CORRIGENDUM TO THEOREM 2.10 OF "COMMUTATORS ON $(\sum \ell_q)_{\ell_p}$" [STUDIA MATH. 206 (2011), NO.2, 175-190]

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ABSTRACT. We give a corrected proof of Theorem 2.10 in our paper "Commutations on $(\sum \ell_q)_{\ell_p}$" [Studia Math. 206 (2011), no.2, 175-190] for the case $1 < q < p < \infty$. The case when $1 = q < p < \infty$ remains open. As a consequence, the Main Theorem and Corollary 2.17 in [CJZh] are only valid for $1 < p, q < \infty$.

Throughout this note, "small perturbation" means using the image of the subspace under an operator that is close to the identity. The notation is as in [CJZh]. We thank Eugenio Spinu for spotting the error in the last line of the proof of Theorem 2.10 in [CJZh], where it is claimed "Then it is easy to see that $\Sigma_{n=0}^{\infty} R^n T L^n$ is strongly convergent if $\Sigma_n \varepsilon_n < \infty$".

Theorem 0.1. *Let $1 < p < q < \infty$. Let $T : Z_{p,q} \to Z_{p,q}$ be $Z_{p,q}$-strictly singular. Then for all $\epsilon > 0$ there is a 1-complemented subspace Y of $Z_{p,q}$ which is isometric to $Z_{p,q}$ and $\|T|_Y\| < \epsilon$.*

Lemma 0.2. *Let $S : \ell_q \to Z_{p,q}$ $(1 < p < q < \infty)$. Then $\forall \epsilon > 0$, there is an $N \in \mathbb{N}$ such that $\|P_{[N,\infty)} S\| < \epsilon$.*

Proof. Suppose not. Then there is an $\epsilon > 0$ so that for any $N \in \mathbb{N}$, $\|P_{[N,\infty)} S\| \geq \epsilon$. So by a standard perturbation argument, there is a normalized block basis (x_i) of ℓ_q whose image sequence (Tx_i) is equivalent to the unit vector basis of ℓ_p. Since $1 < p < q < \infty$, this contradicts the boundedness of T. □

Lemma 0.3. *Let $S : Z_{p,q} \to \ell_q$ $(1 < p < q < \infty)$. Then $\forall \epsilon > 0$ there is a subspace Y of $Z_{p,q}$, such that Y is isometric to ℓ_q, Y is 1-complemented in $Z_{p,q}$, and $\|S|_Y\| < \epsilon$.*

Dongyang Chen was a participant in the NSF Workshop in Analysis and Probability, Texas A&M University.

The research of William B. Johnson is supported in part by NSF DMS-1301604.

The research of Bentuo Zheng is supported in part by NSF DMS-1068838.

Bentuo Zheng was a participant in the NSF Workshop in Analysis and Probability, Texas A&M University.

Proof. Let $(e_{i,j})$ be the natural unit vector basis of $Z_{p,q}$, where $(e_{i,j})_j$ is the unit vector basis of the ith ℓ_q. By passing to appropriate subsequences of $(e_{i,j})$ and perturbing S slightly, we may assume that $(Se_{i,j})$ are disjointly supported in ℓ_q. Since $1 < p < q < \infty$, we can pick an $N \in \mathbb{N}$ so large that $N^{1/q-1/p} < \epsilon/\|T\|$. Let $x_j = N^{-1/p}\sum_{i=1}^N e_{i,j}$. Then (x_j) is 1-equivalent to the unit vector basis of ℓ_q. Let Y be the closed linear span of (x_j). Then Y is 1-complemented in $Z_{p,q}$ and $\|S|_Y\| < \epsilon$. □

Proof of Theorem 0.1. Fix $\epsilon > 0$. Let (ϵ_i) be a sequence of positive reals decreasing to 0 fast so that $\sum \epsilon_i < \min\{\epsilon/4, 1/4\}$. We write $Z_{p,q} = (\sum \ell_q^{(n)})_{\ell_p}$. Let $X_1 = \ell_q^{(1)}$. By Lemma 0.2, there is $N_1 \in \mathbb{N}$, $P_{[N_1,\infty)}T|_{X_1} < \epsilon_1$. By Lemma 0.3, there are $N_2 \in \mathbb{N}$ and $X_2 \subset P_{[N_1,N_2)}Z_{p,q}$ so that $X_2 \equiv \ell_q$, X_2 is 1-complemented in $Z_{p,q}$, and $\|P_{[1,N_1)}T|_{X_2}\| < \epsilon_2/2$. By using Lemma 0.2 again and increasing N_2, we may also assume that $\|P_{[N_2,\infty)}T|_{X_2}\| < \epsilon_2/2$.

So by induction we get an increasing sequence (N_i) of positive integers and a sequence of subspaces (X_i) so that

- $X_i \equiv \ell_q$;
- X_i is 1-complemented in $Z_{p,q}$;
- $X_i \subset P_{[N_{i-1},N_i)}Z_{p,q}$ (where $N_0 = 1$);
- $\|(I - P_{[N_{i-1},N_i)})T|_{X_i}\| < \epsilon_i$.

We claim that for all but finitely many $i \in \mathbb{N}$, there is a subspace Y_i of X_i so that $Y_i \equiv \ell_q$, Y_i is 1-complemented in X_i, and $\|T|_{Y_i}\| < \epsilon$. Suppose not. Then there is an infinite subset $I \subset \mathbb{N}$ so that for all $i \in I$ and for every 1-complemented subspace Y_i of X_i that is isometric to ℓ_q we have $\|T|_{Y_i}\| \geq \epsilon$. Therefore, for each $i \in I$ there is a normalized block basis $(x_{i,j})_j$ of X_i so that $\|Tx_{i,j}\| \geq \epsilon$. By passing to a subsequence of $(x_{i,j})_j$ and doing a small perturbation, we may assume that $(Tx_{i,j})_j$ is disjointly supported in $Z_{p,q}$. Since $Z_{p,q}$ has a lower q-estimate with constant 1, $(Tx_{i,j})_j$ is $\|T\|/\epsilon$-equivalent to $(x_{i,j})_j$. For each $i \in I$, let Y_i be the closed linear span of $(x_{i,j})_j$. Then $\sum_{i\in I} Y_i$ is isometric to $Z_{p,q}$. Next we show that $T|_{\sum_{i\in I} Y_i}$ is an isomorphism. To see this, let $(y_i)_{i\in I} \in \sum_{i\in I} Y_i$ with $\sum_{i\in I} \|y_i\|^p = 1$.

Then we have

$$\begin{aligned}\|T((y_i)_{i\in I})\| &\geq \|\sum_{i\in I} P_{[N_{i-1},N_i)}Ty_i\| - \sum_{i\in I}\|(I-P_{[N_{i-1},N_i)})Ty_i\| \\ &\geq (\sum_{i\in I}(1-\epsilon_i)^p\|Ty_i\|^p)^{1/p} - \sum_{i\in I}\epsilon_i\|y_i\| \\ &\geq 3\epsilon/4 - \sum_{i\in I}\epsilon_i \\ &> \epsilon/2.\end{aligned}$$

This contradicts the hypothesis that T is $Z_{p,q}$-strictly singular.

Now we have our claim. Without loss of generality, we assume for all $i \in \mathbb{N}$, there is a subspace Y_i of X_i so that $Y_i \equiv \ell_q$, Y_i is 1-complemented in X_i and $\|T|_{Y_i}\| < \epsilon$. Let $Y = \sum Y_i$. Then Y is isometric to $Z_{p,q}$ and 1-complemented in $Z_{p,q}$. Let $(y_i) \in S_Y$. We have

$$\begin{aligned}\|T((y_i))\| &\leq \|\sum P_{[N_{i-1},N_i)}Ty_i\| + \sum\|(I-P_{[N_{i-1},N_i)})Ty_i\| \\ &\leq (\sum\|Ty_i\|^p)^{1/p} + \sum\epsilon_i\|y_i\| \\ &< \epsilon + \sum\epsilon_i \\ &< 5\epsilon/4.\end{aligned}$$

Since ϵ is arbitrary, we are done.

□

Lemma 0.4. *Let $1 < p, q < \infty$ and $n \in \mathbb{N}$. Set $Z := (\sum_{k=1}^n X_n)_p$ with each X_n isometrically isomorphic to ℓ_q. Suppose that X is a subspace of Z. Then for each $\epsilon > 0$ there is a subspace Y of X so that Y is $1+\epsilon$-isomorphic to ℓ_q and Y is $1+\epsilon$-complemented in Z.*

Proof. By the principle of small perturbations we can assume that X contains a sequence (x_k) that is disjointly supported with respect to the canonical basis $(e_{i,j})_{i=1,j=1}^{\infty,\ n}$. By passing to a subsequence of (x_k) and making another small perturbation, we can assume for every $j = 1, \dots, n$ that there is a scalar a_j so that for each $k \in \mathbb{N}$ we have $\|P_j x_k\| = a_j$, so that $\sum_{j=1}^n a_j^p = 1$. One checks easily that (x_k) is 1-equivalent to the unit vector basis of ℓ_q. Indeed, if $z = \sum_k b_k x_k$, then

$$\begin{aligned}\|z\|^p &= \sum_{j=1}^n \|P_j z\|^p = \sum_{j=1}^n \|\sum_k b_k P_j x_k\|^p \\ &= \sum_{j=1}^n (a_j(\sum_k |b_k|^q)^{1/q})^p = (\sum_{j=1}^n a_j^p)(\sum_k |b_k|^q)^{p/q}).\end{aligned}$$

To see that $[x_k]$ is norm one complemented in Z, assume WLOG that no a_j is zero and let $x^*_{k,j}$ be the unique norm one functional in $Z^* = (\sum_{k=1}^n X^*_n)_{p'}$ for which $\langle x^*_{k,j}, P_j x_k\rangle = a_j$. So $x^*_{k,j}$ has the same support as $P_j x_k$ and for each j, the sequence $(x^*_{k,j})_k$ is 1-equivalent to the unit vector basis of $\ell_{q'}$. Define $x^*_k := \sum_{j=1}^n a_j^{p-1} x^*_{k,j}$. Then the sequence (x^*_k) is 1-equivalent to the unit vector basis for $\ell_{q'}$ and is biorthogonal to the sequence (x_k). This implies that $Px := \sum_k \langle x^*_k, x\rangle x_k$ defines a norm one projection from Z onto $[x_k]$. □

Lemma 0.5. *$Z_{p,q}$ is complementably homogeneous for $1 < p < q < \infty$.*

Proof. Let $X = (\sum X_k)$ be a subspace of $Z_{p,q}$ isomorphic to $Z_{p,q}$ so that each X_k is isomorphic to ℓ_q. Let (ϵ_i) be a sequence of positive reals decreasing to 0 fast. Let Y_1 be a subspace of X_1 which is $1+\epsilon_1$-isomorphic to ℓ_q. By Lemma 0.2 and a small perturbation, we may assume that there is $N_1 \in \mathbb{N}$ so that $\|P_{[N_1,\infty)}|_{Y_1}\| = 0$. By Lemma 0.2, Lemma 0.3, stability of ℓ_q, and a small perturbation, we may assume that there is a subspace Y_2 of X so that Y_2 is $1+\epsilon_2$-isomorphic to ℓ_q and $(I - P_{[N_1,N_2)})|_{Y_2} = 0$. Inductively, we get a sequence of subspaces (Y_k) of X and a sequence of increasing positive integers (N_k) so that Y_k is $1+\epsilon_k$-isomorphic to ℓ_q and $Y_k \subset P_{[N_{k-1},N_k)} Z_{p,q}$. By Lemma 0.4 and passing to subspaces of each Y_k, we may assume that Y_k is $1+\epsilon_k$-complemented in $P_{[N_{k-1},N_k)} Z_{p,q}$. Let $Y = \sum Y_k$. Then Y is $1+\epsilon$-isomorphic to $Z_{p,q}$ and $1+\epsilon$-complemented in $Z_{p,q}$ if $\sum \epsilon_k < \epsilon$.

□

Theorem 0.6. *Let $1 < q < p < \infty$. Let $T : Z_{p,q} \to Z_{p,q}$ be $Z_{p,q}$-strictly singular. Then there is a 1-complemented subspace Y of $Z_{p,q}$ which is isometric to $Z_{p,q}$ so that $\|P_Y T\| < \epsilon$, where P_Y is a norm 1 projection from $Z_{p,q}$ onto Y.*

Proof. This follows immediately by applying Theorem 0.1 for T^* and Lemma 0.5. □

Corrected proof of Theorem 2.10 in [CJZh] *for $1 < q < p < \infty$.* By [D, Theorem 8], it is enough to show that there is an ℓ_p−decomposition $\{X_i\}$ of $Z_{p,q}$ into uniformly isomorphic copies of ℓ_q so that

$$\lim_{n\to\infty} \|(\sum_{k\geq n} P_k)T\| = \lim_{n\to\infty} \|T(\sum_{k\geq n} P_k)\| = 0, \tag{0.1}$$

where P_k is the natural projection from $Z_{p,q}$ onto X_k.

By the original proof of Theorem 2.10 in [CJZh], we can get a sequence of subspaces $(X_n)_{n=1}^\infty$ of $(\sum_{n=0}^\infty Z_{p,q})_{\ell_p}$ such that

(1) X_n is isometric to $Z_{p,q}$ and 1-complemented in $Z_{p,q}$;
(2) $\|T|_{X_n}\| < \varepsilon_n$;
(3) $\|\Sigma_{n=1}^{\infty} x_n\| = (\Sigma_{n=1}^{\infty}\|x_n\|^p)^{\frac{1}{p}}$, for all $x_n \in X_n$;
(4) $Z_{p,q} = (\Sigma_{n=1}^{\infty} X_n)_p \oplus X_0$ and X_0 is isomorphic to $Z_{p,q}$.

By Theorem 0.6 and passing to subspaces X_n' of each X_n $(n \geq 1)$ (absorbing the complements of X_n' in X_n into X_0), we may assume one additional condition.

(5) $\|P_n T\| < \epsilon_n$ $(n \geq 1)$, where P_n is the norm one projection from $Z_{p,q}$ onto X_n.

Now Equation 0.1 clearly holds if $\lim_{n\to\infty} \sum_{k\geq n} \epsilon_k \to 0$.

□

References

[D] D. Dosev, *Commutators on ℓ_1*, Journal of Functional Analysis, 256 (2009), no. 11, 3490-3509.

[CJZh] D. Chen, W. B. Johnson and Bentuo Zheng, *Commutators on* $(\sum \ell_q)_{\ell_p}$, Studia Math., 206 (2011), no. 2, 175-190.

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