QUOTIENTS OF ESSENTIALLY EUCLIDEAN SPACES

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Abstract. A precise quantitative version of the following qualitative statement is proved: If a finite dimensional normed space contains approximately Euclidean subspaces of all proportional dimensions, then every proportional dimensional quotient space has the same property.

1. Introduction

Given a function λ from $(0,1)$ into the positive reals, call a finite dimensional normed space E λ essentially Euclidean provided that for every $\epsilon > 0$ there is a subspace E_ϵ of E that has dimension at least $(1-\epsilon)\dim E$ and the Euclidean distortion $c_2(E_\epsilon)$ of E_ϵ is $\le \lambda(\epsilon)$; that is, E_ϵ is $\lambda(\epsilon)$-isomorphic to the Euclidean space of its dimension. A family $\mathcal{F}$ of finite dimensional spaces is λ essentially Euclidean provided that each space in $\mathcal{F}$ is λ essentially Euclidean, and $\mathcal{F}$ is called essentially Euclidean if it is λ essentially Euclidean for some function λ as above. Litwak, Milman, and Tomczak-Jaegermann [**?**] considered the concept of essentially Euclidean, but what we are calling an essentially Euclidean family they would term a 1-*ess-Euclidean* family. The most studied essentially Euclidean families are the class of all finite dimensional spaces that have cotype two constant less than some numerical constant, and the set of all finite dimensional subspaces of a Banach space that has weak cotype two [**?**, Chapter 10]. However, if one is interested in the proportional subspace theory of finite dimensional spaces, cotype two and weak cotype two are unnecessarily strong conditions because they are conditions on all subspaces rather than on just subspaces of proportional dimension. For example, let $0 < \alpha < 1$ and let $\mathcal{F}_\alpha$ be the family $\{\ell_2^{n-n^\alpha} \oplus_2 \ell_\infty^{n^\alpha} : n = 1, 2, 3, \dots\}$ (throughout we use the convention, standard in the local theory of Banach spaces, that when a specified dimension is not a postive integer, it should be adjusted to the next larger or smaller positive integer, depending on

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context). The cotype two constants of the spaces in $\mathcal{F}_\alpha$ are obviously unbounded and it is also well known that the family does not live in any weak cotype two space. Computing that $\mathcal{F}_\alpha$ is $\lambda_\alpha(\epsilon)$ essentially Euclidean with $\lambda_\alpha(\epsilon) \le (1/\epsilon)^{\alpha/2(1-\alpha)}$ is straightforward: First, when $n^\alpha \le \epsilon n$, the space $\ell_2^{n-n^\alpha} \oplus_2 \ell_\infty^{n^\alpha}$ has a subspace of dimension at least $(1-\epsilon)n$ that is isometrically Euclidean. On the other hand, if $\epsilon n < n^\alpha$, then the entire space $\ell_2^{n-n^\alpha} \oplus_2 \ell_\infty^{n^\alpha}$ is $n^{\alpha/2}$-Euclidean since that is the isomorphism constant between $\ell_\infty^{n^\alpha}$ and $\ell_2^{n^\alpha}$, and $n^{\alpha/2} \le (1/\epsilon)^{\alpha/2(1-\alpha)}$.

It is also simple to check that the essentially Euclidean property passes to proportional dimensional subspaces. Suppose that E is an n-dimensional space that is λ essentially Euclidean, F is a subspace of E that has dimension αn, and $\epsilon > 0$. Take a subspace E_1 of E of dimension $(1-\epsilon\alpha)n$ with $c_2(E_1) \le \lambda(\epsilon\alpha)$. Then $\dim E_1 \cap F = (1-\epsilon\alpha)n + \alpha n - n = (1-\epsilon)\alpha n$, which implies that F is λ_F essentially Euclidean with $\lambda_F(\epsilon) \le \lambda(\epsilon\alpha)$. It is, however, not obvious that the essentially Euclidean property passes to proportional dimensional quotients; the main result of this note is that it does.

We use standard notation. We just mention that if A is a set of vectors in a normed space, $[A]$ denotes the closed linear space of A, and e_i denotes the ith unit basis vector in a sequence space.

2. Main Result

The main tool we use is Milman's subspace of quotient theorem [?], [?, Chapters 7 & 8]. In [?] this theorem is not used directly, but the ingredients of its proof are. The theorem says that there is a function $M : (0,1) \to \mathbb{R}^+$ such that for every n and every $0 < \delta < 1$, if $\dim \mathrm{E} = \mathrm{n}$ then there is a subspace F of some quotient of E so that $\dim \mathrm{F} = (1-\delta)\mathrm{n}$ and $c_2(F) \le M(\delta)$. It is known that $M(\delta) \le (C/\delta)(1 + |\log C\delta|)$ as $\delta \to 0$ [?, Theorem 8.4].

In this section we prove

Theorem 2.1. *Suppose that E is λ essentially Euclidean, $0 < \alpha < 1$, and Q is a quotient mapping from E onto a space F. Let $n = \dim \mathrm{E}$ and assume that $\dim \mathrm{F} = \alpha\mathrm{n}$. Then F is γ essentially Euclidean, where $\gamma(\epsilon) \le \lambda(\epsilon\alpha/4)M(\epsilon/4)$; in fact, for each $\epsilon > 0$ there is a subspace E_2 of E and operators $A : \ell_2^{(1-\epsilon)\alpha n} \to E_2$ and $B : QE_2 \to \ell_2^{(1-\epsilon)\alpha n}$ such that BQA is the identity on $\ell_2^{(1-\epsilon)\alpha n}$ and $\|A\| \cdot \|B\| \le \lambda(\epsilon\alpha/4)M(\epsilon/4)$.*

Proof: Set $n := \dim E$ and fix $0 < \epsilon < 1$. Let R be a quotient mapping from F onto a space G that has a subspace G_2 of dimension $(1-\epsilon/4)\alpha n$ such that $c_2(G_2) \le M(\epsilon/4)$. We want to find a subspace E_2 of E with $\dim E_2 \ge (1-\epsilon)\alpha n$ so that $RQE_2 \subset G_2$ and RQ is a "good" isomorphism on E_2, which implies that Q is also a "good" isomorphism on E_2. Since $\|R\| = \|Q\| = 1$, "good" means that $\|RQx\|$ is bounded away from zero for x in the unit sphere of E_2.

Since E is λ essentially Euclidean, there is a subspace E_0 of E with $\dim E_0 = (1 - \alpha\epsilon/4)n$ such that $c_2(E_0) \leq \lambda(\alpha\epsilon/4)$. Put Euclidean norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on E_0 and G_2, respectively, to satisfy for all $x \in E_0$ and all $y \in G_2$ the inequalities

$$(1) \quad \|x\| \leq \|x\|_1 \leq \lambda(\alpha\epsilon/4)\|x\| \quad \text{and} \quad \mathrm{M}(\epsilon/4)^{-1}\|\mathrm{y}\| \leq \|\mathrm{y}\|_2 \leq \|\mathrm{y}\|.$$

Define $E_1 := E_0 \cap (RQ)^{-1}G_2$, so that $\dim E_1 := m \geq (1 - \epsilon\alpha/2)n$.

Now take an orthonormal basis $e_1, e_2, \ldots, e_m$ for the Euclidean space $(E_1, \|\cdot\|_1)$ so that $RQe_1, RQe_2, \ldots, RQe_m$ is orthogonal in the Euclidean space $(G_2, \|\cdot\|_2)$ and ordered so that $\|RQe_1\|_2, \|RQe_2\|_2, \ldots, \|RQe_m\|_2$ is decreasing. We next compute that $\|RQe_j\|_2$ is large for $j := (1 - \epsilon)\alpha n$. Now $\|RQe_j\|_2$ is the norm of the restriction to $E_3 := [e_j, e_{j+1}, \ldots, e_m]$ of the operator RQ when it is considered as an operator from the Euclidean space $(E_1, \|\cdot\|_1)$ to the Euclidean space $(G_2, \|\cdot\|_2)$, and $\dim E_3 = m - j + 1 \geq (1-\alpha)n + \epsilon\alpha n/2 + 1$, which is strictly larger than the dimension of the kernel of RQ, because it has dimension at most $(1-\alpha)n + \alpha\epsilon n/4$. By the definition of quotient norms, the norm of $RQ_{|E_3}$ when RQ is considered as an operator from E_1 to G_2 under their original norms is the maximum over points x in the unit sphere of E_3 of the distance from x to the kernel of RQ. By a well-known consequence of the Borsuk-Ulam antipodal mapping theorem, (first observed in [**?**]; see also [**?**]), this distance is one. In view of the relationship (**??**), we deduce that $\|RQe_j\|_2 \geq \lambda(\alpha\epsilon/4)^{-1}M(\epsilon/4)^{-1}$. Also by (**??**), the norm of RQ is at most one when considered as an operator from $(E_1, \|\cdot\|_1)$ to $(G_2, \|\cdot\|_2)$. Finally, set $E_2 := [e_1, e_2, \ldots, e_j]$ and let U_1 be the restriction to E_2 of RQ, considered as a mapping onto RQE_2. We have just shown that the identity on ℓ_2^j factors through U_1 with factorization constant at most $\lambda(\alpha\epsilon/4)^{-1}M(\epsilon/4)^{-1}$, hence it factors with the same constant through the restriction of Q to E_2, considered as an operator from E_2 to QE_2. ∎

Theorem **??** gives an improvement of the qualitative version of Theorem **??** when $E = \ell_p^n$, $1 \leq p < 2$. For $S \subset \{1, \ldots, n\}$, let ℓ_p^S be the span in ℓ_p^n of the unit vector basis elements $\{e_i : i \in S\}$.

Theorem 2.2. *There is a function $g : (0,1)^2 \to (1, \infty)$ so that for all $1 \leq p < 2$, all natural numbers n, and all $\epsilon \in (0,1)$ the following is true. If Q is a quotient mapping from ℓ_p^n onto a normed space F and $\dim F = \alpha n$, then there is a subset S of $1, 2, \ldots, n$ of cardinality $(1 - \epsilon)\alpha n$ such that $\|(Q_{|\ell_p^S})^{-1}\| \leq g(\alpha, \epsilon)$.*

Sketch of proof: The main point is the observation made in [**?**, Theorem 2.1] that the proof of [**?**, Theorem 2.1] by Bourgain, Kalton, and Tzafriri shows that there is a constant $c > 0$ so that if Q is a quotient mapping from ℓ_p^n, $1 \le p < 2$, onto a space of dimension at least βn, then there is a subset S of $1, 2, \dots, n$ of cardinality at least $c^{1/\beta} n$ so that $\|(Q_{|\ell_p^S})^{-1}\| \le c^{-1/\beta}$. Given a quotient mapping Q on ℓ_p^n whose range has dimension αn and given $0 < \epsilon < 1$, apply the observation iteratively with $\beta := (1-\epsilon)\alpha$. At step one set $Q_1 := Q$ and get $S_1 \subset \{1, 2, \dots, n\}$ of cardinality at least $c^{1/\beta} n$ so that $\|((Q_1)_{|\ell_p^{S_1}})^{-1}\| \le c^{-1/\beta}$. At step two take the quotient mapping Q_2 on ℓ_p^n whose kernel is the span of the kernel of Q_1 and $\{e_i\}_{i \in S_1}$ and get $S_2 \subset \{1, 2, \dots, n\}$ of cardinality at least $c^{1/\beta} n$ so that $\|((Q_2)_{|\ell_p^{S_2}})^{-1}\| \le c^{-1/\beta}$. Necessarily S_1 and S_2 are disjoint. More importantly, from the definition of the norm in a quotient space we see that in $Q\ell_p^n$, the norm of the projection from $Q[e_i]_{i \in S_1 \cup S_2}$ onto $Q[e_i]_{i \in S_1}$ that annihilates $Q[e_i]_{i \in S_1}$ has norm controlled by $c^{-1/\beta}$, which implies that the norm of $(Q_{|\ell_p{}^{S_1 \cup S_2}})^{-1}$ is also controlled. Then let Q_3 be the quotient mapping on ℓ_p^n whose kernel is the span of the kernel of Q and $\{e_i\}_{i \in S_1 \cup S_2}$ and use the observation to get S_3. The iteration stops once the dimension of the kernel of Q_k is larger than $(1 - \beta n)$, which happens after fewer than $c^{-1/\beta}$ steps; say, after k steps. By construction you can estimate the basis constant of $(Q[e_i]_{i \in S_m})_{m=i}^{k-1}$, so that Q will be a good isomorphism on $[e_i : i \in \cup_{m=1}^{k-1} S_m]$ because it is a good isomorphism on each $[e_i : i \in S_m]$ for $1 \le m < k$. ■

Remark 2.3. It is interesting to have the best estimates for γ in Theorem **??** and for g in Theorem **??**. In Theorem **??** we gave the estimate for $\gamma(\epsilon)$ that the method gives and think that this might be the order of the best estimate. We did not do the same in Theorem **??** because we think that a different argument is probably needed to obtain the best estimate for $g(\alpha, \epsilon)$.

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