SOME APPROXIMATION PROPERTIES OF BANACH SPACES AND BANACH LATTICES

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ABSTRACT. The notion of the bounded approximation property $= BAP$ (resp. the uniform approximation property $= UAP$) of a pair [Banach space, its subspace] is used to prove that if X is a $\mathcal{L}_\infty-$space, Y a subspace with the BAP (resp. UAP), then the quotient X/Y has the BAP (resp. UAP). If $Q : X \to Z$ is a surjection, X is a $\mathcal{L}_1-$space and Z is a $\mathcal{L}_p-$space ($1 \le p \le \infty$) then $\ker Q$ has the UAP. A complemented subspace of a weakly sequentially complete Banach lattice has the separable complementation property = SCP. A criterion for a space with GL-l.u.st. to have the SCP is given. Spaces which are quotients of weakly sequentially complete lattices and are uncomplemented in their second duals are studied. Examples are given of spaces with the SCP which have subspaces that fail the SCP. The results are applied to spaces of measures on a compact Abelian group orthogonal to a fixed Sidon set and to Sobolev spaces of functions of bounded variation on $\mathbb{R}^n$.

To the memory of Lior Tzafriri

INTRODUCTION AND NOTATION

Typical problems studied in this paper are the following: Suppose that $Q : X \to Z$ is a surjection acting between Banach spaces (resp. $J : Y \to X$ is an isomorphic embedding). Assume that X is a nice space, for instance a Banach lattice. Having some information on Z (resp. on Y) we would like to obtain some information on $\ker Q$ (resp. on the quotient X/Y).

In Section 1 we introduce the notion of the bounded approximation property (resp. the uniform approximation property) for pairs consisting of a Banach space and a fixed subspace. As an easy application we

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obtain the existence of some new phenomena concerning approximation properties of some subspaces of c_0, l_1 and S_1 (cf. Corollaries 1.13 and 1.14).

In Section 2, influenced by a construction of Lusky [Lu], we apply that notion to show that a quotient of a $\mathcal{L}_\infty-$space has the bounded approximation property = BAP (resp. the uniform approximation property = UAP) provided that the kernel of the quotient map has the same property. By duality we obtain a corresponding criterion for subspaces of $\mathcal{L}_1-$spaces. As a corollary we get that if Q is a surjection from a $\mathcal{L}_1-$space onto a $\mathcal{L}_p-$space ($1 \leq p \leq \infty$) then $\ker Q$ has the UAP. On the other hand it follows from a result of [J2] that $\ker Q$ fails GL-l.u.st.

In Section 3 we discuss an invariant of non-separable Banach spaces - the separable complementation property = SCP. A Banach space X has the SCP provided that every separable subspace of X is contained in a separable subspace which is complemented in X. The property goes back to [AL], who proved that weakly compactly generated = WCG spaces have the SCP. It was observed in [GS] that a weakly sequentially complete Banach lattice has the SCP. We prove that a weakly sequentially complete complemented subspace of a Banach lattice has the SCP. As a corollary we get that a dual Banach space of finite cotype with GL-l.u.st. has the SCP.

In Section 4 we isolate an isomorphic invariant of Banach spaces called property (k). It appears implicitly in [J2] and it is a modification of an invariant invented by S. Kwapień (cf. [KP, pp.142-144]). There is some analogy between property (k) and Grothendieck's characterization [Gr1] of the Dunford - Pettis property. Combining property (k) with the lattice variant of the factorization theorem for weakly compact operators we give a criterion for kernels of quotient maps of Banach lattices to be uncomplemented in their second duals. At the end of the section we give an example which shows that the SCP need not be preserved when passing to subspaces. This answers a question posed in [PlY]. More examples of that sort are presented in Section 5.

In Section 5 we examine two important examples of spaces failing the SCP: the space of all Borel measures on a compact Abelian group orthogonal to an infinite Sidon set, and the Sobolev space of functions of bounded variation on an open subset of $\mathbb{R}^n$.

In Section 6 we mention open problems that are suggested by this research.

Terminology and notation. We employ the standard terminology and notation used in Banach space theory [LT, JL]. B_X denotes the open unit ball and S_X the unit sphere of a Banach space X. By $\mathrm{cl}\, A$ we denote the norm closure of a subset A of a normed space X. A (short) exact sequence of Banach spaces is a diagram

$$0 \to Y \xrightarrow{J} X \xrightarrow{Q} Z \to 0,$$

where X, Y, Z are Banach spaces, $J : Y \to X$ is an isomorphic embedding, $Q : X \to Z$ is a linear surjection, and $J(Y) = \ker Q$. By $L_p[0,1]$, $1 \le p \le \infty$, we denote the usual L_p-space on the interval $[0,1]$ with respect to the normalized Lebesgue measure.

1. The bounded approximation property of pairs

Definition 1.1. Let X be a Banach space, let $Y \subseteq X$ be a closed linear subspace, let $\lambda \ge 1$. The pair (X, Y) is said to have the $\lambda - BAP$ if for each $\lambda' > \lambda$ and each subspace $F \subseteq X$ with $\dim F < \infty$, there is a finite rank operator $u : X \to X$ such that $||u|| < \lambda'$, $u(x) = x$ for $x \in F$ and $u(Y) \subseteq Y$. If moreover for each $\lambda' > \lambda$ there is a function $n \to \phi_{\lambda'}(n)$ for $n \in \mathbb{N}$ such that in addition u can be chosen so that $\dim u(X) < \phi_{\lambda'}(\dim F)$ then the pair (X, Y) is said to have the $\lambda - UAP$. If $Y = X$ we say that X has the $\lambda - BAP$ (resp. $\lambda - UAP$).

X has the BAP (resp. UAP) iff X has the $\lambda{-}BAP$ (resp. $\lambda{-}UAP$) for some $\lambda \ge 1$.

We begin with some simple consequences of Definition 1.1.

Corollary 1.2. *If a pair (X,Y) has the $\lambda{-}BAP$ (resp. $\lambda{-}UAP$) then X, Y and the quotient space X/Y have the $\lambda{-}BAP$ (resp. $\lambda{-}UAP$).*

Proof. Let $q : X \to X/Y$ be the quotient map. Pick a finite-dimensional subspace $\widetilde{F} \subset X/Y$. Then there is a subspace $F \subset X$ such that $\dim F = \dim \widetilde{F}$ and $q(F) = \widetilde{F}$. Since (X, Y) has the $\lambda{-}BAP$, for every $\lambda' > \lambda$ there is a finite rank operator $u : X \to X$ satisfying the requirement of Definition 1.1. In particular $u(Y) \subseteq Y$ which implies the existence of a unique $\widetilde{u} : X/Y \to X/Y$ such that $qu = \widetilde{u}q$. Clearly $\widetilde{u}$ has finite rank, $\|\widetilde{u}\| \le \|u\| < \lambda'$ and $\widetilde{u}(\widetilde{x}) = \widetilde{x}$ for $\widetilde{x} \in \widetilde{F}$. The argument for the $\lambda{-}UAP$ is similar. □

Corollary 1.3. *Let X be a Banach space, $q : X \to Z$ a quotient map, and let $\dim \ker q < \infty$. Then the assumption that X has the $\lambda{-}BAP$ (resp. $\lambda{-}UAP$) implies that Z has the same property.*

Proof. If a subspace $F \subseteq X$ is finite dimensional then the subspace $E := \ker q + F$ is also finite dimensional. Thus the assumption on X implies that for each $\lambda' > \lambda$ there is a finite rank operator $u : X \to X$ with $\|u\| < \lambda'$ which is the identity on E. In particular $u(\ker q) = \ker q$.

Thus the pair $(X, \ker q)$ has the $\lambda{-}BAP$. Now we apply Corollary 1.2. The argument for the $\lambda{-}UAP$ is similar. $\square$

Corollary 1.4. *Let X be a Banach space, Y a subspace of finite codimension.*

(i) *If the dual space X^* has the $\lambda{-}BAP$ then Y has the $\lambda{-}BAP$.*
(ii) *If X has the $\lambda{-}UAP$ then Y has the $\lambda{-}UAP$.*

Proof. (i). Let $Y^\perp = \{x^* \in X^* : x^*(y) = 0 \quad \text{for } y \in Y\}$. Clearly, $\dim Y^\perp = \dim X/Y < \infty$. The dual space Y^* is naturally isometric to $X^*/Y^\perp$. Thus, by Corollary 1.3 the space Y^* has the $\lambda{-}BAP$. This yields that Y has the $\lambda{-}BAP$ (see, e.g., [C, Proposition 3.5 and Theorem 3.3]).

(ii). X has the $\lambda{-}UAP$ if and only if X^* has the $\lambda{-}UAP$ [H, M]. Thus the proof of (ii) reduces to the previous case. $\square$

Lemma 1.5. *Let Y be a closed linear subspace of the Banach space X and let $\lambda < \infty$. Suppose that for every finite dimensional subspace E of X and every $\varepsilon > 0$ there is a finite rank operator $T : X \to X$ such that $T(Y) \subseteq Y$, $\|T\| \le \lambda$ and $\|Tx - x\| \le \varepsilon\|x\|$ for $x \in E$. Then the pair (X, Y) has the $\lambda{-}BAP$.*

Proof. Fix numbers $\varepsilon > 0$, $\lambda' > \lambda$ and a subspace $E \subset X$ with $\dim E < \infty$. Pick a bounded linear projection $P : X \to X$ such that $P(X) = E$ and $P(Y) \subseteq Y$. Let $\alpha > 0$ be a small number to be specified later. Choose a finite rank operator $T : X \to X$ such that $T(Y) \subseteq Y$, $\|T\| \le \lambda$ and $\|Tx - x\| \le \alpha\|x\|$ for $x \in E$. Put $S = T + P - TP = T + (I - T)P$. Clearly, $S(Y) \subseteq Y$, $Sx = x$ for $x \in E$ and $\operatorname{rank} S \le \operatorname{rank} T + \operatorname{rank} P < \infty$. Furthermore, one has $\|S\| \le \|T\| + \alpha\|P\|$. Thus we will get the estimate $\|S\| < \lambda'$ if α has been chosen so that $\alpha\|P\| < \lambda' - \lambda$. $\square$

Proposition 1.6. *Let X be a Banach space and $\lambda < \infty$. Then X^* has the $\lambda{-}BAP$ if and only if the pair (X, Y) has the $\lambda{-}BAP$ for each finite codimensional subspace $Y \subseteq X$.*

Proof. Our proof of the "only if" part uses Lemma 1.5 and the full force of Proposition 3.5 in [C], which yields that if X^* has the $\lambda{-}BAP$ then X has the so called $\lambda{-}$duality bounded approximation property. The latter property means that for each $\varepsilon > 0$ and each pair of finite dimensional subspaces E of X and F of X^*, there is a finite rank operator T on X with $\|T\| \le \lambda$, $\|Tx - x\| \le \varepsilon\|x\|$, for all $x \in E$ and $\|T^*x^* - x^*\| \le \varepsilon\|x^*\|$, for all $x^* \in F$.

Let $\lambda' > \lambda$, $\varepsilon > 0$ and $\alpha > 0$ be fixed (the number α will be specified later on). We also fix a subspace $Y \subseteq X$ of finite codimension and let

$F = Y^{\perp}$. Since F is a weak* closed subspace of X, there is a weak* continuous projection $Q = P^*$ from X^* onto F.

Since X has the $\lambda-$duality bounded approximation property, we can choose a finite rank operator T on X with $\|T\| \leq \lambda$, $\|Tx - x\| \leq \alpha\|x\|$, for all $x \in E$ and $\|T^*x^* - x^*\| \leq \alpha\|x^*\|$, for all $x^* \in F$. Put $U = T + P - PT = T + P(I - T)$. Observe that $U : X \to X$, $\operatorname{rank} U \leq \operatorname{rank} T + \operatorname{rank} P < \infty$, and $U^* = T^* + (I^* - T^*)Q$. Thus for every $x^* \in F$ we have $Qx^* = x^*$ and hence $U^*x^* = T^*x^* + (I^* - T^*)Qx^* = x^*$. It follows that for $y \in Y$ we have $Uy \in Y$, because for every $x^* \in F$ one has $x^*(Uy) = (U^*x^*)y = x^*y = 0$.

Clearly, for each $x^* \in X^*$ one has $\|(I^* - T^*)P^*x^*\| \leq \alpha\|Q\|\|x^*\|$, which implies that $\|U - T\| = \|P(I - T)\| = \|(I^* - T^*)P^*\| \leq \alpha\|Q\|$. This yields $\|U\| \leq \lambda'$, if α has been chosen so that $\alpha\|Q\| \leq \lambda' - \lambda$.

Using the identity $x - Ux = x - Tx - P(x - Tx) = (I - P)(x - Tx)$ which holds true for each $x \in X$, we obtain $\|x - Ux\| \leq \alpha(1 + \|Q\|)\|x\|$ for every $x \in E$. If α is chosen so that $\alpha(1 + \|Q\|) < \varepsilon$, then we obtain the estimate $\|x - Ux\| \leq \varepsilon\|x\|$ for every $x \in E$.

Now, thanks to Lemma 1.5 we can conclude that the pair (X, Y) has the $\lambda'-BAP$ for every $\lambda' > \lambda$. By Definition 1.1 this yields that the pair (X, Y) has the $\lambda-BAP$.

To prove the "if" part, let $F \subseteq X^*$, $\dim F = n < \infty$. Choose a finite dimensional $E \subset X$ such that $F \cap E^{\perp} = \{0\}$. Fix $\varepsilon > 0$. Put $Y = F_{\perp} := \{y \in X : fy = 0 \quad \text{for } f \in F\}$. Since (X, Y) has the $\lambda-BAP$, there is an operator $u : X \to X$ such that $u(Y) \subseteq Y$, $ue = e$ for $e \in E$, $\operatorname{rank} u < \infty$ and $\|u\| < \lambda + \varepsilon$.

We claim that the operator $u^* : X^* \to X^*$ satisfies the condition $u^*f = f$ for $f \in F$. For, observe that if $f \in F$, then $u^*f \in Y^{\perp}$, because for each $y \in Y$ we have $(u^*f)y = f(uy) = 0$. Since $\dim F < \infty$, F is weak* closed in X^* and hence $Y^{\perp} = (F_{\perp})^{\perp} = F$ by the bipolar theorem. It follows that $g := u^*f - f \in F$. Observe that $g \in E^{\perp}$ because for $e \in E$ we have $ge = (u^*f)e - fe = f(ue) - fe = 0$. It follows that $g \in F \cap E^{\perp} = \{0\}$ and hence $u^*f - f = g = 0$. Since $\|u^*\| = \|u\| < \lambda + \varepsilon$, this completes the proof that X^* has the $\lambda-BAP$. □

Remark 1.7. If X is a Banach space with the $\lambda-UAP$ and $Y \subseteq X$ is a closed subspace of finite codimension, then the pair (X, Y) has the $\lambda-UAP$. Indeed, since X^* has the $\lambda-UAP$ [H, M], the proof can just follow the lines of that of Proposition 1.6 (with appropriate estimates of the ranks of finite rank approximations used in the process).

Proposition 1.8. *Let X be a Banach space and let $Y \subseteq X$ be a closed subspace such that* $\dim X/Y = n < \infty$ *and Y has the $\lambda-BAP$ (resp.*

Y has the λ–UAP). Then the pair (X,Y) has the 3λ–BAP (resp. 3λ–UAP).

Proof. Let $\varepsilon > 0$. Choose a subspace E of X to $(1+\varepsilon)$-norm $Y^\perp$ so that $\dim E \le f(n,\varepsilon)$. Define $Q : E \oplus_1 Y \to X$ by

$$Q(e,y) = e + y.$$

Claim: Q is a $3(1+\varepsilon)$-quotient map.

To prove the claim observe first that the quotient map $R : X \to X/Y$ has the property that $R(B_X) \subseteq (1+\varepsilon)R(B_E)$. For, if that were false then there would exist an $x \in B_X$ such that $Rx \notin (1+\varepsilon)R(B_E)$. Let $\eta \in (X/Y)^* = Y^\perp$ be a separating linear functional with $\eta(Rx) = 1$. Then we would obtain that $1 > (1+\varepsilon)\sup\{|\eta(Re)| : e \in B_E\} \ge 1$, because E $(1+\varepsilon)$-norms $Y^\perp$. This contradiction proves the inclusion.

Now, given any $x \in X$, we can find an $e \in E$ such that $Re = Rx$ and $\|e\| \le (1+\varepsilon)\|x\|$. Observe that $x - e \in Y$, because $R(x-e) = Rx - Re = 0$. Letting $y = x - e$, we have $y \in Y$ and $\|y\| \le \|x\| + \|e\| \le (2+\varepsilon)\|x\|$. Since $Q(e,y) = e + (x-e) = x$ and $\|e\| + \|y\| \le (3+2\varepsilon)\|x\|$, we have proved the claim.

Now suppose $G \subseteq X$ is finite dimensional and $G \supseteq E$, where E is from the claim. We may and do assume that $G \cap Y \ne \{0\}$. Take a finite rank $T : Y \to Y$ with $\|T\| \le \lambda + \varepsilon$ and $Ty = y$ for $y \in G \cap Y$. Define S on $E \oplus_1 Y \to E \oplus_1 Y$ by $S(e,y) = (e,Ty)$. Then $\|S\| = \|T\|$, S is a projection if T is, and $S(e,y) = (e,y)$ if $e \in E$ and $y \in G \cap Y$. Since $G \supseteq E$, S is the identity on the kernel of Q and hence S induces an operator $\widetilde{S} : X \to X$ defined by $\widetilde{S}x = e + Ty$ whenever $Q(e,y) = x$. Then $\|\widetilde{S}\| \le 3(1+\varepsilon)(\lambda+\varepsilon)$ and $\widetilde{S}$ is the identity on G. □

Remark 1.9. Clearly $\widetilde{S}$ is a projection if T is and is weak* continuous if X is a dual space, Y is weak* closed, and T is weak* continuous. If $T_1 : Y \to Y$ is a finite rank operator commuting with T then $\widetilde{S}_1$ commutes with $\widetilde{S}$ where $\widetilde{S}_1 : X \to X$ is constructed for T_1 in the same way as $\widetilde{S}$ for T.

Remark 1.10. We do not know whether in the statement of Proposition 1.8 one can replace 3λ by a smaller quantity. However, the number 3 which appears in the claim made in the proof of that proposition cannot be replaced by a smaller one, even in the simple case where $Y = c_0 \subset c = X$ and $E \subset X$ is any subspace of finite dimension. We omit the easy verification of the latter fact.

Corollary 1.11. *If X is a Banach space and Y has the λ–BAP for every finite codimensional subspace $Y \subseteq X$, then X^* has the 3λ–BAP.*

Proof. This follows from Proposition 1.6, because every pair (X, Y) has the $3\lambda-BAP$ by Proposition 1.8. $\square$

Remark 1.12. If X is a Banach space with the BAP such that X^* fails the BAP, then by Corollary 1.11 for every $\lambda < \infty$ there is a finite codimensional subspace $Y_\lambda \subset X$ which fails the $\lambda-BAP$. This observation shows that one cannot replace X^* by X in the statement of Corollary 1.4 (i). Also it leads to a somewhat simpler proof of the main result of [FJ]. In fact, by applying Corollary 1.4 (i) to some specific Banach spaces X we construct here some new examples with rather surprising properties.

Corollary 1.13. *Each of the spaces c_0 and l_1 has a subspace Y which has the approximation property but Y fails the BAP.*

Proof. In the case of c_0 we use an old result of G. Schechtman and the second author of the present paper (see [JO], Corollary JS), who proved that there is a subspace X of the space c_0 which has the BAP while X^* fails the BAP. By Corollary 1.11, for each positive integer n we choose a finite codimensional subspace of X which fails the $n-BAP$. Obviously, all the X_n's have the approximation property. Hence if Y is the c_0-sum of the sequence (X_n) then Y is isometric to a subspace of c_0 and has the approximation property but Y fails the BAP.

Similarly, in the case of l_1 we start with a subspace X of l_1 which fails the approximation property found in [Sz1]. As shown in [J1], from the existence of such a space it follows that if we let Z be the l_1-sum of a dense sequence (X_n) of finite dimensional subspaces of X, then Z^* fails the BAP and yet Z has the approximation property. We conclude as in the previous paragraph (now Y will be the l_1-sum of a suitable sequence of finite codimensional subspaces of Z). $\square$

By the result of A. Szankowski [Sz2] the space $B(H)$ fails the BAP. Since $B(H)$ is the dual space of the trace class S_1, using again Corollary 1.11 we obtain

Corollary 1.14. *Every finite codimensional subspace of the trace class S_1 contains for each $\lambda < \infty$ a finite codimensional subspace Y_λ such that Y_λ fails the $\lambda-BAP$.*

Of course, S_1 has the $1-BAP$ and it is the simplest non commutative L_1-space. On the other hand, Corollary 1.4 implies that every finite codimensional subspace of a commutative L_1-space has the $1-BAP$.

Remark 1.15. One may ask whether the analogue of Proposition 1.8 for quotient spaces is true. Namely, let Y be a Banach space with the

$\lambda-BAP$. Consider those Banach spaces X for which there is a quotient map from X onto Y whose kernel is finite dimensional. When is it true that there exists $\mu < \infty$ such that X has the $\mu-BAP$ for all those X? The answer is that if Y^* fails the BAP then no such μ exists, while if Y^* has the $\mu-BAP$ then X has the $3\mu-BAP$ for all those X. The $3\mu-$estimate follows from Proposition 1.8, while the nonexistence can be demonstrated using the construction applied in the proof of that proposition.

2. Bounded and uniform approximation properties of quotients of $\mathcal{L}_\infty-$spaces (resp. subspaces of $\mathcal{L}_1-$spaces).

The main result of this section is

Theorem 2.1. *Let*

$$0 \to Y \xrightarrow{J} X \xrightarrow{Q} Z \to 0$$

be an exact sequence of Banach spaces. Then

(a) *If X is a $\mathcal{L}_\infty-$space and Y has the BAP (resp. Y has the UAP) then Z has the BAP (resp. Z has the UAP).*
(b) *If X is an $\mathcal{L}_1-$space and the dual space Z^* has the BAP (resp. Z has the UAP) then Y^* and Y have the BAP (resp. Y has the UAP).*
(c) *Let $1 \le p \le \infty$. If X is a $\mathcal{L}_\infty-$space (resp. X is a $\mathcal{L}_1-$space) and Y is a $\mathcal{L}_p-$space (resp. Z is a $\mathcal{L}_p-$space) then Z has the UAP (resp. Y has the UAP).*

Proof. (a) By equipping X with an equivalent norm [KP, Lemma 1.1] we can assume that J is an isometric embedding and Q is a quotient map. Suppose Y has the $\lambda-BAP$ (resp. $\lambda-UAP$). By Corollary 1.2, it suffices to establish

$$(\star) \qquad (X,Y) \text{ has the } \mu-BAP \text{ (resp. } \mu-UAP);$$

($\mu \ge 1$ depends only on λ and X).

To verify $(\star)$ fix a finite-dimensional subspace $F \subseteq X$. Put $Y_1 = Y + F$. Clearly Y_1 is a closed subspace of X and $\dim Y_1/Y \le \dim F < \infty$. By Proposition 1.8, for every $\lambda' > \lambda$ there exists a finite rank operator $u : Y_1 \to Y_1$ such that

$$\|u\| < 3\lambda',\ u(x) = x \quad \text{for } x \in F, u(Y) \subseteq Y.$$

Moreover, in the case of UAP, $\dim u(Y_1) < \phi_{\lambda'}(\dim F)$.

Since X is a $\mathcal{L}_\infty-$space, there are $c > 1$ and a function $f : \mathbb{N} \to \mathbb{N}$ depending only on X such that for each finite-dimensional subspace F_1

of X there exists a subspace $E \subseteq X$ such that

$$E \supseteq F_1, \tag{2.1}$$

$$\dim E < f(\dim F_1), \tag{2.2}$$

$$\text{the Banach-Mazur distance } d(E, \ell_\infty^{\dim E}) \leq c. \tag{2.3}$$

Now specify $F_1 := u(Y_1)$. By (2.1), we can regard u as an operator with values in E. Hence, by (2.3), u admits an extension $\widetilde{u} : X \to E$ with $\|\widetilde{u}\| \leq c\|u\| < 3c\lambda'$ because $\ell_\infty^{\dim E}$ is a 1-injective space. Clearly $\widetilde{u}(Y) = u(Y) \subseteq Y$ because $\widetilde{u}$ extends u. Clearly $\widetilde{u}$ is of finite rank because $\widetilde{u}(X) \subseteq E$. By (2.2) it follows that, in the case of the UAP, $\dim E$ is controlled by $\dim F$ because in that case $\dim E \leq \dim F + \dim u(Y_1) < \dim F + \phi_{\lambda'}(\dim F)$.

(b) If X is a $\mathcal{L}_1-$space then X^* is a $\mathcal{L}_\infty-$space [JL, p.59]. The assumption that Z^* has BAP allows us to apply the already established case (a) to the dual diagram

$$0 \to Z^* \xrightarrow{Q^*} X^* \xrightarrow{J^*} Y^* \to 0$$

to conclude that Y^* has the BAP. Then by [C, Proposition 3.5] we infer that Y has the BAP.

In the case of the UAP it is enough to assume that Z has the UAP. We use the result of Heinrich [H, M] that a Banach space has the UAP if and only if its dual has the UAP.

(c) By [PR], every $L_p(\mu)$ space has the UAP. By [LR], if $1 < p < \infty$ then every $\mathcal{L}_p-$space is complemented in some $L_p(\mu)$ and if either $p = 1$ or $p = \infty$ then some dual of every $\mathcal{L}_p-$space is complemented in some $L_p(\mu)$. By [H] if the dual of a space has the UAP then the space itself has the UAP. Thus every $\mathcal{L}_p-$space has the UAP for $1 \leq p \leq \infty$. The desired conclusion follows now from assertions (a) and (b). $\square$

Remark 2.2. Recently Szankowski [Sz3] proved that there exists a subspace X of $L_\infty[0,1]$ (resp. $C[0,1]$) which fails the approximation property while $L_\infty[0,1]/X$ has the $1-BAP$ (resp. $C[0,1]/X$ has a basis). The result extends to $L_p[0,1]$ for $p > 2$, while for $1 \leq p < 2$ the dual result holds

Lusky [Lu, Corollary 3 (b)] proved

(L) *If* $0 \to X \xrightarrow{J} Y \xrightarrow{Q} Z \to 0$ *is an exact sequence of Banach spaces such that* Y *is a separable* L_1 *space and* Z *is not reflexive and has a basis then* X *has a basis.*

Our next result is a consequence of (L). It gives additional information for non-separable spaces. For the terminology and properties of abstract $L-$ and $M-$spaces we refer to [K1, K2] and [LT, Vol II].

It holds for abstract $L-$ and $M-$spaces over both real and complex scalars.

Proposition 2.3. *Let* $0 \to X \xrightarrow{J} Y \xrightarrow{Q} Z \to 0$ *be an exact sequence of Banach spaces. Assume that* Y *is isomorphic to an abstract* $L-$*space and* Z *is isomorphic to an abstract* $M-$*space with unit. Then every separable subspace of* X *is contained in a separable subspace of* X *with a basis.*

Proof. We use the following well known facts.

(j) Every separable subspace of an abstract $L-$space is contained in a separable subspace which is an abstract $L-$space.

(jj) Every separable subspace of an abstract $M-$space with a unit is contained in a separable subspace which is an abstract $M-$space with the unit.

(jjj) There is an $a < \infty$ such that if F is a separable subspace of Z then there is a separable subspace E of Y such that $\operatorname{cl} aQ(B_E) \supset B_F$.

((jjj) is a simple consequence of the proof of the Open Mapping Principle.)

Let E be an infinite-dimensional separable subspace of X. Choose $y_0 \in Y$ so that Qy_0 corresponds via an isomorphism to the unit of an abstract $M-$space isomorphic to Z. Using (j) - (jjj) we define inductively increasing sequences $(E_n)_{n=1}^\infty$ and $(Y_n)_{n=1}^\infty$ of subspaces of Y, and $(F_n)_{n=1}^\infty$ and $(Z_n)_{n=1}^\infty$ of separable subspaces of Z such that

$$E_1 \supset J(E) \cup \{y_0\}; \tag{2.4}$$

$$Y_n \text{ is isomorphic to a separable abstract } L-\text{space}; \tag{2.5}$$

$$Z_n \text{ is isomorphic to a separable abstract } M-\text{space}; \tag{2.6}$$

$$E_n \subset Y_n; F_n \subset Z_n; \operatorname{cl} Q(Y_n) = F_n; \tag{2.7}$$

$$\operatorname{cl} aQ(B_{E_{n+1}}) \supset B_{Z_n}. \tag{2.8}$$

The isomorphisms in (2.5) (resp. in (2.6)) are restrictions of the same isomorphism of Y onto an abstract $L-$space (resp. of Z onto an abstract $M-$space).

Put

$$Y_\infty := \operatorname{cl} \bigcup_{n=1}^{\infty} Y_n;\ Z_\infty := \operatorname{cl} \bigcup_{n=1}^{\infty} Z_n;\ X_0 := J^{-1}(J(X) \cap Y_\infty).$$

By (2.5) and the comment after (2.4)-(2.8), Y_∞ is isomorphic to an abstract $L-$space. By (2.6) and the comment, Z_∞ is isomorphic to an abstract $M-$space which has the unit because, by (2.4), $Q(y_0) \in Z_\infty$.

Since the Y_n's and the Z_n's are separable so are Y_∞, Z_∞ and X_0. It follows from (2.7) and (2.8) that

$$aQB_{Y_\infty} \supset \bigcup_{n=1}^{\infty} aQB_{Y_n} \supset \bigcup_{n=1}^{\infty} aQB_{E_n} \supset \bigcup_{n=2}^{\infty} B_{Z_{n-1}} = \bigcup_{n=1}^{\infty} B_{Z_n}.$$

Since the sequence $(B_{Z_n})_{n=1}^{\infty}$ is increasing, $\operatorname{cl}\bigcup_{n=1}^{\infty} B_{Z_n} = \operatorname{cl} B_{Z_\infty}$. Hence aQB_{Y_∞} is norm dense in B_{Z_∞}. Thus the restriction $Q|Y_\infty$ maps Y_∞ onto Z_∞. Therefore $\ker Q|Y_\infty = \ker Q \cap Y_\infty = J(X) \cap Y_\infty = J(X_0)$.

Now we consider the exact sequence

$$0 \to X_0 \overset{J|X_0}{\to} Y_\infty \overset{Q|Y_\infty}{\to} Z_\infty \to 0.$$

Y_∞ is isomorphic to a separable abstract $L-$space and thus by [K1] is isomorphic to a separable $L_1(\mu)$. The space Z_∞ is isomorphic to a $C(K)$ space on a compact metric K because Z_∞ is isomorphic to a separable abstract $M-$space with unit [K2]. Obviously Z_∞, being infinite-dimensional, is non-reflexive. By a result of Vakher [V, MP], for every compact metric space K the space $C(K)$ has a basis. Thus, by [Lu], X_0 has a basis. Obviously X_0 contains E. $\square$

3. Banach spaces with the separable complementation property and complemented subspaces of Banach lattices.

Definition 3.1. (cf. [C]) A Banach space X has the SCP (= *separable complementation property*) if every separable subspace Y of X is contained in a separable complemented subspace Z of X. If always Z can be chosen to be the range of a projection of norm at most λ, we say that X has the λ-SCP.

A routine argument shows that a space with the SCP must have the λ-SCP for some $\lambda < \infty$.

Recall that the following classes of Banach spaces have the SCP

(a) Abstract $L-$spaces [K1].
(b) Preduals of von Neumann algebras (Haagerup [GGMS, Appendix]).
(c) Weakly compactly determined spaces, in particular weakly compactly generated spaces, in particular reflexive spaces (cf. [DGZ, Chapt.VI]).
(d) Weakly sequentially complete Banach lattices [GS].
(e) Banach spaces with the commuting bounded approximation property (Casazza, Kalton, Wojtaszczyk [C, Theorem 9.3]).

It is not known whether a complemented subspace of a space with the SCP must have the SCP. However, we have

Proposition 3.2. *A weakly sequentially complete Banach space which is isomorphic to a complemented subspace of a Banach lattice has the SCP.*

Proof. By [LT, Vol.II,1.c.6] it is enough to show that if X is a weakly sequentially complete Banach lattice and Y is the range of a projection $P : X \to X$ then Y has the SCP. We need the following observations.

(i) A separable subspace of a Banach lattice is contained in a separable sublattice;

(ii) - a precise form of (d). If a Banach lattice X is weakly sequentially complete then every separable subspace of X is contained in the range of a contractive projection of X whose range is a separable sublattice.

(iii) A weakly sequentially complete Banach lattice is complemented in its second dual [LT, Vol.II,1.c.4].

Let E be a fixed separable subspace of Y. Using (i) and (ii) we construct inductively increasing sequences of separable subspaces of X, (E_n), (S_n), (G_n) and a sequence $p_n : X \to X$ of contractive projections with separable ranges such that

$$E_1 = E,\ E_n \subseteq Y \quad \text{for } n = 1, 2. \dots; \tag{3.1}$$

$$S_n \quad \text{is a sublattice of } X \text{ and } E_n \subseteq S_n \text{ for } n = 1, 2, \dots; \tag{3.2}$$

$$S_n \subseteq G_n := p_n(X) \text{ for } n = 1, 2, \dots; \tag{3.3}$$

$$E_{n+1} := \mathrm{cl}(P(G_n)). \tag{3.4}$$

Put $F := \mathrm{cl}(\bigcup_{n=1}^{\infty} E_n)$ and $S := \mathrm{cl}(\bigcup_{n=1}^{\infty} S_n)$. Clearly, F and S are separable and $E \subseteq F \subseteq Y$. Observe that $P(S) \subseteq F$, because for each n one has $P(S_n) \subseteq P(G_n) \subseteq E_{n+1} \subseteq F$.

Since S a separable weakly sequentially complete sublattice, the canonical image of S in S^{**} is the range of a contractive band projection $p_S : S^{**} \to S$ (by (iii)). We shall define a projection π from X onto F as the composition $\pi = Pp_Sp$, where p is defined as follows.

Let LIM be a fixed Banach limit, i.e., a positive linear functional on l_∞ such that $\mathrm{LIM}(a) = \lim_n a_n$ for each convergent scalar sequence $a = (a_n)$ (cf. [B, II.3.3], [DS, II.4.22]). We define a linear operator $p : X \to S^{**}$ by the formula $(px)(s^*) = \mathrm{LIM}_n\, s^*(p_n(x))$ for every $x \in X$ and $s^* \in S^*$.

Note that p_Sp is a projection from X onto S, because for every fixed n and $z \in S_n$ one has $z \in G_m$ for $m \geq n$. Hence $p_m(z) = z$ for $m \geq n$.

Thus $p_S p(z) = z$. Since $\bigcup_{n=1}^{\infty} S_n$ is dense in S and $||p|| = \sup_n ||p_n|| = 1$, we infer that $p_S p(z) = z$ for $z \in S$.

Now, if $f \in F$, then $\pi(f) = P(p_S p(f)) = P(f) = f$, because $f \in Y$. Also, if $x \in X$, then $\pi(x) = P(p_S p(x)) = P(s)$, where $s = p_S p(x) \in S$. Since $P(S) \subseteq F$, it follows that π projects X down onto F. Thus the restriction of π to Y is the desired projection in Y with separable range containing E. □

For the sake of completeness we include

Proof of (ii). Let E be a separable subspace of a weakly sequentially complete Banach lattice X. Hence X is order continuous. Then, by a result of Kakutani (cf. [LT, Vol.II,1.a.9]) there is a sublattice X_0 of X with a weak unit such that $X_0 \supset E$ and there is a contractive projection, say P_0, from X onto X_0. Since X_0 has a weak unit and is order continuous, a result of Nakano (cf.[LT, Vol.II,1.b.16]) and the remark after the proof of [LT, Vol.II,1.b.16]) gives that X_0 is weakly compactly generated. Thus, by a result of Amir and Lindenstrauss [AL], (cf. [DGZ, Chapt. VI, Theorem 2.5]), there is a contractive projection $P_1 : X_0 \to X_0$ whose range is separable and contains E. The desired projection is the composition $P_1 \circ P_0$. □

Remark 3.3. A Banach space X has the SEP=*separable extension property* provided that for every separable subspace $E \subset X$ there exists an operator $u : X \to X$ with separable range such that $u(x) = x$ for $x \in E$. Clearly the SCP implies the SEP. A slight modification of the proof of Proposition 3.2 shows that for weakly sequentially complete Banach lattices the SEP implies the SCP.

Recall [JL, section 8] that a Banach space E has *finite cotype*, equivalently E *does not contain* ℓ_∞^n *uniformly*, provided that

$$\sup_n \inf_{F \subset E, \dim F = n} d(F, \ell_\infty^n) = \infty.$$

Here $d(\cdot, \cdot)$ denotes the Banach-Mazur multiplicative distance. Recall [JL, section 9] that a Banach space Y has GL-l.u.st. provided that Y^{**} is isomorphic to a complemented subspace of a Banach lattice.

Corollary 3.4. *If a Banach space Y of finite cotype has GL-l.u.st. then Y^{**} has the SCP. Moreover if Y is isomorphic to a dual Banach space then Y has the SCP.*

Proof. By the local reflexivity principle Y^{**} has finite cotype. Thus Y^{**} does not contain subspaces isomorphic to c_0 and hence is weakly sequentially complete [LT, Vol.II,1.c.4]. Remembering that Y has GL-l.u.st., we infer that the weakly sequentially complete space Y^{**} is isomorphic to a complemented subspace of a Banach lattice. Hence, by Proposition 3.2, Y^{**} has the SCP. Moreover if Y is a dual Banach space then Y is complemented in Y^{**}; again we use Proposition 3.2. $\square$

4. Property (k)

Recall that (y_m) is a CCC sequence of (x_n) of elements of a linear space provided that there are a sequence (c_n) of non-negative numbers and a strictly increasing sequence of positive integers (n_m) such that

$$y_m = \sum_{n=n_m}^{n_{m+1}-1} c_n x_n, \quad \sum_{n=n_m}^{n_{m+1}-1} c_n = 1 \quad (m = 1, 2, \dots). \tag{4.1}$$

CCC - stands for "consecutive convex combinations".

Definition 4.1. A Banach space X has property (k) provided that for every sequence $(x_n^*) \subset X^*$ which weak* converges to 0 there exists a CCC sequence (y_m^*) of (x_n^*) such that for each linear operator $u : L_1[0,1] \to X$,

$$\lim_m y_m^*(uf_m) = 0$$

for every sequence (f_m) in $L_1[0,1]$ which weakly converges to 0 such that $\sup_m ||f_m||_\infty < +\infty$.

Remark 4.2. We thank Professor Eve Oja who has pointed out that if X is a Banach space with the Radon-Nikodým property then X has property (k). For, by a result of Lewis and Stegall (cf. [DU, Chapt. III, Theorem 8, p. 66]) every operator $u : L_1[0,1] \to X$ factors through ℓ_1. Hence if $(f_n) \subset L_1[0,1]$ is a weakly null sequence then $\lim_n ||uf_n||_X = 0$. Therefore $\lim_n x_n^*(uf_n) = 0$ for every bounded sequence $(x_n^*) \subset X^*$. $\square$

Remark 4.3. It follows from Sobczyk's theorem [LT, Vol.I,2.f.5] that every weak* convergent to 0 sequence in the dual of a subspace of a separable space extends to a weak* convergent to 0 sequence in the dual to the whole space. Thus a subspace of a separable space with property (k) again has property (k). Clearly if X has property (k) then every complemented subspace of X has property (k). Therefore if X has the SCP and property (k) then every separable subspace of X has property (k).

Remark 4.4. In Definition 4.1 the requirement $\sup_m \|f_m\|_\infty < \infty$ is superfluous in view of

FACT. If (f_n) is a weakly-null sequence in $L_1[0,1]$ then for every $\varepsilon > 0$ there exists a weakly-null sequence $(f'_n) \subset L_1[0,1]$ such that $\sup_n \|f'_n\|_\infty < \infty$ and $\sup_n \|f_n - f'_n\|_1 < \varepsilon$.

Proof (S. Kwapień). For $f \in L_1[0,1]$ let $S_m(f)$ denote the m-th partial sum of the expansion of f with respect to the Haar basis $(m = 1, 2, \dots)$. Since S_m is a conditional expectation with respect to the sigma field generated by the first m functions of the Haar basis, $\|S_m(f)\|_1 \leq \|f\|_1$ and $\|S_m(f)\|_\infty \leq \|f\|_\infty$. Since (f_n) is weakly-null, $\lim_n \|S_m(f_n)\|_1 = 0$ for $m = 1, 2. \dots$. Thus there is a non-decreasing sequence of indices (m_n) with $\lim_n m_n = \infty$ such that $\lim_n \|S_{m_n}(f_n)\|_1 = 0$. Now given $\varepsilon > 0$ the equi-integrability of the set of elements of the weakly-null sequence (f_n) yields the existence of $M < \infty$ and $(\widetilde{f_n}) \subset L_1[0,1]$ such that $\sup_n \|\widetilde{f}_n\|_\infty < M$ and $\sup_n \|f_n - \widetilde{f}_n\|_1 < \varepsilon/2$. Put

$$f'_n := \widetilde{f}_n + S_{m_n}(f_n - \widetilde{f}_n) = \widetilde{f}_n - S_{m_n}(\widetilde{f}_n) + S_{m_n}(f_n) \quad (n = 1, 2, \dots).$$

Note that $(\widetilde{f}_n - S_{m_n}(\widetilde{f}_n))$ is a weakly-null sequence in $L_1[0,1]$ because $\sup_n \|\widetilde{f}_n - S_{m_n}(\widetilde{f}_n)\|_\infty < 2M$ and $\lim_n S_m(\widetilde{f}_n - S_{m_n}(\widetilde{f}_n)) = 0$ for $m = 1, 2, \dots$. Thus (f'_n) is weakly-null. Finally $\|f_n - f'_n\|_1 \leq \|f_n - \widetilde{f}_n\|_1 + \|S_{m_n}(f_n - \widetilde{f}_n)\|_1 < \varepsilon$ for $n = 1, 2, \dots$. □

Proposition 4.5.

(a) *A separable subspace of a weakly sequentially complete Banach lattice has property* (k).
(b) *A weakly sequentially complete Banach lattice with a weak unit has property* (k).

We need the following variant of the factorization theorem for weakly compact operators [DFJP].

Lemma 4.6. *Let L be a Köthe function space on a probability measure space (Ω, Σ, μ) (cf.* [LT, Definition Vol. II,1.b.17]*). Assume that L as a Banach space is weakly sequentially complete. Then there is a reflexive Köthe function space R on (Ω, Σ, μ) such that*

$$L_\infty := L_\infty(\Omega, \Sigma, \mu) \overset{I_{\infty,R}}{\hookrightarrow} R \overset{I_{R,L}}{\hookrightarrow} L,$$

i.e. the set theoretical inclusion $I_{\infty,L}$ admits the factorization through the set theoretical inclusions $I_{\infty,L} = I_{R,L} \circ I_{\infty,R}$.

Proof. Let Y be a function space on (Ω, Σ, μ) which contains constant functions and is a module over the algebra L_∞. Call a norm p on Y *monotone* provided that

$$p(\phi f) \leq p(f)||\phi||_\infty \quad (f \in Y, \phi \in L_\infty).$$

Then Y under the norm p is a Köthe function space on (Ω, Σ, μ).

It is enough to show that if $f \in Y$ then $|f| \in Y$ and $p(|f|) = p(f)$. To prove this, define

$$\phi(x) = \begin{cases} |f(x)|/f(x) & \text{for } f(x) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Note that $||\phi||_\infty = ||\phi^{-1}||_\infty$. Thus the identities $|f| = f\phi$ and $f = |f|\phi^{-1}$ imply $p(|f|) \leq p(f)$ and $p(f) \leq p(|f|)$. Thus $p(f) = p(|f|)$.

Next observe that since L is weakly sequentially complete and L_∞ is isomorphic to a $C(K)-$space, $I_{\infty,L}$ is weakly compact [DS, Theorem VI,§7.6]. Put $W = I_{\infty,L}(B_{L_\infty})$. Then W is a convex weakly relatively compact subset of L (In fact W is weakly compact). For each $n = 1, 2, \ldots$ let p_n be the gauge functional of the set $2^n W + 2^{-n} B_L$. Define, for $x \in L$, $p(x) = (\sum_{n=1}^\infty p_n^2(x))^{1/2}$, let $R = \{x \in L : p(x) < \infty\}$. Clearly R is a function space on (Ω, Σ, μ) which is a module over L_∞. It is shown in [DFJP] that R under the norm p is a reflexive Banach space and we have the factorization $I_{\infty,L} = I_{R,L} \cdot I_{\infty,R}$. It remains to establish that R is a Köthe space on (Ω, Σ, μ), which reduces to verifying that the norm p is monotone. For some $n \in \mathbb{N}$ pick $f \in 2^n W + 2^{-n} B_L$. Then $f = g + h$ with $||g||_\infty \leq 2^n$ and $||h||_L \leq 2^{-n}$. Hence for every $\psi \in L_\infty$ with $||\psi||_\infty \leq 1$ we have $f\psi = g\psi + h\psi$ with $||g\psi||_\infty \leq 2^n$ and $||h\psi||_L \leq 2^{-n}$. Thus $f\psi \in 2^n W + 2^{-n} B_L$. Therefore p_n is monotone. Now, in a similar way, we infer that p is monotone. $\square$

Proof of Proposition 4.5. Recall that every separable subspace of a Banach lattice is contained in a separable sublattice. Consequently, a Banach lattice is weakly sequentially complete iff every separable sublattice of X is weakly sequentially complete. Every separable lattice has a weak unit. Hence case (a) reduces to (b). Therefore one can assume that the lattice in question, say L, has a weak unit. Thus, by [LT, Vol II,Theorem 1.b.14], there exists a probability measure space (Ω, Σ, μ) such that L can be represented as a Köthe space on (Ω, Σ, μ) and the inclusion $I_{\infty,L}$ has a dense range. By Lemma 4.6, there is a reflexive Köthe space R on (Ω, Σ, μ) such that $I_{\infty,L} = I_{R,L} \circ I_{\infty,R}$. Now we repeat an argument of [J2]. Since R is reflexive, so is R^*. Thus the adjoint $(I_{R,L})^* : L^* \to R^*$ is weakly compact, and it is injective because $I_{R,L}$ has a dense range. Thus if $(x_n^*) \subset L^*$ weak* converges to 0 then $(I_{R,L})^*(x_n^*)$ converges to 0 weakly in R^*. Therefore, by S. Mazur's theorem, there is a CCC sequence (y_m^*) of (x_n^*) such that $\lim_m ||(I_{R,L})^*(y_m^*)||_{R^*} = 0$. Thus also the sequence $(|(I_{R,L})^*(y_m^*)|)$ converges to 0 in the norm $||\cdot||_{R^*}$. But $|(I_{R,L})^*(y_m^*)| = (I_{R,L})^*(|y_m^*|)$ because $(I_{R,L})^*$ is a lattice homomorphism. But then the only possible weak* cluster point in L^* of $(|y_m^*|)$ is 0, so $(|y_m^*|)$ must converge weak* in L^* to 0.

Next note that for every weakly sequentially complete Banach lattice L every operator $u : L_1[0,1] \to L$ takes the unit ball of $L_\infty[0,1]$, regarded as a subset of $L_1[0,1]$, into a lattice bounded subset of L (cf. [AB, p.24]). Thus for every weakly convergent to 0 sequence $(f_m) \subset L_1[0,1]$ with $\sup_m ||f_m||_\infty < \infty$ there is a positive $x \in L$ such that $|uf_m| \le x$ for $m = 1, 2, \ldots$. Therefore $\lim_m y_m^*(uf_m) = 0$ because

$$|y_m^*(uf_m)| \le |y_m^*|(|uf_m|) \le |y_m^*|(x) \quad (m = 1, 2 \ldots)$$

and $(|y_m^*|)$ weak* converges to 0. □

Proposition 4.5 implies in particular that separable subspaces of abstract $L-$spaces have property (k). The next proposition extends this result to some preduals of von Neumann algebras.

Proposition 4.7. *A separable subspace of the predual of a von Neumann algebra has property* (k).

Proof. A precise version of a result of Haagerup [GGMS, Appendix] states that every separable subspace of the predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ is contained in a separable complemented subspace of $\mathcal{M}_*$ which is itself the predual of a von Neumann algebra. Thus without loss of generality we can assume in the sequel that $\mathcal{M}_*$ is separable. Let $\mathcal{M}_*^+$ denote the positive cone of $\mathcal{M}_*$. Since $\mathcal{M}_*$ is separable there is in $\mathcal{M}_*$ a sequence (ω_r) whose elements form a dense set in $\mathcal{M}_*^+$. Now let $(a_n) \subset \mathcal{M}$ be a weak*-null sequence (= a sequence convergent to 0 in the $\sigma(\mathcal{M}, \mathcal{M}_*)-$topology). Then for every $\omega \in \mathcal{M}_*^+$ the sequence (a_n) converges weakly to 0 in the Hilbert space $(\mathcal{M}, \omega)$. The space $(\mathcal{M}, \omega)$ is the completion of $\mathcal{M}$ in the norm $\|\cdot\|_\omega$ where $\|g\|_\omega = \omega(g^*g + gg^*)^{1/2}$ for $g \in \mathcal{M}$. Thus applying Mazur's theorem on convex combinations we construct a CCC sequence (b_m) of the sequence (a_n) so that for $m = 1, 2, \ldots,$

$$(4.2) \qquad \omega_r(b_m b_m^* + b_m^* b_m)^{1/2} < 1/m \quad \text{for } r = 1, 2, \ldots, m.$$

Next let $u : L_1[0,1] \to \mathcal{M}_*$ be a bounded linear operator and let $(f_m) \subset L_1[0,1]$ be a weakly convergent to 0 sequence such that $\sup_m \|f_m\|_\infty < \infty$. Then the sequence (uf_m) converges to 0 weakly in $\mathcal{M}_*$. Therefore, by [T, Theorem III.5.4 (iv)], there exists $\omega \in \mathcal{M}_*^+$ with the property that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\omega(bb^* + b^*b)^{1/2} < \delta$ then $|b(uf_m)| \|b\| < \varepsilon$ for $m = 1, 2, \ldots$. Since (ω_r) is norm dense in $\mathcal{M}_*^+$, it follows from (4.2) that for some k large enough $\omega_k(bb^* + b^*b)^{1/2} < \delta$ implies $|b(uf_m)| < \varepsilon$ for $m = 1, 2 \ldots$. Therefore $\lim_m b_m(uf_m) = 0$. □

The next example shows that in Proposition 4.5 the assumption on the existence of a weak unit is essential even for abstract $L-$spaces.

Example 4.I. Let T be the set of all strictly increasing sequences $(n_m)_{m=1}^{\infty}$ of integers with $n_1 = 1$. Define

$$X := (\sum_{t\in T} L_1(\mu_t))_1,$$

where for all $t \in T$ the measure μ_t is the normalized Lebesgue measure on $[0, 1]$. Then

$$X^* = (\sum_{t\in T} L_\infty(\mu_t))_\infty.$$

Let (r_j) denote the Rademacher functions. Define $(g_n^*) \subset X^*$ by

$$g_n^*(t) = r_{j(n,t)},$$

$$\text{where } j(n,t) = m \text{ for } n_m \le n < n_{m+1} \quad (n = 1, 2, \ldots \quad t \in T).$$

Then $g_n^* \to 0$ weak* because, for all $t \in T$, $g_n(t) \to 0$ weak* in $L_\infty(\mu_t) = (L_1(\mu_t))^*$ as $n \to \infty$ and $\sup_n \|g_n^*\|_{X^*} = 1$.

Suppose now that (h_m^*) is a CCC sequence of (g_n^*), say

$$h_m^* = \sum_{j=n_m^o}^{j=n_{m+1}^o-1} c_j g_j^*, \quad \sum_{j=n_m^o}^{j=n_{m+1}^o-1} c_j = 1, \quad c_j \ge 0 \quad (m, j = 1, 2, \ldots).$$

Let $t_o = (n_m^o) \in T$. Define $u : L_1[0, 1] \to X$ to be identity on the t_o coordinate and 0 otherwise. Clearly $r_{j(m,t_o)} \to 0$ weakly in $L_1[0, 1]$ as $m \to \infty$, while $h_m^*(ur_{j(m,t_o)}) = 1$ for $m = 1, 2, \ldots.$ Thus X fails property (k).

Recall the Lindenstrauss Lifting Principle [L], [KP]

(LLP) Let $Q : X \to V$ be a surjection ($X, V-$Banach spaces). Assume that $\ker Q$ is complemented in $(\ker Q)^{**}$. Then for every $\mathcal{L}_1-$space Z, every linear operator $u : Z \to V$ admits a lifting $\widetilde{u} : Z \to X$, i.e. $Q\widetilde{u} = u$.

Corollary 4.8. *Let*

$$0 \to W \stackrel{J}{\to} X \stackrel{Q}{\to} V \to 0 \tag{4.3}$$

be an exact sequence of Banach spaces. If W is complemented in W^{**} and X has property (k) then V has property (k).

Proof. Let a sequence $(v_n^*) \subset V^*$ weak* converge to 0. Then $(Q^*v_n^*)$ weak* converges to 0. Since X has property (k), there is a CCC sequence (y_m^*) of $(Q^*v_n^*)$ which satisfies the requirement of Definition 4.1. Let $(w_m^*) \subset V^*$ be a unique CCC sequence of (v_n^*) such that $Q^*w_m^* = y_m^*$ for $m = 1, 2 \ldots.$ Since $\ker Q = W$ is complemented in W^{**}, (LLP) implies that every $u : L_1[0, 1] \to V$ admits a lifting

$\widetilde{u}: X \to V$. Since X has property (k), for every weakly convergent to 0 sequence $(f_k) \subset L_1[0,1]$ with $\sup_k \|f_k\| < \infty$ one has

$$\lim_m w_m^*(uf_m) = \lim_m w_m^*(Q\widetilde{u}(f_m)) = \lim_m y_m^*(\widetilde{u}(f_m)) = 0.$$

□

Proposition 4.9. *If a Banach space V contains a complemented subspace isomorphic to c_0, then V fails property (k).*

Proof. Let $w : c_0 \to V$ be an isomorphic embedding onto a complemented subspace of V. Then there is a operator $p : V \to c_0$ such that $pw : c_0 \to c_0$ is the identity. Let (e_n) (resp. (e_n^*)) denote the unit-vector basis of c_0 (resp. of ℓ_1). Then (e_n^*) weak* converges to 0 in ℓ_1 regarded as the dual space of c_0. Hence $(p^*(e_n))$ weak* converges to 0 in V^*. Let (y_m^*) be an arbitrary CCC sequence of $(p^*(e_n^*))$. Define $\Delta_m \subset c_0$ for $m = 1, 2, \dots$ by $\Delta_m = \sum_{n=n_m}^{n_{m+1}-1} e_n$ where (n_m) is the sequence of positive integers satisfying (4.1). Define $u : L_1[0,1] \to V$ by

$$uf = w\Big(\sum_m (\int_0^1 f\bar{f}_m)\,\Delta_m\Big) \quad \text{for } f \in L[0,1]$$

where (f_m) is any orthonormal sequence in $L_2[0,1]$ with $\|f_m\|_\infty = 1$ for $m = 1, 2, \dots$. Clearly (f_m) converges weakly to 0 in $L_1[0,1]$ and it is easy to see that $uf_m = w\Delta_m$ for $m = 1, 2, \dots$ and $\|u\| = \|w\|$. We have for $m = 1, 2, \dots$,

$$\begin{aligned} y_m^*(uf_m) &= (\sum_{n=n_m}^{n_{m+1}-1} c_n p^* e_n^*)(w\Delta_m) \\ &= (\sum_{n=n_m}^{n_{m+1}-1} c_n e_n^*)(pw\Delta_m) \\ &= (\sum_{n=n_m}^{n_{m+1}-1} c_n e_n^*)(\Delta_m) = \sum_{n=n_m}^{n_{m+1}-1} c_n = 1. \end{aligned}$$

Thus V fails property (k). □

Note that ℓ_∞ contains c_0 but has property (k) because in ℓ_∞^* every weak* convergent sequence converges weakly (cf. [Gr1]).

Corollary 4.10. *Suppose that we are given an exact sequence* (4.3). *Assume that X has property (k) and V contains a complemented subspace isomorphic to c_0. Then W is not complemented in W^{**}, in particular W is not isomorphic to a dual Banach space.* □

Applying results on property (k) and some results of previous sections to Banach lattices we get

Corollary 4.11. *Suppose that we are given an exact sequence* (4.3). *Assume that V contains a subspace isomorphic to c_0 and either*

(i) *X is a separable subspace of a weakly sequentially complete Banach lattice,*

or

(ii) *X is a complemented subspace of a weakly sequentially complete Banach lattice with a weak unit,*

or

(iii) *X is a separable subspace of a Banach space of finite cotype with GL-l.u.st.*

Then W is not complemented in W^{**}, in particular W is not isomorphic to a dual Banach space. □

For (iii) note that a Banach space of finite cotype with GL-l.u.st. is isomorphic to a subspace of a weakly sequentially complete lattice.

If in (4.3) X is a weakly sequentially complete Banach lattice without a weak unit and V contains an uncomplemented subspace isomorphic to c_0 then sometimes W is not complemented in W^{**} (see Example 4.II), and sometimes W is a dual Banach space (see Example 4.III).

Example 4.II. If Γ is a set of cardinality continuum then there is a surjection $Q : \ell_1(\Gamma) \to \ell_\infty$. Then $W = \ker Q$ is not complemented in W^{**}.

Proof. Define $u : L_1[0,1] \to c_0 \subset \ell_\infty$ by $uf = (\int_0^1 f\bar{f}_n dx)$ for $f \in L_1[0,1]$ where (f_n) is any orthonormal uniformly bounded sequence. Then u does not lift to $\ell_1(\Gamma)$ because (f_n) weakly converges to 0, while $\|uf_n\|_\infty = 1$ for $n = 1, 2, \ldots$ and in $\ell_1(\Gamma)$ every weakly convergent sequence converges in norm. Thus in view of (LLP), W is not complemented in W^{**}. □

In the last example of this section the kernel of the surjection is a dual Banach space. The example shows that the SCP may fail when passing to subspaces. This answers in the negative a question posed in [PlY].

Example 4.III. Let $Q : L \to c_0(\Gamma)$ be a surjection where L is a separable lattice of finite cotype and Γ is an infinite set. Then $Q^{**} : L^{**} \to \ell_\infty(\Gamma) = c_0(\Gamma)^{**}$ is also a surjection. By Proposition 3.2, L^{**} has the SCP because L^{**} has the same cotype as L; hence it is a weakly sequentially complete lattice. Clearly $\ker Q^{**}$ is a weak*-closed subspace of L^{**}, hence it is a dual Banach space, in particular $\ker Q^{**}$ is complemented in its second dual.

Proposition 4.12. $\ker Q^{**}$ *fails the SCP; precisely if* $\Gamma_0 \subset \Gamma$ *is a countable infinite set,* L_0 *is a separable subspace of* L *such that* $Q(L_0) = c_0(\Gamma_0)$ *and* E *is a separable subspace of* $\ker Q^{**}$ *with* $\ker Q \cap L_0 \subset E$, *then there is no projection from* $\ker Q^{**}$ *onto* E.

Proof. Let $X = \mathrm{cl}(L_0 + E)$. Then X is a separable subspace of the weakly sequentially complete Banach lattice L^{**}. Therefore, by Proposition 4.5(a), X has property (k). We show that the quotient space X/E contains c_0. Let (δ_j) be the unit-vector basis of $c_0 = c_0(\Gamma_0)$. Clearly $c_0(\Gamma_0)$ can be regarded as a subspace of $c_0(\Gamma)$. Choose $\xi_j \in L_0$ so that $Q(\xi_j) = \delta_j$ for $j = 1, 2, \ldots$. Identifying $c_0(\Gamma)$ with its canonical image in $\ell_\infty(\Gamma)$ and L with its canonical image in L^{**} we infer that $Q^{**}(\xi_j) = \delta_j$ for $j = 1, 2, \ldots$. Since $Q(L_0) = c_0(\Gamma_0)$ and $Q^{**}(L^{**}) = \ell_\infty(\Gamma)$, there are positive constants a and b such that for arbitrary $n = 1, 2, \ldots$ and scalars $t_1, t_2, \ldots, t_n$ one has

$$a\|\sum_{j=1}^{n} t_j\delta_j\|_{c_0(\Gamma_0)} \geq \inf_{\xi\in\ker Q\cap L_0} \|\sum_{j=1}^{n} t_j\xi_j + \xi\| \geq \inf_{\xi^{**}\in E} \|\sum_{j=1}^{n} t_i\xi_j + \xi^{**}\|$$
$$= \|\sum_{j=1}^{n} t_j\xi_j\|_{X/E} \geq \inf_{\xi^{**}\in\ker Q^{**}} \|\sum_{j=1}^{n} t_j\xi_j + \xi^{**}\| \geq b\|\sum_{j=1}^{n} t_j\delta_j\|_{\ell_\infty}.$$

Thus $(q\xi_j)$ is equivalent to the unit-vector basis of c_0, where $q : X \to X/E$ denotes the quotient map. Therefore, by Corollary 4.11(i), E is not complemented in E^{**} and hence is not isomorphic to a complemented subspace of any dual space, which implies that there is no projection from $\ker Q^{**}$ onto E (cf. [D, L]). □

Remark 4.13. Analyzing the proof of Proposition 4.12 we obtain
If $\widetilde{L}$ *is a subspace of* $\ker Q^{**}$ *such that* $\ker Q \subset \widetilde{L}$ *and* $\widetilde{L}$ *is complemented in* $(\widetilde{L})^{**}$ *then* $\widetilde{L}$ *fails the SCP.*

5. Application to Sidon sets and Sobolev spaces of functions of bounded variation

5.1. **Spaces of measures on a compact Abelian group orthogonal to a fixed Sidon set.** Let G be a compact Abelian group, Γ its dual. Let $L_1(G)$ denote the space of all complex-valued functions on G absolutely integrable with respect to the Haar measure of G. Let $M(G)$ denote the space of all complex-valued regular Borel measures μ on G with bounded variation $\mathrm{var}(\mu)$ with the norm $\|\mu\| = \mathrm{var}(\mu(G))$.

For $S \subset \Gamma$ put $S^\perp = \Gamma \setminus S$. Put

$$L_1^{S^\perp}(G) = \{f \in L_1(G) : \textstyle\int_G f(g)\gamma^{-1}(g)dg = 0 \quad \text{for } \gamma \in S\};$$
$$M^{S^\perp}(G) = \{\mu \in M(G) : \textstyle\int_G \gamma^{-1}(g)d\mu = 0 \quad \text{for } \gamma \in S\}.$$

Clearly $L_1^{S^\perp}(G)$ can be regarded as subspace of $M^{S^\perp}(G)$. Less obvious is the following fact.

Proposition 5.1. *There is an isometrically isomorphic embedding* $\Upsilon : M^{S^\perp}(G) \to (L_1^{S^\perp}(G))^{**}$ *and a contraction* $P : (L_1^{S^\perp}(G))^{**} \to M^{S^\perp}(G)$ *such that* $P \circ \Upsilon = id_{M^{S^\perp}(G)}$*; in other words,* $M^{S^\perp}(G)$ *is isometrically isomorphic to a complemented subspace (via a contractive projection) of the second dual of* $L_1^{S^\perp}(G)$*.*

Outline of the proof. Let $(\Phi_U)_{U\in\mathcal{O}}$ be an approximate identity of the convolution algebra $L_1(G)$, where $\mathcal{O}$ is the set of neighborhoods of the unit of G; $0 \le \Phi_U \in L_1(G)$; $\int_G \Phi_U dg = 1$; $\Phi(g) = 0$ for $g \in G \setminus U$.

Let $\mathcal{U}$ be an ultrafilter on $\mathcal{O}$ such that $\{V \in \mathcal{O} : V \subseteq U\} \in \mathcal{U}$, for $U \in \mathcal{O}$. For $\mu \in M^{S^\perp}(G)$ define $\Upsilon\mu \in (L_1^{S^\perp}(G))^{**}$ by

$$\Upsilon\mu(x^*) = \mathrm{LIM}_{U\in\mathcal{U}} \int_G (\Phi_U \star \mu)\phi_{x^*}(g)dg \quad \text{for } x^* \in (L_1^{S\perp}(G))^*,$$

where $\mathrm{LIM}_{U\in\mathcal{U}}$ denotes the limit with respect to the ultrafilter $\mathcal{U}$, $\phi_{x^*} \in L_\infty(G) = (L_1(G))^*$ is a norm preserving Hahn-Banach extension of the functional x^* and "$\star$" stands for convolution.

To define P first observe that $M^{S^\perp}(G)$ is naturally isometrically isomorphic with the dual space of the quotient $C(G)/(M^{S^\perp}(G))_\perp$ where

$$(M^{S^\perp}(G))_\perp = \{f \in C(G) : \int_G f(g)d\mu = 0 \quad \text{for } \mu \in M^{S^\perp}(G)\}.$$

Define the isometrically isomorphic embedding

$$J : C(G)/(M^{S^\perp}(G))_\perp \to (L_1^{S^\perp}(G))^*$$

as follows: if $[\psi]$ is a coset of a $\psi \in C(G)$ then $J([\psi]) \in L_1^{S^\perp}(G)^*$ is defined by $J([\psi])(f) = \int_G \psi(g)f(g)dg$ for $f \in L_1^{S^\perp}(G)$. Remembering that $(C(G)/(M^{S^\perp}(G))_\perp)^* = M^{S^\perp}(G)$ we put

$$P := J^* : (L_1^{S\perp}(G))^{**} \to M^{S^\perp}(G),$$

in other words Px^{**} is the restriction of the functional x^{**} acting on $L_1^{S^\perp}(G)^*$ to the subspace $J(C(G)/(M^{S^\perp}(G))_\perp)$. We omit a routine verification that Υ and J are isometrically isomorphic embeddings and the proof of the identity $P\Upsilon(\mu) = \mu$ for $\mu \in M^{S^\perp}(G)$. □

Next we list several properties of the spaces $M^{S^\perp}(G)$ and $L_1^{S^\perp}(G)$ for S

being Sidon sets. The properties (vi) and (vii) below were established with different proofs in [J2, Theorem 2.1 and Corollary 2.9].

Theorem 5.2. *Let $S \subset \Gamma$ be an infinite Sidon set in the dual Γ of a compact Abelian group G. Then*

(i) *The spaces $L_1^{S^\perp}(G)$ and $M^{S^\perp}(G)$ have the UAP.*
(ii) *If Γ is countable then $L_1^{S^\perp}(G)$ has a basis.*
(iii) *Every separable subspace of $L_1^{S^\perp}(G)$ (resp. $M^{S^\perp}(G)$) is contained in a separable subspace of $L_1^{S^\perp}(G)$ (resp. $M^{S^\perp}(G)$) with a basis.*
(iv) *$M^{S^\perp}(G)$ fails the SCP.*
(v) *$M^{S^\perp}(G)$ fails the commuting bounded approximation property.*
(vi) *$M^{S^\perp}(G)$ fails GL-l.u.st.*
(vii) *$L_1^{S^\perp}(G)$ fails GL-l.u.st.*

Proof. Recall that S is a Sidon set iff the map $Q : f \to (\int_G f(g)\gamma^{-1}(g)dg)_{\gamma\in S}$ is a surjection from $L_1(G)$ onto $c_0(S)$ (resp. the map $\mu \to (\int_G \gamma^{-1}(g)d\mu)_{\gamma\in S}$ is a surjection from $M(G)$ onto $\ell_\infty(S)$ (cf. [HR, Theorem 37.4]). Of course, $L_1(G)$ and $M(G)$ are $\mathcal{L}_1-$spaces. Thus applying Theorem 2.1 (c) we get (i) because $c_0(S)$ and $\ell_\infty(S)$ are $\mathcal{L}_\infty-$spaces.

(ii) is due to Lusky and it follows directly from (L) (see Section 2).

(iii) follows from Proposition 2.3 because for S infinite $c_0(S)$ is isomorphic to the space of all continuous functions on the one-point compactification of the discrete set S, while $\ell_\infty(S)$ is isometrically isomorphic to the space of all continuous functions on the Stone-Čech compactification of S.

(iv) $L_1(G)$ is a Banach lattice of cotype 2. By Proposition 5.1, $M^{S^\perp}(G) \subset \ker Q^{**} = (L_1^{S^\perp}(G))^{**}$. Moreover $M^{S^\perp}(G)$ is complemented in its second dual because it is the dual of $C(G)/(M^{S^\perp}(G))_\perp$. The desired conclusion follows from Proposition 4.12 and Remark 4.13. To apply Remark 4.13 we put $\widetilde{L} = \Upsilon(M^{S^\perp}(G))$, where Υ is defined in the proof of Proposition 5.1.

(v) Combine (iv) with the result of Casazza, Kalton and Wojtaszczyk mentioned in Section 3.

(vi) Combine (iv) with Corollary 3.4. Note that $M^{S^\perp}(G) \subset M(G)$ has cotype 2.

(vii) If a Banach space X has GL-l.u.st. then all complemented subspaces of the even duals of X have GL-l.u.st. Thus (vii) follows from (vi) and Proposition 5.1. □

5.2. **Sobolev spaces of functions of bounded variation.** Let $\Omega \subset \mathbb{R}^n$ be an open non-empty set. For a measurable complex-valued f on

Ω put
$$||f||_{BV} = \int_\Omega |f| d\lambda_n + |Df|(\Omega),$$
where $|Df|$ is a positive regular measure of bounded variation defined for open $\Omega' \subset \Omega$ by
$$|Df|(\Omega') = \sup|\int_{\Omega'} f \operatorname{div} \phi d\lambda_n| = \sup|\int_{\Omega'} (f \sum_j \frac{\partial \phi_j}{\partial x_j}) d\lambda_n|,$$
λ_n denotes the $n-$dimensional Lebesgue measure and the supremum extends over all infinitely many times differentiable complex valued functions $\phi = (\phi_j)_{j=1}^n$ with compact support from $\mathbb{R}^n$ into the unit ball of the $n-$dimensional complex Hilbert space.

Put
$$BV(\Omega) = \{f : \Omega \to \mathbb{C} \text{ - measurable} : ||f||_{BV} < \infty\}.$$
$BV(\Omega)$ is a nonseparable Banach space. It consists of absolutely integrable functions whose distributional partial derivatives exist and are measures with total bounded variation. The Sobolev space $W_1^1(\Omega)$ is isometrically isomorphic with the subspace of $BV(\Omega)$ consisting of those functions whose distributional gradient is a $\mathbb{C}^n$-valued measure absolutely continuous with respect to λ_n.

Theorem 5.3. *If $\Omega \subset \mathbb{R}^n$ is an open non-empty set, then*

(a) *$BV(\Omega)$ isometrically embeds into an abstract L_1-space - the Cartesian power $(\mathcal{M}(\Omega^n))^{n+1}$, where $\mathcal{M}(\Omega)$ is the space of all complex-valued regular Borel measures on Ω with bounded variation.*

(b) *$BV(\Omega)$ is a dual Banach space.*

(c) *If $n \geq 2$ then $BV(\Omega)$ fails the SCP.*

Proof. (a) is obvious.

For (b) observe that $\mathcal{M}(\Omega)^{n+1}$ is the dual space of the space $C_0(\Omega)^{n+1}$ where $C_0(\Omega)$ is the space generated in the uniform norm by all continuous complex-valued functions on Ω with compact support. The space $BV(\Omega)$ is isometrically isomorphic to a weak$^*-$closed subspace of $\mathcal{M}(\Omega)^{n+1}$ (cf.[PW]).

The proof of (c) requires several steps. First, it routinely reduces to the case $\Omega = \mathbb{R}^n$. Indeed, an open non-empty set Ω contains a closed cube. If I^n denotes the interior of the cube then there is a linear extension operator from $BV(I^n)$ into $BV(\Omega)$. Therefore $BV(\Omega)$ contains a complemented subspace isomorphic to $BV(I^n)$; the operator of restriction of functions on Ω to I^n composed with the linear extension is the desired projection. Finally, by a decomposition technique (cf.

[PW, Proof of Theorem 7.1]) one shows that $BV(I^n)$ is isomorphic to $BV(\mathbb{R}^n)$. Thus $BV(\Omega)$ contains a complemented subspace isomorphic to $BV(\mathbb{R}^n)$. To complete the proof of (c) it suffices to establish

Proposition 5.4. *If $n \geq 2$ then $BV(\mathbb{R}^n)$ fails the SCP. More precisely, if E is a separable subspace of $BV(\mathbb{R}^n)$ which contains $W_1^1(\mathbb{R}^n)$ then E is not complemented in $BV(\mathbb{R}^n)$.*

We need several facts about traces.

Let $b \in \mathbb{R}$; $x = (x(j))_{j=1}^n \in \mathbb{R}^n$. Put

$$H_b^- = \{x \in \mathbb{R}^n : x(n) < b\}; \quad H_b^+ = \{x \in \mathbb{R}^n : x(n) > b\};$$
$$H_b = \{x \in \mathbb{R}^n : x(n) = b\}.$$

In the sequel we identify $L_1(H_b, \lambda_{n-1})$ with $L_1(\mathbb{R}^{n-1})$.

Write $f \in CV^b(H_b^-)$ provided that $f \in BV(H_b^-)$ and there is $\delta = \delta(f) > 0$ such that the restriction $f|\{x : -\delta + b < x(n) < b\}$ is λ_n a.e. equal to a C^∞−function uniformly continuous with all its partial derivatives. Note that if $f \in CV^b(H_b^-)$ then f has a unique extension - denoted also by f - to a C^∞−function on $H_b^- \cup H_b$. The definition of $CV^b(H_b^+)$ is similar.

General Gagliardo-Nirenberg Theorem [PW], [AFP]. *There is a unique contractive linear operator $\mathrm{Tr}_b^\sigma : BV(H_b^\sigma) \to L_1(\mathbb{R}^{n-1})$, where $\sigma = +$ or $\sigma = -$, such that if $f \in CV^b(H_b^\sigma)$ then $\mathrm{Tr}_b^\sigma(f) = f|H_b$. Moreover:*

(i) *if $f \in W_1^1(\mathbb{R}^n)$ then $\mathrm{Tr}_b^-(f|H_b^-) = \mathrm{Tr}_b^+(f|H_b^+) =: \mathrm{Tr}_b(f)$;*
(ii) $\mathrm{Tr}_b(W_1^1(\mathbb{R}^n)) = L_1(\mathbb{R}^{n-1})$.

Peetre's Theorem [Pee], [PW]. *The operator $\mathrm{Tr}_b^- : W_1^1(H_b^-) \to L_1(\mathbb{R}^{n-1})$ has no bounded lifting, i.e. there is no linear operator $V : L_1(\mathbb{R}^{n-1}) \to W_1^1(H_b^-)$ such that $\mathrm{Tr}_b^- \circ V = \mathrm{Id}_{L_1(\mathbb{R}^{n-1})}$.*

Theorem on extending traces [PW]. *There is a bounded linear operator*

$$\Lambda_b^- : BV(H_b^-) \to W_1^1(H_b^+)$$

such that

$$\mathrm{Tr}_b^-(f) = \mathrm{Tr}_b^+(\Lambda_b^- f) \quad \forall f \in BV(H_b^-),$$

and there is also $\Lambda_b^+ : BV(H_b^+) \to W_1^1(H_b^-)$ with similar properties.

Proof of Proposition 5.4. Let $E \supset W_1^1(\mathbb{R}^n)$ be a separable subspace of $BV(\mathbb{R}^n)$. It suffices to construct a separable subspace $F \subset BV(\mathbb{R}^n)$ such that

(j) $E \subset F$;
(jj) E is uncomplemented in F.

To this end observe that there exists $b \in \mathbb{R}$ such that

$$|Df|(H_b) = 0 \quad \text{for all } f \in E. \tag{5.1}$$

Indeed if $b_1 \neq b_2$ then $H_{b_1} \cap H_{b_2} = \emptyset$. Thus for every $f \in E$ the set $A_f = \{b \in \mathbb{R} : |Df|(H_b) \neq 0\}$ is countable. Hence if S is a countable dense set in the separable space E then the set $\bigcap_{f \in S}(\mathbb{R} \setminus A_f)$ is uncountable. It is easy to see that every b in the intersection satisfies (5.1).

Next fix $b \in \mathbb{R}$ satisfying (5.1) and put

$$\widetilde{W_1^1(H_b^-)} := \Big\{g \in BV(\mathbb{R}^n) : g|H_b^- \in W_1^1(H_b^-) \text{ and } g|(\mathbb{R}^n \setminus H_b^-) \equiv 0\Big\};$$

$$F := \text{the closure in } BV(\mathbb{R}^n) \text{ of } E + \widetilde{W_1^1(H_b^-)}.$$

Define the operator $Q : F \to L_1(\mathbb{R}^{n-1})$ by $Qf = \operatorname{Tr}_b^-(f|H_b^-) - \operatorname{Tr}_b^+(f|H_b^+)$. Then $Q(F) = L_1(\mathbb{R}^{n-1})$, because $Q(F) \supseteq Q(\widetilde{W_1^1(H_b^-)}) \supseteq L_1(\mathbb{R}^{n-1})$, by the General Gagliardo-Nirenberg Theorem.

Lemma 5.5. $\ker Q = E$.

Proof. If $e \in E$ then $|De|(H_b) = 0$. Thus by [PW, Lemma 3.1.1] (special case $k = 1, \Xi_a = \mathbb{R}^n$), given $\varepsilon > 0$ there is an $f \in BV(\mathbb{R}^n)$ such that $\|f - e\|_{BV(\mathbb{R}^n)} < \varepsilon$ and for some $\delta > 0$ the restriction $f|\{x : |b - x(n)| < \delta\}$ is equal to a $C^\infty-$function λ_n a.e. Clearly $\operatorname{Tr}_b^-(f|H_b^-) - \operatorname{Tr}_b^+(f|H_b^+) = 0$. Hence $|\operatorname{Tr}_b^-(e|H_b^-) - \operatorname{Tr}_b^+(e|H_b^+)| < 2\varepsilon$. Letting $\varepsilon \downarrow 0$ we get $\operatorname{Tr}_b^-(e|H_b^-) - \operatorname{Tr}_b^+(e|H_b^+) = 0$. Therefore $e \in \ker Q$.

To prove that $\ker Q \subseteq E$, fix $f \in \ker Q$ and let $\varepsilon > 0$. Pick $g \in \widetilde{W_1^1(H_b^-)}$ and $e \in E$ with $||f - g - e||_{BV(\mathbb{R}^n)} < \varepsilon$. Since $||Q|| \leq 2$,

$$2\varepsilon > ||Q(f - g - e)||_{L_1(\mathbb{R}^{n-1})} = ||Qg||_{L_1(\mathbb{R}^{n-1})} = ||\operatorname{Tr}_b^-(g)||_{L_1(\mathbb{R}^{n-1})}.$$

Note that $g|H_b^- \in W_1^1(H_b^-)$ because $g \in \widetilde{W_1^1(H_b^-)}$. Thus using the fact that $CV^b(H_b^-)$ is dense in $W_1^1(H_b^-)$ (cf. [PW, Lemma 3.1.1]) we infer that there exists $a < b$ with $2\varepsilon > ||g|\{a < x(n) < b\}||_{W_1^1(\{a<x(n)<b\})}$. Hence, by the extension formula for W_1^1 functions (cf. [A]), there exists $e^* \in W_1^1(\mathbb{R}^n)$ which extends $g|\{a < x < b\}$ with $||e^*||_{W_1^1(\mathbb{R}^n)} \leq C\varepsilon$, where the constant C is independent of ε. Define $g^* \in W_1^1(\mathbb{R}^n) \subset E$ by

$$g^*(x) = \begin{cases} g(x) & \text{for } x \leq a, \\ e^*(x) & \text{for } x > a. \end{cases}$$

Then $\|g-g^*\|_{BV(\mathbb{R}^n)} \le C\varepsilon$. Thus $\|f-g^*-e\|_{BV(\mathbb{R}^n)} < (2+C)\varepsilon$. Hence, letting $\varepsilon \downarrow 0$, we infer that $f \in E$. □

We now complete the proof of Proposition 5.4. Assume that (jj) is false, i.e., there exists a bounded projection $P : F \overset{onto}{\to} E$. Define $S : L_1(\mathbb{R}^{n-1}) \to F$ by $Sg = f - Pf$ whenever $Qf = g$. Then S is well defined, because $\ker Q = E$ by Lemma 5.5 and S is bounded, by the Open Mappping Theorem. Clearly, $QSg = Q(f-Pf) = Qf = g$ for $g \in L_1(\mathbb{R}^{n-1})$.

Let $\Lambda_b^\sigma : BV(H_b^\sigma) \to W_1^1(H_b^{-\sigma})$ be trace preserving operators and let R_σ denote the operator of restriction of functions to H_b^σ, where $\sigma = \mp$. Define $U : F \to W_1^1(H_b^-)$ by

$$Uf = \Lambda_b^+\Lambda_b^- R_- f - \Lambda_b^+ R_+ f \quad (f \in F).$$

We shall show that $V = US$ lifts Tr_b^- (contrary to Peetre's Theorem).

First note that given $f \in F$ the properties of Λ_b^σ imply

$$\mathrm{Tr}_b^-(\Lambda_b^+ R_+ f) = \mathrm{Tr}_b^+(R_+ f),$$
$$\mathrm{Tr}_b^-(\Lambda_b^+\Lambda_b^- R_- f) = \mathrm{Tr}_b^+(\Lambda_b^- R_- f) = \mathrm{Tr}_b^-(R_- f),$$

and hence $\mathrm{Tr}_b^-(Uf) = Qf$.

Now recall that if $g \in L_1(\mathbb{R}^{n-1})$ then $Sg \in F$ and $QSg = g$. Hence

$$\mathrm{Tr}_b^-(USg) = QSg = g.$$

This contradiction completes the proof of Proposition 5.4. □

Remark 5.6. $BV(\mathbb{R}^n)$ has the BAP and every separable subspace of $BV(\mathbb{R}^n)$ is contained in a separable subspace with a basis [ACPP]. We do not know whether $BV(\Omega)$ has the same property for arbitrary open $\Omega \subset \mathbb{R}^n$.

6. Open problems

Problem 6.1. If Y is a subspace of X such that X, Y and X/Y all have the BAP, must the pair (X, Y) have the BAP?

We do not know the answer to Problem 6.1 even when X is reflexive.

Problem 6.2. If X is a $\mathcal{L}_p$ space, $2 < p < \infty$, and Y is a subspace of X with the BAP (resp. UAP), must X/Y have the BAP (resp. UAP)? Equivalently, if X is a $\mathcal{L}_q$ space, $1 < q < 2$, and Y is a subspace of X such that X/Y has the BAP (resp. UAP), must Y have the BAP (resp. UAP)?

Problem 6.3. Does every weakly sequentially complete Banach space with GL-l.u.st. have the SCP?

Problem 6.4. Does every complemented subspace of a space with the SCP have the SCP? Does X have the SCP if X^{**} does?

Proposition 3.2 and Corollary 3.4 give positive answers to special cases of Problem 6.3 and Problem 6.4.

Remark 6.5. The projective tensor product of spaces with property (k) need not have (k). We are indebted to Professor Eve Oja for the following two examples.

The first one is the space $Z = L_1([0,1], \ell_\infty)$ of Bochner integrable ℓ_∞-valued functions. Since Z contains a complemented subspace isomorphic to c_0 (cf. [E]), it fails (k), by Proposition 4.9. Note that $L_1([0,1])\hat{\otimes}\ell_\infty = Z$ (cf. [Gr2, Chap. I §2.2, p. 59]), and both $L_1([0,1])$ and ℓ_∞ have (k).

The second example is provided by the space $X = \mathcal{L}_\lambda[\ell_2]$ constructed in [BP], where $\lambda > 1$. Namely, X has (k) by Remark 4.2, while $X\hat{\otimes}X$ fails (k) by Proposition 4.9 and Sobczyk's theorem. Indeed, it was proved in [BP] that X is a separable $\mathcal{L}_{\infty,\lambda}-$space with the following properties: (i) X has the RNP, (ii) X contains an isometric copy E of l_2, (iii) $X\hat{\otimes}X$ contains an isomorphic copy Y of c_0. □

Problem 6.6. Does a predual of a $\sigma-$finite von Neumann algebra have property (k)? (A von Neumann algebra is said to be $\sigma-$finite if it admits at most countably many mutually orthogonal projections.)

Problem 6.7. Does $BV(\mathbb{R}^n)$ have the UAP?

A well known and important problem is whether X^* has the metric approximation property whenever X^* has the BAP; see [C, Theorem 3.7]. We do not know the answer to the following two special cases of this problem.

Problem 6.8. Does $BV(\mathbb{R}^n)$ have the metric approximation property?

Problem 6.9. Does $M^{S^\perp}(G)$ have the metric approximation property for every Sidon set $S \subset \Gamma$?

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