda \\ \sigma \in \mathfrak{S}_n}} \chi_\pi(\sigma) sgn(q) \bigotimes_{i=1}^n f_{\sigma \cdot \tau \cdot q \cdot p(i)} = 0, \forall\, \tau \in \mathfrak{S}_n.$$

For any $\tau \in \mathfrak{S}_n$,

$$\sum_{\substack{p\in P_\lambda\\ q\in Q_\lambda\\ \alpha\in\mathfrak{S}_n}} \chi_\pi(\alpha\cdot\tau^{-1})sgn(q)\bigotimes_{i=1}^{n} f_{\alpha\cdot q\cdot p(i)} = \sum_{\gamma\in\mathfrak{S}_n}\sum_{\substack{p\in P_\lambda\\ q\in Q_\lambda\\ \alpha\in\mathfrak{S}_n\\ \alpha\cdot q\cdot p=\gamma}} \chi_\pi(\alpha\cdot\tau^{-1})sgn(q)\bigotimes_{i=1}^{n} f_{\gamma(i)}$$

$$= \sum_{\gamma\in\mathfrak{S}_n}[\sum_{\substack{p\in P_\lambda\\ q\in Q_\lambda}} \chi_\pi(\gamma\cdot p^{-1}\cdot q^{-1}\cdot\tau^{-1})sgn(q)]\bigotimes_{i=1}^{n} f_{\gamma(i)}.$$

Let σ,$\tau \in \mathfrak{S}_n$, $\sigma\cdot\tau = \sigma\cdot\tau\cdot\sigma^{-1}\cdot\sigma$, so $\sigma\cdot\tau = \tau^{'}\cdot\sigma$, where $\tau^{'}$ is conjugate to τ in $\mathfrak{S}_n$ by σ. Therefore, we can rewrite the previous equation as:

$$\sum_{\gamma\in\mathfrak{S}_n}[\sum_{\substack{p\in P_\lambda\\ q\in Q_\lambda}} \chi_\pi(\tau^{-1}\cdot\gamma\cdot p^{-1}\cdot q^{-1})sgn(q)]\bigotimes_{i=1}^{n} f_{\gamma(i)}.$$

Therefore, it suffices to show:

$$\sum_{\substack{p\in P_\lambda\\ q\in Q_\lambda}} \chi_\pi(\gamma\cdot p\cdot q)sgn(q) = 0, \forall\, \gamma\in\mathfrak{S}_n$$

This equality holds because the left hand side is the trace of $\gamma\cdot c_\lambda$ as an operator on the space $\mathbb{C}\mathfrak{S}_n\cdot c_\pi$,the group algebra of $\mathfrak{S}_n$, which is a realization of $[\pi]$ in $\mathbb{C}\mathfrak{S}_n$, but this operator is in fact zero:

$$\forall\, \sigma\in\mathfrak{S}_n\ ,\ \gamma\cdot c_\lambda\cdot\sigma\cdot c_\pi = \gamma\cdot\sigma\cdot c_{\lambda'}\cdot c_\pi = 0,$$

where $\lambda^{'}$ is the young tableau of shape $\lambda^{'}$ which is conjugate to λ by σ. This implies that c_λ and c_π are of different type, hence $c_\lambda\cdot c_\pi = 0$, in particular, the trace of this operator is 0. Therefore, $im_\pi \in S_\pi E\otimes S_\pi F$. □

Corollary B.5. *Immanants are linearly independent and form a basis of the space of all homogeneous degree n polynomials preserved by $(T(E)\times T(F))\rtimes\Delta(\mathfrak{S}_n)$.*

Proof. If Q is preserved by $(T(E)\times T(F))\rtimes\Delta(\mathfrak{S}_n)$, then Q is in $\bigoplus_{\lambda\vdash n}\mathbb{C}_\lambda$. By the proposition, immanants form a basis of $\bigoplus_{\lambda\vdash n}\mathbb{C}_\lambda$, the corollary follows. $\square$

C. Stabilizers of immanants

Next, we study stabilizers of immanants in the group $GL(E\otimes F)$.

Example C.1. For $\pi = (1,1,...,1)$ and $\pi = (n)$, $G(im_\pi)$ are well-known: If $\pi = (1,1,...,1)$, then $G(im_\pi) = S(GL(E)\times GL(F))$, and if $\pi = (n)$, then $G(im_\pi) = T(GL(E)\times GL(F)) \rtimes (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes \mathbb{Z}_2$, where $S(GL(E)\times GL(F))$ is a subgroup of $GL(E)\times GL(F)$ consisting of pairs (A,B) with $det(A)det(B) = 1$. For the stabilizer of determinant, see G.Frobenius [25]. For the stabilizer of the permanent, see Botta [2].

Assume $C = (c_{ij})$ with $c_{ij} \neq 0$ and $X = (x_{ij})$ are $n\times n$ matrices. Denote the torus action of C on X by $C * X = Y$, where $Y = (y_{ij})$ is an $n\times n$ matrix with entry $y_{ij} = c_{ij}x_{ij}$. We identify C with a diagonal matrix in GL_{n^2}, then the torus action is just the action of the diagonal matrices in $GL(E\otimes F)$ on the vector space $E\otimes F$. To find $G(im_\pi)$, we need the following result:

Theorem C.2 ([13])**.** *Assume $n \geq 4$, $\pi \neq (1,1,...,1)$ and (n). A linear transformation $T \in GL(E\otimes F)$ preserves the immanant im_π iff $T \in T(GL(E\otimes F)) \rtimes (\mathfrak{S}_n\times\mathfrak{S}_n) \rtimes \mathbb{Z}_2$, and satisfies the relation:*

$$\chi_\pi(\sigma)\prod_{i=1}^{n} c_{i\sigma(i)} = \chi_\pi(\tau_2\sigma\tau_1^{-1}),$$

where σ runs over all elements in $\mathfrak{S}_n$, $T(GL(E\otimes F))$ is the torus of $GL(E\otimes F)$, acting by the torus action described above, $\mathfrak{S}_n$ is the symmetric group in n elements, acting by left and right multiplication, and $\mathbb{Z}_2$ sending a matrix to its transpose.

Sketch of proof. Step 1: Let π be a fixed partition of n. Define a subset $X^{(n-1)}$ of the set $M_n(\mathbb{C})$ of n by n matrices as follows, $X^{(n-1)} := \{A \in M_n(\mathbb{C}) :$ degree of $im_\pi(xA+B) \leqslant 1$, for every $B \in M_n(\mathbb{C})\}$. Geometrically, $X^{(n-1)}$ is the most singular locus of the hypersurface defined by im_π. If A is in $X^{(n-1)}$, and T preserves im_π, then we have that

$T(A) \in X^{(n-1)}$, since the preserver of the hypersurface defined by im_π will preserve the most singular locus as well.

Step 2: Characterize the set $X^{(n-1)}$. To do this, first define a subset R_i (resp. R^i) of $M_n(\mathbb{C})$, consisting of matrices that have nonzero entries only in i-th row (resp. column). Then one proves that $A \in X^{(n-1)}$ if and only if it is in one of the forms:

1. R_i or R^i for some i.
2. The nonzero elements are in the 2×2 submatrix $A[i, h \mid i, h]$, and
$$\chi_\pi(\sigma) a_{ii} a_{hh} + \chi_\pi(\tau) a_{ih} a_{hi} = 0$$
for every σ and τ satisfying $\sigma(i) = i$, $\tau(h) = h$ and $\tau = \sigma(ih)$.
3. $\pi = (2, 1, ..., 1)$ and there are complementary sets of indices $\{i_1, ..., i_p\}$, $\{j_1, ..., j_q\}$ such that the nonzero elements are in $A[i_1, ..., i_p \mid j_1, ..., j_q]$ and the rank of $A[i_1, ..., i_p \mid j_1, ..., j_q]$ is one.
4. $\pi = (n-1, 1)$, the nonzero elements are in a 2 by 2 submatrix $A[u, v \mid r, s]$, and the permanent of this submatrix is zero.

Step 3: Characterize T by sets R_i and R^j. $\square$

We will start from this theorem. From this theorem, we know that $G(im_\pi)$ is contained in the group $T(GL(E \otimes F)) \rtimes (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes \mathbb{Z}^2$, and subject to the relation in Theorem C.2.

Now instead of considering n^2 parameters, we can consider $n!$ parameters, $i.e$, consider the stabilizer of an immanant in the bigger group $\mathbb{C}^{*n!} \rtimes (\mathfrak{S}_n \times \mathfrak{S}_n) \rtimes \mathbb{Z}^2$, where $\mathbb{C}^*$ is the multiplicative group of nonzero complex numbers. We can ignore the $\mathbb{Z}^2$-part of this group. The action of this group on the weight-zero-space of $S^n(E \otimes F)$ spanned by $\{x_{1\sigma(1)}x_{2\sigma(2)}\cdots x_{n\sigma(n)} \mid \sigma \in \mathfrak{S}_n\}$ is:

$$(\tau_1, \tau_2, (c_\sigma)_{\sigma\in\mathfrak{S}_n}) \cdot (x_{1\sigma(1)}x_{2\sigma(2)}\cdots x_{n\sigma(n)}) = c_\sigma x_{\tau_1(1)\tau_2\sigma(1)}x_{\tau_1(2)\tau 2\sigma(2)}\cdots x_{\tau_1(n)\tau_2\sigma(n)}.$$

Proposition C.3. *The stabilizer of im_π in $\mathbb{C}^{*n!} \rtimes (\mathfrak{S}_n \times \mathfrak{S}_n)$ is determined by equations*

$$c_{\tau_2^{-1}\tau_1\sigma}\chi_\pi(\tau_2^{-1}\tau_1\sigma) = \chi_\pi(\sigma), \forall\, \sigma \in \mathfrak{S}_n \tag{3.3}$$

Proof. The action of this group on im_π is:

$$\begin{aligned}
(\tau_1, \tau_2, (c_\sigma)_{\sigma\in\mathfrak{S}_n}) \cdot im_\pi &= \sum_{\sigma\in\mathfrak{S}_n} \chi_\pi(\sigma)c_\sigma \prod_{i=1}^n x_{\tau_1(i),\tau_2\sigma(i)} \\
&= \sum_{\sigma\in\mathfrak{S}_n} \chi_\pi(\sigma)c_\sigma \prod_{i=1}^n x_{i,\tau_2\sigma\tau_1^{-1}(i)} \\
&= \sum_{\sigma\in\mathfrak{S}_n} \chi_\pi(\tau_2^{-1}\sigma\tau_1)c_{\tau_2^{-1}\sigma\tau_1} \prod_{i=1}^n x_{i,\sigma(i)} \\
&= \sum_{\sigma\in\mathfrak{S}_n} \chi_\pi(\tau_2^{-1}\tau_1(\tau_1^{-1}\sigma\tau_1))c_{\tau_2^{-1}\tau_1(\tau_1^{-1}\sigma\tau_1)} \prod_{i=1}^n x_{i,\sigma(i)}
\end{aligned}$$

If $(\tau_1, \tau_2, (c_\sigma))$ stabilizes im_π, then

$$\chi_\pi(\tau_2^{-1}\tau_1(\tau_1^{-1}\sigma\tau_1))c_{\tau_2^{-1}\tau_1(\tau_1^{-1}\sigma\tau_1)} = \chi_\pi(\sigma) = \chi_\pi(\tau_1^{-1}\sigma\tau_1), \forall\, \sigma \in \mathfrak{S}_n.$$

Therefore, we have: $c_{\tau_2^{-1}\tau_1\sigma}\chi_\pi(\tau_2^{-1}\tau_1\sigma) = \chi_\pi(\sigma), \forall\, \sigma \in \mathfrak{S}_n$. □

Our next task is to find τ_1 and τ_2, such that the equation (3.3) has a solution for $(c_\sigma)_{\sigma\in\mathfrak{S}_n}$. For convenience, in the equation (3.3), set $\tau_2^{-1}\tau_1 = \tau$, so we get a new equation:

$$c_{\tau\sigma}\chi_\pi(\tau\sigma) = \chi_\pi(\sigma), \forall\, \sigma \in \mathfrak{S}_n \tag{3.4}$$

Lemma C.4. *If the equation (3.4) has a solution then $\tau \in \mathfrak{S}_n$ satisfies:*

1. *If $\chi_\pi(\sigma) = 0$, then $\chi_\pi(\tau\sigma) = 0$;*

2. *If* $\chi_\pi(\sigma) \neq 0$, *then* $\chi_\pi(\tau\sigma) \neq 0$;

Proof. Clear. $\square$

Definition C.5. : For a fixed partition $\pi \vdash n$, define:

$$K := \{\sigma \in \mathfrak{S}_n \mid \chi_\pi(\sigma) = 0\},$$

$$L := \{\sigma \in \mathfrak{S}_n \mid \chi_\pi(\sigma) \neq 0\}.$$

$$G := (\cap_{\sigma \in K} K\sigma) \cap (\cap_{\sigma \in L} L\sigma).$$

Lemma C.6. *If the equation* (3.4) *has a solution, then* $\tau \in G$.

Proof. It suffices to show that the 2 conditions in Lemma $C.4$ imply $\tau \in G$. If τ satisfies conditions 1 and 2, then

$$\tau\sigma \in K,\ \forall \sigma \in K; \tau\sigma^{'} \in L,\ \forall \sigma^{'} \in L,$$

therefore

$$\tau \in K\sigma^{-1},\ \forall\ \sigma \in K; \tau \in L\sigma^{'-1},\ \forall\ \sigma^{'} \in L,$$

so

$$\tau \in G.$$

$\square$

Example C.7. We can compute G directly for small n. If $n = 3$, then we have three representations $M_{(3)}$, $M_{(1,1,1)}$, $M_{(2,1)}$, $G = \mathfrak{S}_3, A_3, A_3$, respectively. If $n = 4$, then we have five representations $M_{(4)}$, $M_{(1,1,1,1)}$, $M_{(3,1)}$, $M_{(2,1,1)}$, $M_{(2,2)}$, and $G = \mathfrak{S}_4$, A_4, $\{(1), (12)(34), (13)(24), (14)(23)\}$, $\{(1), (12)(34), (13)(24), (14)(23)\}$, A_4, respectively. If the partition $\pi = (1, 1, ..., 1)$, then $G = A_n$. If the partition $\pi = (n)$, then $G = \mathfrak{S}_n$. Note that in these examples, G is a normal subgroup. In fact, this holds in general.

The following proposition is due to Coelho, M. Purificação in [10], using Murnagham-Nakayama Rule. One can give a different proof using Frobenius character formula for cycles (see, for example, [17]).

Proposition C.8 ([10])**.** *For any $n \geq 5$ and partition $\pi \vdash n$, G is a normal subgroup of $\mathfrak{S}_n$. Moreover, if $\pi \neq (1, 1, ..., 1)$ or (n), then G is the trivial subgroup (1) of $\mathfrak{S}_n$.*

Sketch. It is easy to show that G is a normal subgroup of $\mathfrak{S}_n$. And then one can prove $G \neq \mathfrak{S}_n$ by computing character χ_π. Then assume $G = A_n$, one can show that $L = A_n$ and $K = \mathfrak{S}_n - A_n$. If such a partition π exists, then it must be symmetric. But then one can construct cycles σ contained in A_n case by case (using Murnagham-Nakayama rule or Frobenius character formula) such that $\chi_\pi(\sigma) = 0$. It contradicts that $L = A_n$. □

Now we return to the equation (see Theorem C.2):

$$\chi_\pi(\sigma) \prod_{i=1}^{n} c_{i\sigma(i)} = \chi_\pi(\tau_2 \sigma \tau_1^{-1}),\ \forall\ \sigma \in \mathfrak{S}_n.$$

By Proposition C.8, we can set $\tau_1 = \tau_2$ in the above equation then we have equations for c_σ's:

$$\prod_{i=1}^{n} c_{i\sigma(i)} = 1,\ \forall\ \sigma \in \mathfrak{S}_n \text{ with } \chi_\pi(\sigma) \neq 0. \tag{3.5}$$

So elements in $G(im_\pi)$ can be expressed as triples $(\tau, \tau, (c_{ij}))$ where matrices (c_{ij}) is determined by equation(3.5).

Remark C.9. The coefficients of those linear equations are $n \times n$ permutation matrices. If we ignore the restriction $\chi_\pi(\sigma) \neq 0$,then we get all $n \times n$ permutation matrices.

Lemma C.10. *The permutation matrices span a linear space of dimension $(n-1)^2 + 1$ in $Mat_{n\times n} \cong \mathbb{C}^{n^2}$.*

Proof. We consider the action of $\mathfrak{S}_n$ on $\mathbb{C}^n$ induced by permuting a fixed basis, then $\sigma \in \mathfrak{S}_n$ is an element in $End(\mathbb{C}^n)$, corresponding to a permutation matrix, and vice versa. Now as $\mathfrak{S}_n$ modules, $\mathbb{C}^n \cong M_{(n-1,1)} \oplus \mathbb{C}$, where $\mathbb{C}$ is the trivial representation of $\mathfrak{S}_n$. So we have a decomposition of vector spaces:

$$\begin{aligned} End(\mathbb{C}^n) \cong & End(M_{(n-1,1)} \oplus \mathbb{C}) \\ \cong & End(M_{(n-1,1)}) \oplus End(\mathbb{C}) \oplus Hom(M_{(n-1,1)}, \mathbb{C}) \oplus Hom(\mathbb{C}, M_{(n-1,1)}). \end{aligned}$$

Since $\mathbb{C}$ and $M_{(n-1,1)}$ are $\mathfrak{S}_n$ modules, $\mathfrak{S}_n \hookrightarrow End(M_{(n-1,1)}) \oplus End(\mathbb{C})$. Note that $dim(End(M_{(n-1,1)}) \oplus End(\mathbb{C})) = (n-1)^2 + 1$, so it suffices to show that the image of $\mathfrak{S}_n$ in $End(\mathbb{C}^n)$ spans $End(M_{(n-1,1)}) \oplus End(\mathbb{C})$, but this is not hard to see because we have an algebra isomorphism:

$$\mathbb{C}[\mathfrak{S}_n] \cong \bigoplus_{\lambda \vdash n} End([\lambda])$$

Hence, we have:

$$\mathbb{C}[\mathfrak{S}_n] \twoheadrightarrow End(M_{(n-1,1)}) \oplus End(\mathbb{C}).$$

$\square$

Remark C.11. Lemma C.10 shows that the dimension of the stabilizer of an immanant is at least $2n-2$. We will show next that for any partition π of $n \geq 5$ except $(1, ..., 1)$ and (n), the dimension of the stabilizer $G\ (im_\pi)$ is exactly $2n-2$.

We compute the Lie algebra of the stabilizer of $G(im_\pi)$. Since $G(im_\pi) \subset GL(E \otimes F)$, the Lie algebra of $G(im_\pi)$ is a subalgebra of $\mathfrak{gl}(E \otimes F)$. Let $\mathfrak{sl}(E)$ (resp. $\mathfrak{sl}(F)$) be the

Lie algebra of $SL(E)$ (resp. $SL(F)$). We have the decomposition of $\mathfrak{gl}\ (E \otimes F)$

$$\begin{aligned}
\mathfrak{gl}(E \otimes F) &= End(E \otimes F) \\
&\cong (E \otimes F)^* \otimes (E \otimes F) \\
&\cong E^* \otimes E \otimes F^* \otimes F \\
&\cong (\mathfrak{sl}_R(E) \oplus I_E \oplus T(E)) \otimes (\mathfrak{sl}_R(F) \oplus I_F \oplus T(F)) \\
&\cong \mathfrak{sl}_R(E) \otimes \mathfrak{sl}_R(F) \oplus \mathfrak{sl}_R(E) \otimes I_F \oplus I_E \otimes \mathfrak{sl}_R(F) \oplus I_E \otimes I_F \\
&\oplus T(E) \otimes T(F) \oplus T(E) \otimes I_F \oplus I_E \otimes T(F)
\end{aligned}$$

Here $\mathfrak{sl}_R(E)$ is the root space of $\mathfrak{sl}(E)$, $T(E)$ is the torus of $\mathfrak{sl}(E)$, and I_E is the space spanned by identity matrix. The similar notation is used for F. We will show that the Lie algebra of $\{C \in M_{n\times n} \mid im_\pi(C * X) = im_\pi(X)\}$ is $T(E) \otimes I_F \oplus I_E \otimes T(F)$. Let $\{e_i \mid i = 1, ..., n\}$ be a fixed basis of E and $\{\alpha^i \mid i = 1, ..., n\}$ be the dual basis. Let $H^E_{1i} = \alpha^1 \otimes e_1 - \alpha^i \otimes e_i$. Then $\{H_{1i} \mid i = 2, ..., n\}$ is a basis of $T(E)$. We use H^F for F and define $A_{ij} = H^E_{1i} \otimes H^F_{1j}$ for all $i \geq 2$, $j \geq 2$.

Now consider the action of A_{ij} on variable x_{pq}.

$$C^{i,j}_{p,q} := A_{ij}(x_{pq}) = (\delta^1_p - \delta^i_p)(\delta^1_q - \delta^j_q)$$

$$C^{i,j}_{p,q} = \begin{cases} 1,\ p = q = 1 \\ -1,\ p = i, q = 1 \\ -1,\ p = 1, q = j \\ 1,\ p = i, q = j \\ 0,\ otherwise \end{cases} \tag{3.6}$$

Equation (3.5) implies that the matrices $C = (c_{ij})$ that stabilize im_π is contained in the torus of $GL(E \otimes F)$, hence the Lie algebra of the set of such matrices is contained in the torus of $\mathfrak{gl}(E \otimes F)$, that is, it is contained in $t := I_E \otimes I_F \oplus T(E) \otimes T(F) \oplus T(E) \otimes I_F \oplus I_E \otimes T(F)$. Now let L be an element of t, then L can be expressed as the linear combination of A_{ij}'s and $I_E \otimes I_F$. Hence:

$$L = (aI_E \otimes I_F + \sum_{i,j>1} a_{ij} A_{ij})$$

for some a, $a_{ij} \in \mathbb{C}$.

Then

$$L(x_{pq}) = \begin{cases} (a + \sum_{i,j>1} a_{ij})x_{11}, & p = q = 1 \\ (a - \sum_{i>1} a_{iq})x_{1q}, & p = 1, q \neq 1 \\ (a - \sum_{j>1} a_{pj})x_{p1}, & p \neq 1, q = 1 \\ (a + a_{pq})x_{pq}, & p \neq 1, q \neq 1 \end{cases} \tag{3.7}$$

Now for a permutation $\sigma \in \mathfrak{S}_n$, the action of L on the monomial $x_{1\sigma(1)}\ x_{2\sigma(2)}\ ...\ x_{n\sigma(n)}$ is: if $\sigma(1) = 1$,

$$L(\prod_{p=1}^{n} x_{p\sigma(p)}) = (na + \sum_{i,j>1} a_{ij} + \sum_{p=2}^{n} a_{p\sigma(p)}) \prod_{p=1}^{n} x_{p\sigma(p)}. \tag{3.8}$$

if $\sigma(1) \neq 1$ and $\sigma(k) = 1$,

$$L(\prod_{p=1}^{n} x_{p\sigma(p)}) = (na + \sum_{p\neq 1,k} a_{p\sigma(p)} - \sum_{i>1} a_{i\sigma(1)} - \sum_{j>1} a_{kj}) \prod_{p=1}^{n} x_{p\sigma(p)}. \tag{3.9}$$

Lemma C.12. *For any solution of the system of linear equations*

$$a_{ij} + a_{jk} + a_{km} = \quad a_{ik} + a_{kj} + a_{jm}, \text{where } \{i, j, k, m\} = \{2, 3, 4, 5\} \tag{3.10}$$

$$a_{ij} + a_{jk} = \quad a_{ij'} + a_{j'k}, \text{where } \{i, j, k, j'\} = \{2, 3, 4, 5\} \tag{3.11}$$

there exists a number γ such that for any permutation μ of the set $\{2, 3, 4, 5\}$ moving l

elements,

$$\sum_{\substack{i=2 \\ \mu(i)\neq i}}^{5} a_{i\mu(i)} = l\gamma \tag{3.12}$$

Proof. check by solving this linear system. □

Lemma C.13. *let $n \geq 6$ be an integer, π be a fixed partition of n, which is not $(1, ..., 1)$ or (n). Assume that there exists a permutation(so a conjugacy class) $\tau \in \mathfrak{S}_n$ such that:*

1. *$\chi_\pi(\tau) \neq 0$;*
2. *τ contains a cycle moving at least 4 numbers;*
3. *τ fixes at least 1 number.*

Also assume that $L(im_\pi) = 0$. Then under the above assumptions, $a = a_{ij} = 0$ for all $i, j > 1$.

Proof. $L(im_\pi) = 0$ means that $L(\prod_{p=1}^{n} x_{p\sigma(p)})$ for all $\sigma \in \mathfrak{S}_n$ such that $\chi_\pi(\sigma) \neq 0$. Consider permutations (2345...) ... (...) and (2435...) ... (...)(all cycles are the same except the first one, and for the first cycle, all numbers are the same except the first 4), from formula (3.8), we have:

$$na + \sum_{i,j>1} a_{ij} + a_{23} + a_{34} + a_{45} + E = 0 \tag{3.13}$$

$$na + \sum_{i,j>1} a_{ij} + a_{24} + a_{43} + a_{35} + E = 0 \tag{3.14}$$

for some linear combination E of a_{ij}'s. Thus

$$a_{23} + a_{34} + a_{45} = a_{24} + a_{43} + a_{35} \tag{3.15}$$

Similarly,

$$a_{ij} + a_{jk} + a_{km} = a_{ik} + a_{kj} + a_{jm}, \text{ for } i, j, k, m \text{ distinct.} \tag{3.16}$$

Next, consider permutations $(1234...k)...(...)$ and $(1254...k)...(...)$, again, from formula (3.9), we obtain:

$$na + a_{23} + a_{34} + E' - \sum_{i>1} a_{i,2} - \sum_{j>1} a_{k,j} = 0 \tag{3.17}$$

$$na + a_{25} + a_{54} + E' - \sum_{i>1} a_{i,2} - \sum_{j>1} a_{k,j} = 0 \tag{3.18}$$

Hence,

$$a_{25} + a_{54} = a_{23} + a_{34} \tag{3.19}$$

and thus

$$a_{ij} + a_{jk} = a_{ij'} + a_{j'k} \text{ ,for all } i, j, j', k \text{ distinct.} \tag{3.20}$$

Now for $2 \leq i < j < k < m \leq n$, we have system of linear equations of the same form as Lemma C.12. So we have relations:

$$\sum_{\substack{m \geq p \geq i \\ \mu(p) \neq p}} a_{p\mu(p)} = l\gamma_{ijkm}$$

where μ is a permutation of the set $\{i, j, k, m\}$, l is the number of elements moved by μ, and γ_{ijkm} is a constant number.

It is easy to see that γ_{ijkm} is the same for different choices of the set $\{2 \leq i < j < k < m \leq n\}$, for example, we can compare $\{i, j, k, m\}$ and $\{i, j, k, m'\}$ to obtain $\gamma_{ijkm} = \gamma_{ijkm'}$. From now on, we write all γ_{ijkm}'s as γ. Hence, given any permutation of the set $\{2, ..., n\}$ moving l elements,

$$\sum_{\substack{n \geq p \geq 2 \\ \mu(p) \neq p}} a_{p\mu(p)} = l\gamma \tag{3.21}$$

Next, we find relations among the a_{ii}'s for $i \geq 2$. For this purpose, consider $\tau_1 = (243...k)...(...)$ and $\tau_2 = (253...k)...(...)$, then

$$na + \textstyle\sum_{i,j>1} a_{ij} + a_{24} + ... + a_{k2} + a_{55} + (\text{sum of } a_{ii}\text{'s for } i \neq 5 \text{ fixed by } \tau_1) = 0 \tag{3.22}$$

$$na + \textstyle\sum_{i,j>1} a_{ij} + a_{25} + ... + a_{k2} + a_{44} + (\text{sum of } a_{ii}\text{'s for } i \neq 4 \text{ fixed by } \tau_1) = 0 \tag{3.23}$$

Combine these two equations and equation (3.21) to obtain

$$a_{44} = a_{55}.$$

The same argument implies that $a_{ii} = a_{55}$ for all $n \geq i \geq 2$. Now we have:

$$\sum_{i,j>1} a_{ij} = \sum_{1<i<j} (a_{ij} + a_{ij}) + 2(n-1)a_{55} = (n-1)(n-2)\gamma + 2(n-1)a_{55}$$

Let $\sigma = (1)$, formula(3.8) implies

$$na + (n-1)(n-2)\gamma + 3(n-1)a_{55} = 0 \tag{3.24}$$

Let $\sigma = (2345...)...(...)(\sigma(1) = 1)$, again by formula (3.8)

$$na + ((n-2)(n-1) + l)\gamma + (3(n-1) - l)a_{55} = 0 \tag{3.25}$$

where l is the number of elements moved by σ. Let $\sigma_1 = (123...p4)...(...)$ and $\sigma_2 = (143...p2)...(...)$. Then formula (3.9) gives:

$$0 = \quad na + (a_{23} + ... + a_{p4}) + \tilde{E} - \textstyle\sum_{i>1} a_{i2} - \sum_{j>1} a_{4j} \tag{3.26}$$

$$0 = \quad na + (a_{43} + ... + a_{p2}) + \tilde{E} - \textstyle\sum_{i>1} a_{i4} - \sum_{j>1} a_{2j} \tag{3.27}$$

Note that $\tilde{E}$ comes from the product of disjoint cycles in σ_1 and σ_2 except the first one, so they are indeed the same, and if we assume that σ_1 moves l' elements, and the first cycle in σ_1 moves r elements, then $\tilde{E} = (l' - r)\gamma + (n - l')a_{55}$. On the other hand,

$$a_{23} + ... + a_{p4} = a_{23} + ... + a_{p4} + a_{42} - a_{42} = r\gamma - a_{42}$$

$$a_{43} + ... + a_{p2} = a_{43} + ... + a_{p2} + a_{24} - a_{24} = r\gamma - a_{24}$$

Equations (3.26) and (3.27) gives:

$$na + (l' + 2n - 5)\gamma + (n - l' + 2)a_{55} = 0 \tag{3.28}$$

Now equations (3.24), (3.25), and (3.28) imply that $\gamma = a = a_{55} = 0$. From equation (3.26), we have:

$$a_{42} = \sum_{j>1} a_{4j} + \sum_{i>1} a_{i2}$$

similarly,

$$a_{k2} = \sum_{j>1} a_{kj} + \sum_{i>1} a_{i2} \text{ for all } n \geq k \geq 2$$

The sum of these equations is:

$$\sum_{i>1} a_{i2} = (n-1)\sum_{i>1} a_{i2} + \sum_{i,j>1} a_{ij} = (n-1)\sum_{i>1} a_{i2}.$$

Hence $\sum_{i>1} a_{i2} = 0$. For the same reason $\sum_{j>1} a_{4j} = 0$, therefore $a_{42} = 0$. By the same argument, $a_{ij} = 0$ for all $n \geq i \neq j \geq 2$, and this completes the proof of the lemma. □

Lemma C.14. *If $n \geq 6$, then for any partition λ of n, except $\lambda = (3,1,1,1)$, $\lambda = (4,1,1)$ and $\lambda = (4,1,1,1)$, there exists a permutation $\tau \in \mathfrak{S}_n$ satisfying conditions (1), (2), (3) in Lemma C.13.*

Proof. Write $\lambda = (\lambda_1, \lambda_2, ..., \lambda_p)$ where $\lambda_1 \geq \lambda_2 \geq ...\lambda_p \geq 1$ and $\sum_{i=1}^{p} \lambda_i = n$. Without loose of generality, we may assume $p \geq \lambda_1$, otherwise, we can consider the conjugate λ' of λ. There exists a largest integer m such that the Young diagram of λ contains an $m \times m$ square.

Now we construct τ using the Murnagham-Nakayama Rule case by case:

1. If $m = 1$ then λ is a hook:$(\lambda_1, 1, ..., 1)$, there are the following cases:
   (a) $p > \lambda_1$ and $\lambda_1 \geq 4$. Take $\tau = (p-1, 1^{n-p+1})$ then $\chi_\lambda(\tau) \neq 0$ by the Murnagham-Nakayama Rule. In this case, $n \geq 8$.
   (b) $p > \lambda_1$ and $\lambda_1 = 1$. This case is trivial.
   (c) $p > \lambda_1$ and $\lambda_1 = 2$ or 3. $\tau = (4, 1^{n-4})$ will work.

(d) $p = \lambda_1$. Take $\tau = (p-1, 1^{n-p+1})$ if $p \geq 6$ is even and $\tau = (p-2, 1^{n-p+2})$ if $p \geq 7$ is odd. In this case $n \geq 11$.

(e) $p = \lambda_1 = 5$. τ exists by checking the character tables.

2. If $m \geq 2$, let ξ be the length of the longest skew hook contained in the young diagram of λ. Then take $\tau = (\xi, 1^{n-\xi})$.

□

Proof of Proposition A.1. For the case $n = 5$, one can check directly. By Lemma C.13 and lemma C.14, we know that for $n \geq 6$ and π not equal to $(3,1,1,1)$ and $(4,1,1,1)$, the Lie algebra of $\{C \in M_{n\times n} \mid im_\pi(C * X) = im_\pi(X)\}$ is $T(E) \otimes I_F \oplus I_E \otimes T(F)$, so the identity component of $\{C \in M_{n\times n} \mid im_\pi(C * X) = im_\pi(X)\}$ is $T(GL(E) \times GL(F))$, and hence the identity component of $G(im_\pi)$ is $T(GL(E) \times GL(F)) \rtimes \Delta(\mathfrak{S}_n) \rtimes \mathbb{Z}_2$. For cases $\pi = (3,1,1,1)$, $\pi = (4,1,1)$ the statement is true by Theorem A.3. □

By investigating the equation (3.5), we can give a sufficient condition for the stabilizer of im_π to be $T(GL(E) \times GL(F)) \rtimes \Delta(\mathfrak{S}_n) \rtimes \mathbb{Z}_2$ as follows:

Lemma C.15. *Let π be a partition of n which is not $(1,...,1)$ or (n). Assume that there exist permutations σ, $\tau \in \mathfrak{S}_n$ and an integer $p \geq 2$, such that $\chi_\pi((i_1...i_p)\sigma) \neq 0$, $\chi_\pi((1i_1...i_p)\sigma) \neq 0$, $\chi_\pi(\tau) \neq 0$ and $\chi_\pi((ij)\tau) \neq 0$, where $(i_1...i_p)$ and $(1i_1...i_p)$ are cycles disjoint from σ, and (ij) is disjoint from τ. Then the stabilizer of im_π is $T(GL(E) \times GL(F)) \rtimes \Delta(\mathfrak{S}_n) \rtimes \mathbb{Z}_2$.*

Proof. For convenience, we will show for the case $\sigma = (1)$ and $p = 2$, the other cases are similar. In equations 3.5, let $\sigma = (ij)$ and $(1ij)$, where $1 < i, j \leqslant n$ and $i \neq j$. Then

$$c_{ji}c_{ij}c_{11} \prod_{k \neq 1,i,j} c_{kk} = 1 \tag{3.29}$$

$$c_{ji}c_{1j}c_{i1} \prod_{k \neq 1,i,j} c_{kk} = 1 \tag{3.30}$$

So $c_{ij} = \frac{c_{i1}c_{1j}}{c_{11}}$. Similarly, the existence of τ will give the relation $c_{ii} = \frac{c_{i1}c_{1i}}{c_{11}}$ for all $1 \leq i \leq n$. Set

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}, \mathbf{B} = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{pmatrix}$$

Where $a_{ii} = c_{i1}$ and $b_{jj} = \frac{c_{1j}}{c_{11}}$. So we have

$$C * X = AXB, \text{ with } det(AB) = 1.$$

$\square$

The following two propositions guarantee the existence of permutations satisfied conditions in Lemma (C.15).

Proposition C.16. *Let $n \geq 3$, and π be a non-symmetric partition of n, then there exists nonnegative integers k_1,..., k_r such that k_1 +...+ $k_r = n-2$, such that $|(\chi_\pi(\tau))| = 1$, where τ is a permutation of type $(k_1,..., k_r, 1^2)$ or $(k_1,..., k_r, 2)$.*

Proof. See Proposition (3.1), Coelho, M. Purificação and Duffner, M. Antónia [11]. $\square$

Proposition C.17. *Let $n > 4$ and π be a non-symmetric partition of n, then there exists nonnegative integers k_1 ,..., k_r and q with $k_r > 1$, $q \geq 1$, and k_1 +...+ k_r $+q = n$, such that $\chi_\pi(\sigma) \neq 0$, where $\sigma \in \mathfrak{S}$ is of type $(k_1, ..., k_r, 1^q)$ or $(k_1, ..., k_r + 1, 1^{q-1})$.*

Proof. See Proposition (3.2), Coelho, M. Purificação and Duffner, M. Antónia [11]. $\square$

Proof of Theorem A.3. Since $n \geq 5$, by propositions C.16 and C.17, there exist permutations satisfying conditions in Lemma C.15, then the theorem follows. $\square$

CHAPTER IV

GEOMETRY OF TENSOR NETWORK STATES

This chapter is based on [22].

A. Introduction

1. Origins in physics

Tensors describe states of quantum mechanical systems. If a system has n particles, its state is an element of $H_1 \otimes \cdots \otimes H_n$ with H_j Hilbert spaces. In numerical many-body physics, in particular solid state physics, one wants to simulate quantum states of thousands of particles, often arranged on a regular lattice (e.g., atoms in a crystal). Due to the exponential growth of the dimension of $H_1 \otimes \cdots \otimes H_n$ with n, any naïve method of representing these tensors is intractable on a computer. Tensor network states were defined to reduce the complexity of the spaces involved by restricting to a subset of tensors that is physically reasonable, in the sense that the corresponding spaces of tensors are only locally entangled because interactions (entanglement) in the physical world appear to just happen locally. Such spaces have been studied since the 1980's. These spaces are associated to graphs, and go under different names: *tensor network states*, *finitely correlated states (FCS)*, *valence-bond solids (VBS)*, *matrix product states (MPS)*, *projected entangled pairs states (PEPS)*, and *multi-scale entanglement renormalization ansatz states (MERA)*, see, e.g., [38, 15, 21, 14, 42, 9] and the references therein. We will use the term tensor network states.

2. Definition

For a graph Γ with edges e_s and vertices v_j, $s \in e(j)$ means e_s is incident to v_j. If Γ is directed, $s \in in(j)$ means e_s is an incoming edge into v_j and $s \in out(j)$ means e_s is an outgoing edge from v_j.

Let $V_1,\dots,V_n$ be complex vector spaces, let $\mathbf{v}_i=\dim V_i$. Let Γ be a graph with n vertices v_j, $1\le j\le n$, and m edges e_s, $1\le s\le m$, and let $\vec{\mathbf{e}}=(\mathbf{e}_1,\dots,\mathbf{e}_m)\in\mathbb{N}^m$. Associate V_j to the vertex v_j and an auxiliary vector space E_s of dimension $\mathbf{e}_s$ to the edge e_s. Make Γ into a directed graph. (The choice of directions will not effect the end result.) Let $\mathbf{V}=V_1{\otimes}\cdots{\otimes}V_n$. Let

$$TNS(\Gamma,\vec{\mathbf{e}},\mathbf{V}):= \tag{4.1}$$
$$\{T\in\mathbf{V}\mid \exists\, T_j\in V_j{\otimes}(\otimes_{s\in in(j)}E_s){\otimes}(\otimes_{t\in out(j)}E_t^*),\ \text{such that } T=Con(T_1{\otimes}\cdots{\otimes}T_n)\},$$

where Con is the contraction of all the E_s's with all the E_s^*'s.

Example A.1. Let Γ be a graph with two vertices and one edge connecting them, then, $TNS(\Gamma,\mathbf{e}_1,V_1{\otimes}V_2)$ is just the set of elements of $V_1{\otimes}V_2$ of rank at most $\mathbf{e}_1$, denoted by $\hat\sigma_{\mathbf{e}_1}(Seg(\mathbb{P}V_1\times\mathbb{P}V_2))$ and called the (cone over the) $\mathbf{e}_1$-st secant variety of the Segre variety. To see this, let $\epsilon_1,\dots,\epsilon_{\mathbf{e}_1}$ be a basis of E_1 and $\epsilon^1,\dots,\epsilon^{\mathbf{e}_1}$ the dual basis of E^*. Assume, to avoid trivialities, that $\mathbf{v}_1,\mathbf{v}_2\ge\mathbf{e}_1$. Given $T_1\in V_1{\otimes}E_1$ we may write $T_1=u_1{\otimes}\epsilon_1+\cdots+u_{\mathbf{e}_1}{\otimes}\epsilon_{\mathbf{e}_1}$ for some $u_\alpha\in V_1$. Similarly, given $T_2\in V_2{\otimes}E_1^*$ we may write $T_1=w_1{\otimes}\epsilon^1+\cdots+w_{\mathbf{e}_1}{\otimes}\epsilon^{\mathbf{e}_1}$ for some $w_\alpha\in V_2$. Then $Con(T_1{\otimes}T_2)=u_1{\otimes}w_1+\cdots+u_{\mathbf{e}_1}{\otimes}w_{\mathbf{e}_1}$.

The graph used to define a set of tensor network states is often modeled to mimic the physical arrangement of the particles, with edges connecting nearby particles, as nearby particles are the ones likely to be entangled.

Remark A.2. The construction of tensor network states in the physics literature does not use a directed graph, because all vector spaces are Hilbert spaces, and thus self-dual. However the sets of tensors themselves do not depend on the Hilbert space structure of the vector space, which is why we omit this structure. The small price to pay is the edges of the graph must be oriented, but all orientations lead to the same set of tensor network states.

B. Grasedyck's question

Lars Grasedyck asked:

Is $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$ Zariski closed? That is, given a sequence of tensors $T_\epsilon \in \mathbf{V}$ that converges to a tensor T_0, if $T_\epsilon \in TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$ for all $\epsilon \neq 0$, can we conclude $T_0 \in TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$?

He mentioned that he could show this to be true when Γ was a tree, but did not know the answer when Γ is a triangle.

Definition B.1. A dimension $\mathbf{v}_j$ is *critical*, resp. *subcritical*, resp. *supercritical*, if $\mathbf{v}_j = \Pi_{s \in e(j)} \mathbf{e}_s$, resp. $\mathbf{v}_j \leq \Pi_{s \in e(j)} \mathbf{e}_s$, resp. $\mathbf{v}_j \geq \Pi_{s \in e(j)} \mathbf{e}_s$. If $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$ is critical for all j, we say $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$ is critical, and similarly for subcritical and supercritical.

Theorem B.2. $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$ *is not Zariski closed for any* Γ *containing a cycle whose vertices have non-subcritical dimensions.*

C. Critical loops

We adopt the convention that $\mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_n)$ acts on $V_1 \otimes \cdots \otimes V_n$ by

$$(Z_1, \ldots, Z_n) \cdot v_1 \otimes \cdots \otimes v_n = (Z_1 v_1) \otimes \cdots \otimes (Z_n v_n).$$

Let $\mathfrak{gl}(V_j)$ denote the Lie algebra of $GL(V_j)$. It is naturally isomorphic to $\mathrm{End}(V_j)$ but $\mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_n)$ acts on $V_1 \otimes \cdots \otimes V_n$ via the Leibnitz rule:

$$\begin{aligned}(X_1, \ldots, X_n) \cdot v_1 \otimes \cdots \otimes v_n = (X_1 v_1) \otimes v_2 \otimes \cdots \otimes v_n + v_1 \otimes (X_2 v_2) \otimes v_3 \otimes \cdots \otimes v_n \\ + \cdots + v_1 \otimes \cdots \otimes v_{n-1} \otimes (X_n v_n).\end{aligned} \quad (4.2)$$

This is because elements of the Lie algebra should be thought of as derivatives of curves in the Lie group at the identity. If $X \subset \mathbf{V}$ is a subset, $\overline{X} \subset \mathbf{V}$ denotes its closure. This closure is the same whether one uses the Zariski closure, which is the common zero set of all polynomials vanishing on X, or the Euclidean closure, where one fixes a metric compatible with the linear structure on $\mathbf{V}$ and takes the closure with respect to limits.

Proposition C.1. *Let* $\mathbf{v}_1 = \mathbf{e}_2\mathbf{e}_3, \mathbf{v}_2 = \mathbf{e}_3\mathbf{e}_1, \mathbf{v}_3 = \mathbf{e}_2\mathbf{e}_1$. *Then* $TNS(\triangle, (\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_1), V_1 \otimes V_2 \otimes V_3)$ *consists of matrix multiplication and its degenerations (and their different expressions after changes of bases), i.e.,*

$$TNS(\triangle, (\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_1), V_1 \otimes V_2 \otimes V_3) = End(V_1) \times End(V_2) \times End(V_3) \cdot M_{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1}.$$

It has dimension $\mathbf{e}_2^2\mathbf{e}_3^2 + \mathbf{e}_2^2\mathbf{e}_1^2 + \mathbf{e}_3^2\mathbf{e}_1^2 - (\mathbf{e}_2^2 + \mathbf{e}_3^2 + \mathbf{e}_1^2 - 1)$.

More generally, if Γ *is a critical loop,* $TNS(\Gamma, (\mathbf{e}_n\mathbf{e}_1, \mathbf{e}_1\mathbf{e}_2, \dots, \mathbf{e}_{n-1}\mathbf{e}_n), V_1 \otimes \cdots \otimes V_n)$ *is* $\mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_n) \cdot M_{\vec{\mathbf{e}}}$, *where* $M_{\vec{\mathbf{e}}} : V_1 \times \cdots \times V_n \to \mathbb{C}$ *is the matrix multiplication operator* $(X_1, \dots, X_n) \mapsto \mathrm{trace}(X_1 \cdots X_n)$.

Proof. For the triangle case, a generic element $T_1 \in E_2 \otimes E_3^* \otimes V_1$ may be thought of as a linear isomorphism $E_2^* \otimes E_3 \to V_1$, identifying V_1 as a space of $\mathbf{e}_2 \times \mathbf{e}_3$-matrices, and similarly for V_2, V_3. Choosing bases $e_s^{u_s}$ for E_s^*, with dual basis $e_{u_s,s}$ for E_s, induces bases $x_{u_3}^{u_2}$ for V_1 etc.. Let $1 \le i \le \mathbf{e}_2$, $1 \le \alpha \le \mathbf{e}_3$, $1 \le u \le \mathbf{e}_1$. Then

$$con(T_1 \otimes T_2 \otimes T_3) = \sum x_\alpha^i \otimes y_u^\alpha \otimes z_i^u$$

which is the matrix multiplication operator. The general case is similar. $\square$

Proposition C.2. *The Lie algebra of the stabilizer of* $M_{\mathbf{e}_n\mathbf{e}_1, \mathbf{e}_1\mathbf{e}_2, \dots, \mathbf{e}_{n-1}\mathbf{e}_n}$ *in* $GL(V_1) \times \cdots \times GL(V_n)$ *is the image of* $\mathfrak{sl}(E_1) \oplus \cdots \oplus \mathfrak{sl}(E_n)$ *under the map*

$$\begin{aligned}\alpha_1 \oplus \cdots \oplus \alpha_n \mapsto & (Id_{E_n} \otimes \alpha_1, -\alpha_1^T \otimes Id_{E_2}, 0, \dots, 0) + (0, Id_{E_1} \otimes \alpha_2, -\alpha_2^T \otimes Id_{E_3}, 0, \dots, 0) \\ & + \cdots + (-\alpha_n^T \otimes Id_{E_1}, 0, \dots, 0, Id_{E_{n-1}} \otimes \alpha_n).\end{aligned}$$

Here T *as a superscript denotes transpose.*

The proof is safely left to the reader.

Large loops are referred to as "1-D systems with periodic boundary conditions" in the physics literature and are often used in simulations. By Proposition C.2, for a critical loop, $\dim(TNS(\Gamma,\vec{\mathbf{e}},\mathbf{V})) = \mathbf{e}_1^2\mathbf{e}_2^2+\cdots+\mathbf{e}_{n-1}^2\mathbf{e}_n^2+\mathbf{e}_n^2\mathbf{e}_1^2-(\mathbf{e}_1^2+\cdots+\mathbf{e}_n^2-1)$, compared with the ambient space which has dimension $\mathbf{e}_1^2\cdots\mathbf{e}_n^2$. For example, when $\mathbf{e}_j=2$ for all j, $\dim(TNS(\Gamma,\vec{\mathbf{e}},\mathbf{V}))=12n+1$, compared with $\dim\mathbf{V}=4^n$.

D. Zariski closure

Theorem D.1. *Let $\mathbf{v}_1=\mathbf{e}_2\mathbf{e}_3, \mathbf{v}_2=\mathbf{e}_3\mathbf{e}_1, \mathbf{v}_3=\mathbf{e}_2\mathbf{e}_1$. Then $TNS(\triangle,(\mathbf{e}_2\mathbf{e}_3,\mathbf{e}_3\mathbf{e}_1,\mathbf{e}_2\mathbf{e}_1),\mathbf{V})$ is not Zariski closed. More generally any $TNS(\Gamma,\vec{\mathbf{e}},\mathbf{V})$ where Γ contains a cycle with no subcritical vertex is not Zariski closed.*

Proof. Were $T(\triangle):=TNS(\triangle,(\mathbf{e}_2\mathbf{e}_3,\mathbf{e}_3\mathbf{e}_1,\mathbf{e}_2\mathbf{e}_1),V_1{\otimes}V_2{\otimes}V_3)$ Zariski closed, it would be

$$\overline{GL(V_1)\times GL(V_2)\times GL(V_3)\cdot M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}}. \tag{4.3}$$

To see this, note that the $G=GL(V_1)\times GL(V_2)\times GL(V_3)$ orbit of matrix multiplication is a Zariski open subset of $T(\triangle)$ of the same dimension as $T(\triangle)$.

We need to find a curve $g(t)=(g_1(t),g_2(t),g_3(t))$ such that $g_j(t)\in GL(V_j)$ for all $t\neq 0$ and $\lim_{t\to 0} g(t)\cdot M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}$ is both defined and not in $End(V_1)\times End(V_2)\times End(V_3)\cdot M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}$.

Note that for $(X,Y,Z)\in GL(V_1)\times GL(V_2)\times GL(V_3)$, we have

$$(X,Y,Z)\cdot M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}(P,Q,R)=\mathrm{trace}(X(P)Y(Q)Z(R))$$

Here$X: E_2^*{\otimes}E_3\to E_2^*{\otimes}E_3$, $Y: E_3^*{\otimes}E_1\to E_3^*{\otimes}E_1$, $Z: E_1^*{\otimes}E_2\to E_1^*{\otimes}E_2$.

Take subspaces $U_{E_2E_3}\subset E_2^*{\otimes}E_3$, $U_{E_3E_1}\subset E_3^*{\otimes}E_1$. Let $U_{E_1E_2}:=Con(U_{E_2E_3},U_{E_3E_1})\subset E_2^*{\otimes}E_1$ be the images of all the $pq\in E_2^*{\otimes}E_1$ where $p\in U_{E_2E_3}$ and $q\in U_{E_3E_1}$ (i.e., the matrix multiplication of all pairs of elements). Take X_0,Y_0,Z_0 respectively to be the projections to $U_{E_2E_3}$, $U_{E_3E_1}$ and ${U_{E_1E_2}}^{\perp}$. Let X_1,Y_1,Z_1 be the projections to complementary spaces (so, e.g., $X_0 + X_1 = Id_{V_1^*}$). For $P \in V_1^*$, write $P_0 = X_0(P)$ and $P_1 = X_1(P)$, and similarly for Q, R.

Take the curve (X_t, Y_t, Z_t) with

$$X_t = \frac{1}{\sqrt{t}}(X_0 + tX_1), Y_t = \frac{1}{\sqrt{t}}(Y_0 + tY_1), Z_t = \frac{1}{\sqrt{t}}(Z_0 + tZ_1).$$

Then the limiting tensor, as a map $V_1^* \times V_2^* \times V_3^* \to \mathbb{C}$, is

$$(P, Q, R) \mapsto \mathrm{trace}(P_0Q_0R_1) + \mathrm{trace}(P_0Q_1R_0) + \mathrm{trace}(P_1Q_0R_0).$$

Call this tensor $\tilde{M}$. First observe that $\tilde{M}$ uses all the variables (i.e., considered as a linear map $\tilde{M} : V_1^* \to V_2 \otimes V_3$, it is injective, and similarly for its cyclic permutations). Thus it is either in the orbit of matrix multiplication or a point in the boundary that is not in $End(V_1) \times End(V_2) \times End(V_3) \cdot M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}$, because all such boundary points have at least one such linear map non-injective.

It remains to show that there exist $\tilde{M}$ such that $\tilde{M} \notin G \cdot M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}$ To prove some $\tilde{M}$ is a point in the boundary, we compute the Lie algebra of its stabilizer and show it has dimension greater than the dimension of the stabilizer of matrix multiplication. One may take block matrices, e.g.,

$$X_0 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \ X_1 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix},$$

and Y_0, Y_1 have similar shape, but Z_0, Z_1 have the shapes reversed. Here one takes any splitting $\mathbf{e}_j = \mathbf{e}_j' + \mathbf{e}_j''$ to obtain the blocks.

For another example, if one takes $\mathbf{e}_j = \mathbf{e}$ for all j, X_0, Y_0, Z_1 to be the diagonal matrices and and X_1, Y_1, Z_0 to be the matrices with zero on the diagonal, then one obtains a stabilizer of dimension $4\mathbf{e}^2 - 2\mathbf{e} > 3\mathbf{e}^2 - 1$. (This example coincides with the previous one when all $\mathbf{e}_j = 2$.)

To calculate the stabilizer of $\tilde{M}$, first write down the tensor expression of $\tilde{M} \in V_1 \otimes V_2 \otimes V_3$ with respect to fixed bases of V_1, V_2, V_3. Then set an equation $(X, Y, Z).\tilde{M} = 0$ where $X \in \mathfrak{gl}(V_1)$, $Y \in \mathfrak{gl}(V_2)$ and $Z \in \mathfrak{gl}(V_3)$ are unknowns. Recall that here the action of (X, Y, Z) on $\tilde{M}$ is the Lie algebra action, so we obtain a collection of linear equations. Finally we solve this collection of linear equations and count the dimension of the solution space. This dimension is the dimension of the stabilizer of $\tilde{M}$ in $GL(V_1) \times GL(V_2) \times GL(V_3)$.

To give an explicit example, let $\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_3 = \mathbf{e}$ and let $X_0 = diag(x_1^1, ..., x_{\mathbf{e}}^{\mathbf{e}})$, $Y_0 = diag(y_1^1, ..., y_{\mathbf{e}}^{\mathbf{e}})$, $Z_0 = diag(z_1^1, ..., z_{\mathbf{e}}^{\mathbf{e}})$, $X_1 = (x_j^i) - X_0$, $Y_1 = (y_j^i) - Y_0$, $Z_1 = (z_j^i) - Z_0$. Then

$$\tilde{M} = \sum_{i,j=1}^{\mathbf{e}} (x_j^i y_j^j + x_i^i y_j^i) z_i^j.$$

Let $X = \sum a_{\binom{k}{l}}^{\binom{i}{j}} X_{\binom{i}{j}}^{\binom{k}{l}}$ be an element of $\mathfrak{gl}(V_1)$, where $\{X_{\binom{i}{j}}^{\binom{k}{l}}\}$ is a basis of $\mathfrak{gl}(V_1)$, and define Y and Z in the same pattern with coefficients $b_{\binom{k}{l}}^{\binom{i}{j}}$'s and $c_{\binom{k}{l}}^{\binom{i}{j}}$'s, respectively. Consider the equation $(X, Y, Z).T = 0$ and we want to solve this equation for $a_{\binom{k}{l}}^{\binom{i}{j}}$'s, $b_{\binom{k}{l}}^{\binom{i}{j}}$'s and $c_{\binom{k}{l}}^{\binom{i}{j}}$'s. For these equations to hold, the coefficients of z_i^j's must be zero. That is, for each pair (j, i) of indices we have:

$$\sum_{k,l=1}^{\mathbf{e}} a_{\binom{k}{l}}^{\binom{i}{j}} x_l^k y_j^j + b_{\binom{k}{l}}^{\binom{j}{j}} x_j^i y_k^l + a_{\binom{k}{l}}^{\binom{i}{i}} x_l^k y_j^i + b_{\binom{k}{l}}^{\binom{i}{j}} x_i^i y_l^k + c_{\binom{j}{i}}^{\binom{l}{k}} (x_l^k y_l^l + x_k^k y_l^k) = 0.$$

For these equations to hold, the coefficients of y_s^r's must be zero. For example, if $s \neq j$, $r \neq s$ then we have:

$$b_{\binom{r}{s}}^{\binom{j}{j}} x_j^i + b_{\binom{r}{s}}^{\binom{i}{j}} x_i^i + c_{\binom{j}{i}}^{\binom{s}{r}} x_r^r = 0$$

Now coefficients of x terms must be zero, for instance, if $i \neq j$ and $i \neq r$, then we have:

$$b_{\binom{r}{s}}^{\binom{j}{j}} = 0, \quad b_{\binom{r}{s}}^{\binom{i}{j}} = 0, \quad c_{\binom{j}{i}}^{\binom{s}{r}} = 0.$$

If one writes down and solves all such linear equations, the dimension of the solution is $4\mathbf{e}^2 - 2\mathbf{e}$.

The same construction works for larger loops and cycles in larger graphs as it is essentially local - one just takes all other curves the constant curve equal to the identity. □

Remark D.2. When $\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_3 = 2$ we obtain a codimension one component of the boundary. In general, the dimension of the stabilizer is much larger than the dimension of G, so the orbit closures of these points do not give rise to codimension one components of the boundary. It remains an interesting problem to find the codimension one components of the boundary.

E. Algebraic geometry perspective

For readers familiar with algebraic geometry, we recast the previous section in the language of algebraic geometry and put it in a larger context. This section also serves to motivate the proof of the previous section.

To make the parallel with the GCT program clearer, we describe the Zariski closure as the cone over the (closure of) the image of the rational map

$$\begin{aligned} \mathbb{P}End(V_1) \times \mathbb{P}End(V_2) \times \mathbb{P}End(V_3) &\dashrightarrow \mathbb{P}(V_1 \otimes V_2 \otimes V_3) \\ ([X],[Y],[Z]) &\mapsto (X,Y,Z)\cdot[M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}]. \end{aligned} \tag{4.4}$$

(Compare with the map ψ in [4, 7.2].)

The indeterminacy locus consists of $([X],[Y],[Z])$ such that for all triples of matrices P, Q, R,

$$\mathrm{trace}(X(P)Y(Q)Z(R)) = 0.$$

In principle one can obtain (4.3) as the image of a map from a succession of blow-ups of $\mathbb{P}End(V_1) \times \mathbb{P}End(V_2) \times \mathbb{P}End(V_3)$. (See, e.g., [19, p. 81] for the definition of a blow-up)

One way to attain a point in the indeterminacy locus is to take $([X_0],[Y_0],[Z_0])$ as described in the proof. Taking a curve in G that limits to this point may or may not give something new. In the proof we gave two explicit choices that do give something new.

A more invariant way to discuss that $\tilde{M} \notin End(V_1) \times End(V_2) \times End(V_3) \cdot M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}$ is to consider an auxiliary variety, called a subspace variety,

$$Sub_{\mathbf{f}_1,\dots,\mathbf{f}_n}(\mathbf{V}) := \{T \in V_1 \otimes \cdots \otimes V_n \mid \exists V_j' \subset V_j, \dim V_j' = \mathbf{f}_j, \mathrm{and T} \in \mathrm{V}_1' \otimes \cdots \otimes \mathrm{V}_n'\},$$

and observe that if $T \in \times_j \operatorname{End}(V_j) \cdot M_{\vec{\mathbf{e}}}$ and $T \notin \times_j GL(V_j) \cdot M_{\vec{\mathbf{e}}}$, then $T \in Sub_{\mathbf{f}_1,,\dots,\mathbf{f}_n}(\mathbf{V})$ where $\mathbf{f}_j < \mathbf{e}_j$ for at least one j.

The statement that "$\tilde{M}$ uses all the variables" may be rephrased as saying that $\tilde{M} \notin Sub_{\mathbf{e}_2\mathbf{e}_3-1,\mathbf{e}_2\mathbf{e}_1-1,\mathbf{e}_3\mathbf{e}_1-1}(V_1 \otimes V_2 \otimes V_3)$

F. Reduction from the supercritical case to the critical case with the same graph

For a vector space W, let $G(k, W)$ denote the Grassmannian of k-planes through the origin in W. Let $\mathcal{S} \to G(k, W)$ denote the tautological rank k vector bundle whose fiber over $E \in G(k, W)$ is the k-plane E. Assume $\mathbf{f}_j \leq \mathbf{v}_j$ for all j with at least one inequality strict. Form the vector bundle $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$ over $G(\mathbf{f}_1, V_1) \times \cdots \times G(\mathbf{f}_n, V_n)$, where $\mathcal{S}_j \to G(\mathbf{f}_j, V_j)$ are the tautological subspace bundles. Note that the total space of $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$ maps to $\mathbf{V}$ with image $Sub_{\vec{\mathbf{f}}}(\mathbf{V})$. Define a fiber sub-bundle, whose fiber over $(U_1 \times \cdots \times U_n) \in G(\mathbf{f}_1, V_1) \times \cdots \times G(\mathbf{f}_n, V_n)$ is $TNS(\Gamma, \vec{\mathbf{e}}, U_1 \otimes \cdots \otimes U_n)$. Denote this bundle by $TNS(\Gamma, \vec{\mathbf{e}}, \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n)$.

The supercritical cases may be realized, in the language of Kempf, as a "collapsing of a bundle" over the critical cases as follows:

Proposition F.1. *Assume* $\mathbf{f}_j := \Pi_{s \in e(j)} \mathbf{e}_s \leq \mathbf{v}_j$. *Then* $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ *is the image of the bundle* $TNS(\Gamma, \vec{\mathbf{e}}, \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n)$ *under the map to* $\mathbf{V}$. *In particular*

$$\dim(TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})) = \dim(TNS(\Gamma, \vec{\mathbf{e}}, \mathbb{C}^{\mathbf{f}_1} \otimes \cdots \otimes \mathbb{C}^{\mathbf{f}_n})) + \sum_{j=1}^{n} \mathbf{f}_j(\mathbf{v}_j - \mathbf{f}_j).$$

Proof. If $\Pi_{s\in e(j)}\mathbf{e}_s \leq \mathbf{v}_j$, then any tensor $T \in V_j \otimes (\otimes_{s\in in(j)} E_s) \otimes (\otimes_{t\in out(j)} E_t^*)$, must lie in some $V_j' \otimes (\otimes_{s\in in(j)} E_s) \otimes (\otimes_{t\in out(j)} E_t^*)$ with $\dim V_j' = \mathbf{f}_j$. The space $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is the image of this subbundle under the map to $\mathbf{V}$. □

This type of bundle construction is standard, see [24, 43]. Using the techniques in [43], one may reduce questions about a supercritical case to the corresponding critical case.

G. Reduction of cases with subcritical vertices of valence one

The subcritical case in general can be understood in terms of projections of critical cases, but this is not useful for extracting information. However, if a subcritical vertex has valence one, one may simply reduce to a smaller graph as we now describe.

Proposition G.1. *Let $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$ be a tensor network state, let v be a vertex of Γ with valence one. Relabel the vertices such that $v = v_1$ and so that v_1 is attached by e_1 to v_2. If $\mathbf{v}_1 \leq \mathbf{e}_1$, then $TNS(\Gamma, \overrightarrow{\mathbf{e}}, V_1 \otimes \cdots \otimes V_n) = TNS(\tilde{\Gamma}, \overrightarrow{\tilde{\mathbf{e}}}, \tilde{V}_1 \otimes V_3 \otimes ... \otimes V_n)$, where $\tilde{\Gamma}$ is Γ with v_1 and e_1 removed, $\overrightarrow{\tilde{\mathbf{e}}}$ is the vector $(\mathbf{e}_2, ..., \mathbf{e}_n)$ and $\tilde{V}_1 = V_1 \otimes V_2$.*

Proof. A general element in $TNS(\Gamma, \overrightarrow{\mathbf{e}}, V_1 \otimes \cdots \otimes V_n)$ is of the form $\sum_{i,j=1}^{\mathbf{e}_1,\mathbf{e}_2} u_i \otimes v_{iz} \otimes w_z$, where $w_z \in V_3 \otimes \cdots \otimes V_n$. Obviously, $TNS(\Gamma, \overrightarrow{\mathbf{e}}, V_1 \otimes \cdots \otimes V_n) \subseteq TNS(\tilde{\Gamma}, \overrightarrow{\tilde{\mathbf{e}}}, \tilde{V}_1 \otimes V_3 \otimes ... \otimes V_n) =: TNS(\tilde{\Gamma}, \overrightarrow{\tilde{\mathbf{e}}}, \tilde{\mathbf{V}})$. Conversely, a general element in $TNS(\tilde{\Gamma}, \overrightarrow{\tilde{\mathbf{e}}}, \tilde{\mathbf{V}}))$ is of the form $\sum_z X_z \otimes w_z$, $X_z \in V_1 \otimes V_2$. Since $\mathbf{v}_1 \leq \mathbf{e}_1$, we may express X_z in the form $\sum_{i=1}^{e_1} u_i \otimes v_{iz}$, where $u_1, ..., u_{v_1}$ is a basis of V_1. Therefore, $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V}) \supseteq TNS(\tilde{\Gamma}, \overrightarrow{\tilde{\mathbf{e}}}, \tilde{\mathbf{V}})$.. □

H. Trees

With trees one can apply the two reductions successively to reduce to a tower of bundles where the fiber in the last bundle is a linear space. The point is that a critical vertex is both sub- and supercritical, so one can reduce at valence one vertices iteratively. Here are a few examples in the special case of chains. The result is similar to the Allman-Rhodes reduction theorem for phylogenetic trees [1].

Example H.1. Let Γ be a chain with 3 vertices. If it is supercritical, $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V}) = V_1 \otimes V_2 \otimes V_3$. Otherwise $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V}) = Sub_{\mathbf{e}_1, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_2}(V_1 \otimes V_2 \otimes V_3)$.

Example H.2. Let Γ be a chain with 4 vertices. If $\mathbf{v}_1 \leq \mathbf{e}_1$ and $\mathbf{v}_4 \leq \mathbf{e}_3$, then, writing $W = V_1 \otimes V_2$ and $U = V_3 \otimes V_4$, by Proposition G.1, $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$ is the set of rank at most $\mathbf{e}_2$ elements in $W \otimes U$ (the secant variety of the two-factor Segre). Other chains of length four have similar complete descriptions.

Example H.3. Let Γ be a chain with 5 vertices. Assume that $\mathbf{v}_1 \leq \mathbf{e}_1$, $\mathbf{v}_5 \leq \mathbf{e}_4$ and $\mathbf{v}_1\mathbf{v}_2 \geq \mathbf{e}_2$ and $\mathbf{v}_4\mathbf{v}_5 \geq \mathbf{e}_3$. Then $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$ is the image of a bundle over $G(\mathbf{e}_2, V_1 \otimes V_2) \times G(\mathbf{e}_3, V_4 \otimes V_5)$ whose fiber is the set of tensor network states associated to a chain of length three.

CHAPTER V

OTHER WORK

A. Second osculating spaces

Definition A.1. Let $X \subset \mathbb{P}^N$ be a smooth submanifold of the projective space $\mathbb{P}^N$ and let x be a point on X. The tangent space of X at x is defined to be

$$T_xX := \{c'(0) \mid c : (-\epsilon, \epsilon) \mapsto X \text{ is a smooth curve through } x \text{ on } X\}.$$

The second osculating space of X at x is defined to be the linear span of $c'(0)$ and $c''(0)$ for all smooth curves $c(t)$ on X passing through x. It is denoted by $T^{(2)}_x X$.

Let $\mathcal{P}erm_n := \overline{GL_{n^2} \cdot [perm_n]}$ be the orbit closure of the permanent polynomial $perm_n$ of degree n. We view $perm_n$ as an element of the vector space $S^n(E \otimes F)$. $[perm_n]$ is the element of $\mathbb{P}S^n(E \otimes F)$ determined by $perm_n$. Since $\mathcal{P}erm_n$ is an orbit closure of $perm_n$, the tangent space and the second osculating space of $\mathcal{P}erm_n$ at the point $[perm_n]$ have the following explicit expression:

$$T := T_{[perm_n]}\mathcal{P}erm_n = \mathfrak{gl}_{n^2}.perm_n,$$

$$T^{(2)} := T^{(2)}_{[perm_n]}\mathcal{P}erm_n = \mathfrak{gl}_{n^2}.\mathfrak{gl}_{n^2}.perm_n.$$

It is known that the stabilizer of $perm_n$ in $GL(E \otimes F)$ is $T(GL(E) \times GL(F)) \rtimes \mathfrak{S}_n \times \mathfrak{S}_n \rtimes \mathbb{Z}_2$, therefore both T and $T^{(2)}$ are $\mathfrak{S}_n \times \mathfrak{S}_n$-modules.

Proposition A.2. *As $\mathfrak{S}_n \times \mathfrak{S}_n$-modules, T is isomorphic to*

$$([(n)] \otimes [(n)]) \oplus ([(n-1,1)] \otimes [(n-1,1)]) \tag{5.1}$$

$$\oplus([(n)] \oplus [(n-1,1)]^{\oplus 2} \oplus [(n-2,2)] \oplus [(n-2,1,1)]) \otimes ([(n-1,1)] \oplus [(n)]) \tag{5.2}$$

$$\oplus([(n-1,1)] \oplus [(n)]) \otimes ([(n)] \oplus [(n-1,1)]^{\oplus 2} \oplus [(n-2,2)] \oplus [(n-2,1,1)]) \tag{5.3}$$

$$\oplus([(n)] \oplus [(n-1,1)]^{\oplus 2} \oplus [(n-2,2)][(n-2,1,1)])^{\otimes 2}. \tag{5.4}$$

Proof. We first identify $E \otimes F$ with $\mathbb{C}^{n^2}$ and we fix bases $\{e_i\}$, $\{f_i\}$ of E and F. Let $X_{\binom{k}{l}}^{\binom{i}{j}}$ be the element of $\mathfrak{gl}(E \otimes F) = End(E \otimes F)$ that sends $e_r \otimes f_s$ to $\delta_{\binom{i}{j}}^{\binom{r}{s}} e_k \otimes f_l$, where $\delta_{\binom{i}{j}}^{\binom{r}{s}}$ is the Kronecker delta. Then we have a decomposition of $End(E \otimes F)$ as a vector space as follows:

$$\begin{aligned} End(E \otimes F) = \; & span\{X_{\binom{i}{j}}^{\binom{i}{j}}\} \\ & \oplus span\{X_{\binom{i}{l}}^{\binom{i}{j}} \mid j \neq l\} \\ & \oplus span\{X_{\binom{k}{j}}^{\binom{i}{j}} \mid i \neq k\} \\ & \oplus span\{X_{\binom{k}{l}}^{\binom{i}{j}} \mid i \neq k, j \neq l\} \end{aligned}$$

We denote by V_1 the space $span\{X_{\binom{i}{j}}^{\binom{i}{j}}\}$. Then $V_1.perm_n$ is an $\mathfrak{S}_n \times \mathfrak{S}_n$ submodule of T. We count the dimension of $V_1.perm_n$. It is easy to see the inclusion

$$V_1.perm_n \subset (S^n E)_0 \otimes (S^n F)_0 \oplus (S_{(n-1,1)}E)_0 \otimes (S_{(n-1,1)}F)_0$$

where $(S^n E)_0$ is the $SL(E)$-weight zero subspace of $S^n E$ and we use similar notations for other modules.

It is known that $(S_\pi E)_0 \cong [(\pi)]$ as $\mathfrak{S}_n$-modules for any partition π of n so we have an inclusion of $\mathfrak{S}_n \times \mathfrak{S}_n$-modules:

$$V_1.perm_n \subset [(n)] \otimes [(n)] \oplus [(n-1,1)] \otimes [(n-1,1)].$$

Now It is easy to see that $dim(V_1.perm_n) = n^2 - (2n-2)$ since the Lie algebra of the stabilizer of $perm_n$ is contained in $sl_0(E \otimes F)$ and has dimension $2n-2$ Therefore we have

$$V_1.perm_n \cong [(n)] \otimes [(n)] \oplus [(n-1,1)] \otimes [(n-1,1)]$$

Let $V_2 = span\{X_{\binom{i}{l}}^{\binom{i}{j}} \mid j \neq l\}$ then $V_2.perm_n$ is again an $\mathfrak{S}_n \times \mathfrak{S}_n$-submodule of T and that $\{X_{\binom{i}{l}}^{\binom{i}{j}} perm_n \mid j \neq l\}$ is a set of basis of $V_2.perm_n$ by counting dimensions. One can prove that $V_2.perm_n$ is isomorphic to the second line of the module in the Proposition by calculating the character of this module. Then same calculation applies to other components and this complete the proof. □

It would be complicated to compute the $\mathfrak{S}_n \times \mathfrak{S}_n$-module structure of $T^{(2)}$. But it is easy to identify the structure of the submodule $V_1.V_1.perm_n$ of $T^{(2)}$, where V_1 is the vector space defined above.

Proposition A.3. *As an $\mathfrak{S}_n \times \mathfrak{S}_n$-module, $V_1.V_1.perm_n$ is isomorphic to*

$$M := [(n)]\otimes[(n)]\oplus[(n-1,1)]\otimes[(n-1,1)]\oplus[(n-2,2)]\otimes[(n-2,2)]\oplus[(n-2,1,1)]\otimes[(n-2,1,1)].$$

Proof. By Pieri's formula, it is easy to see that $V_1.V_1.perm_n$ is contained in M. It suffices to show that $V_1.V_1.perm_n$ contains at least 4 linearly independent $\Delta(\mathfrak{S}_n)$-invariants. We have the following 4 linearly independent invariants:

$$F_1 = \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)}^{\sigma(1)} x_{\sigma(2)}^{\sigma(2)} perm_{n-2}(\sigma(1), \sigma(2) \mid \sigma(1), \sigma(2)),$$

$$F_2 = \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)}^{\sigma(1)} x_{\sigma(3)}^{\sigma(2)} perm_{n-2}(\sigma(1), \sigma(2) \mid \sigma(1), \sigma(3)),$$

$$F_3 = \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(3)}^{\sigma(1)} x_{\sigma(4)}^{\sigma(2)} perm_{n-2}(\sigma(1), \sigma(2) \mid \sigma(3), \sigma(4)),$$

$$F_4 = \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(2)}^{\sigma(1)} x_{\sigma(1)}^{\sigma(2)} perm_{n-2}(\sigma(1), \sigma(2) \mid \sigma(2), \sigma(1)),$$

where $perm_{n-2}(i, j \mid k, l)$ is the permanent of the sub-matrix obtained by removing i, j-th rows and k, l-th columns.

It is easy to verify that these 4 polynomials are linearly independent and this proves the proposition. $\square$

B. Linear subspaces on the hypersurface defined by $perm_n$

We have a more general result:

Proposition B.1. *Let $P \in S^n(E \otimes F)$ be a polynomial of degree n in n^2 variables and assume that p is a $SL(E) \times SL(F)$ weight zero vector. Let $X = Zeroes(p) \subset \mathbb{P}^{n^2-1}$, and let $x = (x^i_j) \in X$ be any point. Then there exists a linear subspace $L \cong \mathbb{P}^{2n-3}$ of $\mathbb{P}^{n^2-1}$ such that $x \in L \subset X$*

Proof. Let $u_j = (a^1_j, ..., a^n_j)^T$ for $j = 1, ..., n$. Then $x = (u_1, ...u_n)$. Let $U_j = (0, ..., 0, u_j, 0, ..., 0)$ for $j = 2, ..., n$ so $U_j \in X$ and U_j is a most singular point of X. Let

$$U_1 = \begin{bmatrix} y^1 & 0 & . & . & . & 0 \\ y^2 & 0 & . & . & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y^n & 0 & . & . & . & 0 \end{bmatrix} \text{ with } y^i\text{'s are unknowns.}$$

Consider

$$\overline{p}(U_1^{i_1}, ..., U_n^{i_n}) = 0,$$

where $i_1 + ... + i_n = n$, $1 \leq i_k \leq n$ and $\overline{p}$ is the polarization of the polynomial p.

Since each U_j is a most singular point of X, we just need to consider

$$\overline{p}(U_1, ...U_n) = 0$$

This is a linear equation with variables $y^1, ..., y^n$ and it has a solution $\begin{bmatrix} a^1_1 & 0 & . & . & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a^n_1 & 0 & . & . & . & 0 \end{bmatrix}$.

So it has at least $n-1$ dimensional solutions. These solutions together with U_2,...,U_n span a linear subspace of dimension $2n-3$ in $\mathbb{P}^{n^2-1}$. $\square$

CHAPTER VI

SUMMARY

A. Summary

In this dissertation, we study two objects: immanants and tensor network states. Both of these two objects are closely related to the Geometric Complexity Theory (GCT) program. In chapter three, we locate immanants as trivial $(SL(E) \times SL(F)) \rtimes \Delta(\mathfrak{S}_n)$-modules contained in the space $S^n(E \otimes F)$ of polynomials of degree n on the vector space $E \otimes F$. We prove that the stabilizer of an immanant associated to any non-symmetric partitions is $T(GL(E) \times GL(F)) \rtimes \Delta(\mathfrak{S}_n) \rtimes \mathbb{Z}_2/N$, where $T(GL(E) \times GL(F))$ is the group of pairs of $n \times n$ diagonal matrices with the product of determinants 1, $\Delta(\mathfrak{S}_n)$ is the diagonal subgroup of $\mathfrak{S}_n \times \mathfrak{S}_n$ and N is the subgroup of $T(GL(E) \times GL(F))$ of pairs of matrices of the form $(\alpha Id_E, \alpha^{-1} Id_F)$ where α is a nonzero complex number. We also prove that the identity component of the stabilizer any immanant is $T(GL(E) \times GL(F))/N$. In chapter four, we answer a question asked by Grasedyck by proving that the tensor network states associated to a triangle is not Zariski closed and we give two reductions of tensor network states from complicated cases to simple cases. In chapter five, we give some results about the tangent space and second osculating space of $\mathcal{P}erm_n$ at $[perm_n]$ and passing through any point of the hyper-surface defined by a weight zero polynomial, we find a linear subspace of dimension $2n-3$ contained in the hypersurface.

APPENDIX A

COMPLEXITY THEORY

L. Valiant defined classes VP and VNP as algebraic analogues of P and NP. VP and VNP are defined in terms of arithmetic circuits.

Definition A.1. An arithmetic circuit C is a finite, acyclic, directed graph with vertices of in-degree 0 or 2 and exactly one vertex of out-degree 0. The vertices of in-degree 0 are labelled by elements of $\mathbb{C} \cup \{x_1, ..., x_n\}$, and those of in-degree 2 are labelled with $+$ or $*$. The size of C is the number of vertices.

Remark A.2. From an arithmetic circuit C, one can construct a polynomial $p_C$in the variables $x_1, ..., x_n$ over $\mathbb{C}$.

Definition A.3. The class VP is the set of sequences (p_n) of polynomials of degree $d(n)$ in $v(n)$ variables, where $d(n)$ and $v(n)$ are bounded by polynomials in n and such that there exists a sequence of arithmetic circuits (C_n) of polynomially bounded size such that C_n computes p_n.

The class VP is closed under linear projections

Proposition A.4. *Let $\pi_n : \mathbb{C}^{v(n)} \mapsto \mathbb{C}^{v'(n)}$ is a sequence of linear projections, and a family (p_n) is in VP, then the family $(\pi_n \circ p_n)$ is also in VP.*

Proof. See [8, Chap.21] or [18]. □

Example A.5. The sequence of determinants (det_n) is a famous example of a sequence in VP

Let $g = (g_n)$ be a sequence of polynomials in variables $x_1, ..., x_n$ of the form

$$g_n = \sum_{e \in \{0,1\}^n} f_n(e) x_1^{e_1} \cdots x_n^{e_n},$$

where $(f_n) \in VP$.

Definition A.6. VNP is the set of all sequences that are projections of sequences of the form g.

Example A.7. The sequence $(perm_n)$ of permanent polynomials is a famous example of a sequence in VNP. See [40] for the proof.

Definition A.8. The class VP_{ws} is the set of sequences (p_n) where $deg(p_n)$ is bounded by a polynomial and such that there exists a sequence of circuits (C_n) of polynomially bounded size such that C_n computes p_n, and such that at any multiplication vertex of C_n, the component of the C_n of one of the two edges coming in is disconnected from the rest of the circuit by removing the multiplication vertex. Such a circuit is called weakly skew.

Example A.9. (det_n) is in VP_{ws}.

Definition A.10. A problem P is hard for a complexity class C if all problems in C can be reduced to P (i.e., there is an algorithm to translate nay instance of a problem in C to an instance of P with comparable input size). A problem P is complete for C if it is hard for C and $P \in C$.

Example A.11. (det_n) is VP_{ws}-complete and $(perm_n)$ is VNP-complete. See [26] for the proof of the first statement and [3] for the proof of the second statement.

Next we give the definition of closures of complexity classes.

Definition A.12. Let C be a complexity class defined in terms of a measure $L_C(p)$ of complexity of polynomials, where a sequence (p_n) is in C if $L_C(p_n)$ is bounded by a polynomial in n. For $p \in S^d\mathbb{C}^v$, write $\overline{L_C}(p) \leq r$ if p is in the Zariski closure of the set $\{q \in S^d\mathbb{C}^v \mid L_C(q) \leq r\}$, and define the class $\overline{C}$ to be the set of sequences p_n such that $\overline{L_C}(p_n)$ is bounded by a polynomial in n

For our purpose, we use the determinantal complexity of a polynomial.

Definition A.13. Let p be a polynomial. Define the determinantal complexity $dc(p)$ of p to be the smallest integer such that p is an affine linear projection of $det_{dc(p)}$. Define $\overline{dc}(p)$ to be the smallest integer δ such that there exists a sequence of polynomials p_t, of constant determinantal complexity δ for all $t \neq 0$ and $\lim_{t \to 0} p_t = p$.

Remark A.14. We can rephrase the Conjecture A.2 as: $\overline{dc}(perm_n)$ is not bounded by any polynomial in n.

REFERENCES

[1] Elizabeth S. Allman and John A. Rhodes, *Mathematical models in biology: an introduction*, Cambridge University Press, Cambridge, 2004. MR 2 013 130

[2] Peter Botta, *Linear transformations that preserve the permanent*, Proc. Amer. Math. Soc. **18** (1967), 566–569. MR MR0213376 (35 #4240)

[3] Jean-Luc Brylinski and Ranee Brylinski, *Complexity and completeness of immanants*, preprint, arXiv:0301024, 2003.

[4] P. Bürgisser, J.M. Landsberg, L. Manivel, and J. Weyman, *An overview of mathematical issues arising in the geometric complexity theory approach to* $\mathbf{VP} \neq \mathbf{VNP}$, SIAM Journal on Computing **40** (2011), 1179–1209.

[5] Peter Bürgisser, *Completeness and reduction in algebraic complexity theory*, Algorithms and Computation in Mathematics, vol. 7, Springer-Verlag, Berlin, 2000. MR MR1771845 (2001g:68030)

[6] ———, *The computational complexity of immanants*, SIAM J. Comput. **30** (2000), no. 3, 1023–1040 (electronic). MR 1778298 (2001k:68154)

[7] ———, *The computational complexity to evaluate representations of general linear groups*, SIAM J. Comput. **30** (2000), no. 3, 1010–1022 (electronic). MR 1778297 (2001m:68185)

[8] Peter Bürgisser, Michael Clausen, and M. Amin Shokrollahi, *Algebraic complexity theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 315, Springer-Verlag, Berlin, 1997, With the collaboration of Thomas Lickteig. MR 99c:68002

[9] J. Ignacio Cirac and Frank Verstraete, *Renormalization and tensor product states in spin chains and lattices*, J. Phys. A **42** (2009), no. 50, 504004, 34. MR 2566331 (2011b:82013)

[10] M. Purificação Coelho, *On linear preservers of immanants*, Linear Algebra Appl. **247** (1996), 265–271. MR 1412753 (97f:15011)

[11] M. Purificação Coelho and M. Antónia Duffner, *On the conversion of an immanant into another*, Linear and Multilinear Algebra **44** (1998), no. 2, 111–130. MR 1674232 (99j:15009)

[12] Stephen Cook, *The complexity of theorem proving procedures*, Proceedings of the Third Annual ACM Symposium on Theory of Computing.

[13] M. Antónia Duffner, *Linear transformations that preserve immanants*, Linear Algebra Appl. **197/198** (1994), 567–588, Second Conference of the International Linear Algebra Society (ILAS) (Lisbon, 1992). MR MR1275631 (95c:15012)

[14] M. Fannes, B. Nachtergaele, and R. F. Werner, *Valence bond states on quantum spin chains as ground states with spectral gap*, J. Phys. A **24** (1991), no. 4, L185–L190. MR 1104168 (92d:82008)

[15] ———, *Finitely correlated states on quantum spin chains*, Comm. Math. Phys. **144** (1992), no. 3, 443–490. MR 1158756 (93i:82006)

[16] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR MR1153249 (93a:20069)

[17] ———, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR MR1153249

(93a:20069)

[18] Joachim von zur Gathen, *Feasible arithmetic computations: Valiant's hypothesis*, J. Symbolic Comput. **4** (1987), no. 2, 137–172. MR MR922386 (89f:68021)

[19] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995, A first course, Corrected reprint of the 1992 original. MR MR1416564 (97e:14001)

[20] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR MR0463157 (57 #3116)

[21] S. Iblisdir, J. I. Latorre, and R. Orús, *Entropy and exact matrix-product representation of the Laughlin wave function*, Phys. Rev. Lett. **98** (2007), no. 6, 060402, 4. MR 2284036 (2007j:82007)

[22] Yang Qi J.M Landsberg, Ke Ye, *On the geometry of tensor network states*, QIC **12** (2012), no. 3,4, 0346–0354.

[23] J. M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, Providence, RI, 2012. MR 2865915

[24] J.M. Landsberg, *Tensors: Geometry and applications*, Graduate Studies in Mathematics, vol. 128, AMS, Providence, 2011.

[25] Dudley E. Littlewood, *The theory of group characters and matrix representations of groups*, AMS Chelsea Publishing, Providence, RI, 2006, Reprint of the second (1950) edition. MR MR2213154 (2006m:20013)

[26] G. Malod and N. Portier, *Characterizing Valiant's algebraic complexity classes*, Journal of Complexity **24** (2008), 16–38.

[27] Thierry Mignon and Nicolas Ressayre, *A quadratic bound for the determinant and permanent problem*, Int. Math. Res. Not. (2004), no. 79, 4241–4253. MR MR2126826 (2006b:15015)

[28] Ketan D. Mulmuley, *Geometric complexity theory: On canonical bases for the non-standard quantum groups*, preprint.

[29] ______, *Geometric complexity theory VI: the flip via saturated and positive integer programming in representation theory and algebraic geometry,*, Technical Report TR-2007-04, computer science department, The University of Chicago, May, 2007.

[30] ______, *Geometric complexity theory VII: Nonstandard quantum group for the plethysm problem*, preprint.

[31] Ketan D. Mulmuley and H. Narayaran, *Geometric complexity theory V: On deciding nonvanishing of a generalized Littlewood-Richardson coefficient*, Technical Report TR-2007-05, computer science department, The University of Chicago, May, 2007.

[32] Ketan D. Mulmuley and Milind Sohoni, *Geometric complexity theory III: on deciding positivity of Littlewood-Richardson coefficients*, preprint cs.ArXiv preprint cs.CC/0501076.

[33] ______, *Geometric complexity theory IV: quantum group for the Kronecker problem*, preprint available at UC cs dept. homepage.

[34] ______, *Geometric complexity theory. I. An approach to the P vs. NP and related problems*, SIAM J. Comput. **31** (2001), no. 2, 496–526 (electronic). MR MR1861288 (2003a:68047)

[35] ______, *Geometric complexity theory. II. Towards explicit obstructions for embeddings among class varieties*, SIAM J. Comput. **38** (2008), no. 3, 1175–1206. MR

MR2421083

[36] Claudio Procesi, *Lie groups*, Universitext, Springer, New York, 2007, An approach through invariants and representations. MR MR2265844 (2007j:22016)

[37] Bruce E. Sagan, *The symmetric group*, second ed., Graduate Texts in Mathematics, vol. 203, Springer-Verlag, New York, 2001, Representations, combinatorial algorithms, and symmetric functions. MR 1824028 (2001m:05261)

[38] A.W. Sandvik and G. Vidal, *Variational quantum monte carlo simulations with tensor-network states*, PHYSICAL REVIEW LETTERS **99** (2007), no. 22, 589–591.

[39] Michael Sipser, *The history and status of the p versus np question*, STOC '92 Proceedings of the twenty-fourth annual ACM symposium on Theory of computing (1992), 603–618.

[40] L. G. Valiant, *The complexity of computing the permanent*, Theoret. Comput. Sci. **8** (1979), no. 2, 189–201. MR MR526203 (80f:68054)

[41] Leslie G. Valiant, *Completeness classes in algebra*, Proc. 11th ACM STOC, 1979, pp. 249–261.

[42] F. Verstraete, M. M. Wolf, D. Perez-Garcia, and J. I. Cirac, *Criticality, the area law, and the computational power of projected entangled pair states*, Phys. Rev. Lett. **96** (2006), no. 22, 220601, 4. MR 2231431 (2006m:81082)

[43] Jerzy Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, vol. 149, Cambridge University Press, Cambridge, 2003. MR MR1988690 (2004d:13020)

[44] Ke Ye, *The stabilizer of immanants*, Linear Algebra Appl. **435** (2011), no. 5, 1085–1098. MR 2807220

VITA

Name:	Ke Ye
Address:	1100 Hensel Drive, Apt#Y4B,
	College Station, TX,77840
Email Address:	kye@math.tamu.edu
Education:	B.S., Sichuan University, 2007
	Ph.D, Texas A&M University,2012

The typist for this thesis was Ke Ye.