Finite dimensional subspaces of L_p

William B. Johnson [1]

Department of Mathematics,
Texas A&M University,
College Station, Texas, USA

Gideon Schechtman [2]

Department of Mathematics,
Weizmann Institute of Science,
Rehovot, Israel

We discuss the finite dimensional structure theory of L_p; in particular, the theory of restricted invertibility and classification of subspaces of ℓ_p^n.

Contents

[1] Supported in part by NSF DMS-9623260 and DMS-9900185, Texas Advanced Research Program 010366-163, and the U.S.-Israel Binational Science Foundation.
[2] Supported in part by the U.S.-Israel Binational Science Foundation.

1 Introduction

1.1 The good, the bad, the natural, and the complemented

There are many interesting problems about infinite dimensional subspaces of $L_p(:= L_p[0,1])$ which have finite dimensional analogues. For example, it has long been a central problem in Banach space theory to classify the complemented subspaces of L_p up to isomorphism; the finite dimensional analogue is to find for any given C a description of the finite dimensional spaces which are C-isomorphic to C-complemented subspaces of L_p. A lot is known about both the infinite dimensional (see [1]) and finite dimensional (see section 5) versions of this complemented subspaces of L_p problem, but in neither case does a classification seem to be close at hand.

It sometimes happens that the finite dimensional version of an infinite dimensional problem leads to a theory which is much more interesting than the infinite dimensional theory. Take, for example, the problem of describing the subspaces of L_p which embed isomorphically into a "smaller" L_p space; namely, ℓ_p; for which there is a fairly simple answer (see [1]). Now it is clear that a finite dimensional subspace X of L_p embeds, with isomorphism constant $1+\epsilon$, into ℓ_p^N if $N = N(\epsilon, X)$ is sufficiently large. The attempt to estimate well N in terms of ϵ and X (or the dimension of X) has led to a deep theory (see sections 2.1 and 5.1). A sideline of this investigation also led to deeper understanding of how certain natural subspaces of L_p (such as the span of a sequence of independent Gaussian random variables) are situated in L_p (see section 2.2).

Besides being an interesting subject in its own right, the study of finite dimensional subspaces of L_p is often needed in order to understand properties of infinite dimensional subspaces. For example, the easiest and best way to obtain subspaces of L_p, $2 < p < \infty$, which fail GL-l.u.st. (cf. [28, section 9]) is to show that random large dimensional subspaces of ℓ_p^n are bad in a certain sense (see section 3.2).

The topic discussed herein which has the most applications is that of restricted invertibility (see section 4). Basically the theorem says that an n by n matrix which has ones on the diagonal and is of norm M, say, as operator on ℓ_p^n, must be invertible on a coordinate subspace of dimension at least $\delta(M)n$. One of the many consequences of this result is that certain finite dimensional subspaces X of L_p contain well-complemented ℓ_p^n subspaces with n proportional to the dimension of X.

1.2 The role of change of density

Generally the structure of the L_p spaces is described for a fixed value of p. However, proofs of many of the results about L_p for a fixed p use the entire scale of L_r spaces, $1 \leq p \leq \infty$. Consider, for example, the proof [28, section 4] that a subspace X of L_p, $2 < p < \infty$, is either isomorphic to a Hilbert space or contains a subspace which is complemented in L_p and is isomorphic to ℓ_p. There one needs only to compare the L_2 norm and the L_p norm on X.

In proofs of other theorems about L_p it is necessary to change the measure before making a comparison between the L_p norm and another norm. Since the technique of changing the measure ("making a change of density") is used in the proofs of most of the results we discuss in this article, we chose to devote this section to describing the change of densities that arise later. It turns out that the framework in which this technique is most naturally used is that of an $L_p(\mu)$ space when μ is a probability. For us there is no loss of generality in restricting to that case since the space ℓ_p^N is isometric to $L_p(\mu)$ when μ is any probability on $\{1, \ldots, N\}$ for which $\mu(\{n\}) > 0$ for each $1 \leq n \leq N$. For such a measure μ we denote $L_p(\mu)$ by $L_p^N(\mu)$, or just L_p^N if μ assigns mass $1/N$ to each integer n, $1 \leq n \leq N$.

A *density* on a probability space (Ω, μ) is a strictly positive μ-measurable function g on Ω for which $\int g \, \mathrm{d}\mu = 1$. Such a density g induces for fixed $0 < p < \infty$ an isometry $M = M_{g,p}$ from $L_p(\mu)$ onto $L_p(g \, \mathrm{d}\mu)$ defined by $Mf = g^{-1/p} f$. Sometimes a gain is achieved by making such a *change of density;* that is, by replacing $L_p(\mu)$ by its isometric copy $L_p(g \, \mathrm{d}\mu)$. The gain usually occurs because for some subspace E of $L_p(\mu)$ and some value of r different from p, the space $M_{g,p}E$ is better situated with respect to $L_r(g\,\mathrm{d}\mu)$ than E is to $L_r(\mu)$. For example, it follows from the Pietsch factorization theorem [28, section 10] that if an operator T from some space X into $L_1(\mu)$ has p-summing adjoint, then by replacing the original L_1 space by another isometrically equivalent L_1 space, TX is actually contained in L_p. Formally,

Proposition 1 *If $T : X \to L_1(\mu)$ (μ a probability measure) has p-summing adjoint, then there is a change of density g and an operator $\tilde{T} : X \to L_p(g\,\mathrm{d}\mu)$ so that $M_{g,p}T$ is the composition of $\tilde{T}$ followed by the canonical injection from $L_p(g\,\mathrm{d}\mu)$ into $L_1(g\,\mathrm{d}\mu)$. Moreover, $\|\tilde{T}\| = \pi_p(T^*)$ as long as TX has full support; i.e., there is no subset $\Omega_0 \subset \Omega$ with $\mu(\Omega_0) < 1$ so that $Tx = 1_{\Omega_0}Tx$ μ-a.e. for every x in X.*

To prove this factorization theorem, assume for simplicity that μ is a regular Borel measure on a compact space Ω and that TX has full support. Get a Pietsch measure ν for the restriction of T^* to $C(K)$, which means that ν is a regular probability measure on K such that $\|T^*f\|^p \leq \pi_p(T^*)^p \int |f|^p \,\mathrm{d}\nu$ for all f in $C(K)$. This same inequality is true if ν is replace by its Radon-Nikodyḿ derivative with respect to μ, so one can assume that ν is of the form $g\,\mathrm{d}\mu$ with $g \geq 0$ and $1 = \nu\Omega = \int g\,\mathrm{d}\mu$. It is easily checked that this g is the desired density provided that g is strictly positive μ-a.e., which it must be since TX has full support. (When TX does not have full support, reason the same way but at the end add a small constant function to g and renormalize.)

The next change of density result, due to D. Lewis [43], gives useful information about finite dimensional subspaces of L_p.

Theorem 2 *Let μ be a probability measure and let E be a k dimensional subspace of $L_p(\mu)$, $0 < p < \infty$, with full support. Then there is a density g so that $M_{g,p}E$ has a basis $\{f_1, \ldots, f_k\}$ which is orthonormal in $L_2(g\,\mathrm{d}\mu)$ and such that $\sum_{n=1}^{k} |f_i|^2 \equiv k$.*

For a proof when $1 \leq p < \infty$ see [43]. The first step is to apply Lewis' lemma ([28, section 8]) to get an operator T from ℓ_2^N onto E for which $\pi_p(T) = 1$ and $I_{p^*}(T^{-1}) = N$. The rest of the proof involves checking that the choices $g := \left(\sum |Te_i|^2\right)^{1/2}$ and $f_i := \sqrt{k}g^{-1/p}Te_i$ satisfy the conditions of Theorem 2. Another proof, for the entire range $0 < p < \infty$, is contained in [55].

Recall that if $T : X \to Y$ is an operator, $\gamma_2(T)$ is the infimum of $\|T\|\|U\|$ over all factorizations $T = SU$ with $U : X \to \ell_2$ and $S : \ell_2 \to Y$.

Theorem 3 *If E is a k-dimensional subspace of $L_p(\mu)$ then there is a projection P from $L_p(\mu)$ onto E with $\gamma_2(P) \leq k^{|1/p-1/2|}$.*

Since the Banach-Mazur distance from ℓ_p^k to ℓ_2^k is $k^{|1/p-1/2|}$ [28, section 8], Theorem 3 implies that the distance of a k dimensional subspace of $L_p(\mu)$ to ℓ_2^k is maximized when the subspace is ℓ_p^k.

To prove Theorem 3, observe first that the case $p=\infty$ follows trivially from the fact proved in [28, section 10] that $\pi_2(I_E)=\sqrt{k}$ for every k dimensional space E. When $p<\infty$, in view of the comments made in [28, section 10], there is no loss in generality in assuming the μ is a probability measure and that E has full support. Theorem 2 says that we can further assume that E has a basis $\{f_1,\dots,f_k\}$ which is orthonormal in $L_2(\mu)$ and such that $\sum_{n=1}^{k}|f_i|^2\equiv k$. Let P be the orthogonal projection onto E. If $p<2$, a simple computation shows that $\|P : L_p(\mu)\to L_2(\mu)\|\le k^{1/p-1/2}$ and so also $\gamma_2(P : L_p(\mu)\to L_p(\mu))\le k^{1/p-1/2}$. The case $p>2$ is even easier.

For a generalization of Theorem 3 to spaces of type $p>1$ see [59, 27.4,28.6].

The change of density in Theorem 2 is used in section 2.1 to show that a k dimensional subspace of L_p well embeds into ℓ_p^n with n not too large. There what is needed is the relation between the L_∞ and the $L_p(g\,\mathrm{d}\mu)$ norms on $M_{g,p}E$. The relevant estimate follows from a trivial observation which we record for later reference.

Lemma 4 *Let μ be a probability measure and let E be a k dimensional subspace of $L_p(\mu)$ which has a basis $\{f_1,\dots,f_k\}$ which is orthonormal in $L_2(\mu)$ and such that $\sum_{n=1}^{k}|f_i|^2\equiv k$. Then for each f in E, $\|f\|_\infty\le k^{1/p}\|f\|_p$ if $0<p\le 2$ and $\|f\|_\infty\le k^{1/2}\|f\|_p$ if $2\le p<\infty$.*

There is a pretty characterization, due to Maurey [44], [62, III.H.10] of subsets of $L_p(\mu)$ which are bounded in L_q after a change of density.

Theorem 5 *Let μ be a probability measure, $0<p<q<\infty$, and S a subset of $L_p(\mu)$ of full support. Then there is a density g so that $M_{g,p}S\subset B_{L_q(g\,\mathrm{d}\mu)}$ if and only if for all finite subsets S_0 of S and $\{a_x : x\in S_0\}\subset[0,1]$,*

$$\|(\sum_{x\in S_0}|a_x x|^q)^{1/q}\|_{L_p(\mu)}\le(\sum_{x\in S_0}|a_x|^q)^{1/q}. \tag{1}$$

Assuming the Pietsch factorization theorem, for the case $p=1$ and when $S=TB_X$ for some operator $T : X\to L_1(\mu)$, Corollary 6 is little more than a restatement of Theorem 5. Recall ([28, Section 5]) that when T is an operator into a Banach lattice, $M^{(q)}(T)$ denotes the q-convexity norm of T.

Corollary 6 *Suppose T is an operator from a Banach space X into an L_1 space. Then $M^{(p)}(T)=\pi_{p^*}(T^*)$.*

A consequence of Theorem 5 that we shall need in section 4 is the result of [27] that an operator on L_p is bounded on L_2 after an appropriate change of density:

Theorem 7 *If $0 < p < \infty$, μ is a probability measure, and T is an operator on $L_p(\mu)$, then $\|M_{g,p} T M_{g,p}^{-1} : L_2(g\,\mathrm{d}\mu) \to L_2(g\,\mathrm{d}\mu)\| \le 2K_G\|T\|$ for some density $g \ge 1/2$, where K_G is the constant in Grothendieck's inequality [28, section 10].*

If one does not wish to make a change to the measure $g\,\mathrm{d}\mu$, then by throwing away the part of the measure space where $g \ge 2$ one gets:

Corollary 8 *If $0 < p < \infty$, μ is a probability measure, and T is an operator on $L_p(\mu)$, then $\|R_A T R_A : L_2(\mu) \to L_2(\mu)\| \le 2K_G\|T\|$ for some set A with $\mu A \ge 1/2$, where R_A is the restriction operator defined by $R_A f := 1_A f$.*

Theorem 7 is a fixed point version of the following factorization consequence (due to Maurey [44]) of Theorem 5.

Theorem 9 *Let μ be probability measure and T be an operator from a Banach lattice X into $L_p(\mu)$, $0 < p \le 2$. Then there is a change of density g and an operator $\tilde{T}$ from X into $L_2(g\,\mathrm{d}\mu)$ so that $M_{g,p}T = I_{2,p}\tilde{T}$, where $I_{2,p}$ is the identity mapping from $L_2(g\,\mathrm{d}\mu)$ into $L_p(g\,\mathrm{d}\mu)$. Moreover, $\|\tilde{T}\| \le K_G M^{(2)}(X)\|T\|$ if TX has full support.*

For the proof of Theorem 9, let $\{x_i\}_{i=1}^n$ be in X and assume that TX has full support. Then using a consequence of Grothendieck's inequality (see [28, section 10]) in the first step we have

$$\begin{aligned} \left\| \left(\textstyle\sum_{i=1}^n |Tx_i|^2\right)^{1/2} \right\|_{L_p(\mu)} &\le K_G\|T\| \left\| \left(\textstyle\sum_{i=1}^n |x_i|^2\right)^{1/2} \right\| \\ &\le K_G\|T\| M^{(2)}(X) \left(\textstyle\sum_{i=1}^n \|x_i\|^2\right)^{1/2}. \end{aligned}$$

Now apply Theorem 5 with S the image under T of the unit sphere of X to get the inequality in the conclusion of Theorem 9. A trivial perturbation argument now gives Theorem 9 when TX does not have full support. In the general case, given $\epsilon > 0$, the density g can be chosen so that $\|\tilde{T}\| \le (K_G M^{(2)}(X) + \epsilon)\|T\|$.

We now deduce Theorem 7 in the range $1 < p < \infty$ from Theorem 9. By duality we can assume that $2 < p < \infty$, and also suppose that $TL_p(\mu)$ has full support. If h is a density and we apply Theorem 9 to the adjoint of the operator

$$L_p(\mu) \xrightarrow{M_{p,h}T} L_p(h\mathrm{d}\mu) \xrightarrow{I_{p,2}} L_2(h\mathrm{d}\mu) \tag{2}$$

we get a density g so that for every f in $L_p(\mu)$,

$$\int |Tf|^2 h^{(p-2)/p} \mathrm{d}\mu \leq K_G^2 \|T\|^2 \int |f|^2 g^{(p-2)/p} \mathrm{d}\mu. \tag{3}$$

Set $g_0 := 1$ and get densities $g_1, g_2, \ldots$ so that for each n, (3) is satisfied with $h := g_n$ and $g := g_{n+1}$. Define $\tilde{g} := \sum_{n=0}^{\infty} 2^{-n-1} g_n^{(p-2)/p}$. This series is absolutely convergent in $L_{(p-2)/p}(\mu)$ to a function whose norm is at most one, so the function $g := \|\tilde{g}\|_{L_1(\mu)} \tilde{g}$ is a density with $g \geq 1/2$ which satisfies, for arbitrary f in $L_p(\mu)$,

$$\int |Tf|^2 g^{(p-2)/p} \mathrm{d}\mu \leq 2K_G^2 \|T\|^2 \int |f|^2 g^{(p-2)/p} \mathrm{d}\mu.$$

This gives Theorem 7 with a slightly better constant when $TL_p(\mu)$ has full support and hence also Theorem 7 as stated in the range $1 < p < \infty$.

For $0 < p \leq 1$, Theorem 7 follows via interpolation from Theorem 10.

Theorem 10 *If $0 < p \leq 1$, $a > 1$, μ is a probability measure, and T is an operator on $L_p(\mu)$, then $\|M_{g,p} T M_{g,p}^{-1} : L_\infty(g\,\mathrm{d}\mu) \to L_\infty(g\,\mathrm{d}\mu)\| \leq a\|T\|$ for some density g.*

To prove Theorem 10, it is enough to find a strictly positive function $\tilde{g}^{1/p}$ in $L_p(\mu)$ so that T maps the order interval $[-\tilde{g}^{1/p}, \tilde{g}^{1/p}]$ into $[-a\|T\|\tilde{g}^{1/p}, a\|T\|\tilde{g}^{1/p}]$. The main point is that every operator T on $L_p(\mu)$, $0 < p \leq 1$, has a modulus $|T|$ which satisfies $\||T|\| = \|T\|$ and $|Tf| \leq |T||f|$ for all f in $L_p(\mu)$ (see, e.g., the remark preceding Theorem 3.2 in [36]). One then defines $\tilde{g}^{1/p} = 1 + \sum_{n=1}^{\infty} a^{-n} \|T\|^{-n} |T|^n 1$. See [27] for details when $p = 1$.

We conclude this section with a change of density lemma due to Pisier [50] which, except for constants, improves Theorem 5. Theorem 11 will be used in section 2.1

Theorem 11 *Let μ be a probability measure, $0 < p < q < \infty$, and S a subset of $L_p(\mu)$. The following statements are equivalent.*

(i) There is a constant C_1 and a density g so that for all measurable sets E and x in S, $\|1_E x\|_{L_p(\mu)} \leq C_1 \left(\int_E g \mathrm{d}\mu\right)^{1/p-1/q}$.

(ii) There is a constant C_2 and a density g so that $M_{g,p} S \subset C_2 B_{L_{q,\infty}(g\,\mathrm{d}\mu)}$.

(iii) There is a constant C_3 so that for all finite subsets S_0 of S and subsets $\{a_x : x \in S_0\}$ of $[0,1]$, $\|\sup_{x \in S_0} |a_x x|\|_{L_p(\mu)} \leq C_3 (\sum_{x \in S_0} |a_x|^q)^{1/q}$.

Moreover, there is a constant $C = C(p,q)$ so that in the implication (i) $\Rightarrow$ (j), $C_j \leq CC_i$.

For a proof of Theorem 11, see [50]. This paper also contains a nice proof of Theorem 5.

2 Subspaces of ℓ_p^n

2.1 Fine embeddings of subspaces of L_p into ℓ_p^n

Let X be a k-dimensional subspace of L_p, $0 < p < \infty$, and let $\epsilon > 0$. What is the smallest n such that X $(1+\epsilon)$-embeds in ℓ_p^n? That is, what is the smallest n such that there is a k-dimensional subspace Y of ℓ_p^n and an isomorphism $T : X \to Y$ with $\|T\|\|T^{-1}\| \leq 1+\epsilon$? Let us denote this n by $N_p(X,\epsilon)$ and the maximal $N_p(X,\epsilon)$, when X ranges over all k-dimensional subspaces of L_p, by $N_p(k,\epsilon)$.

Fixing a basis in X and approximating each of its members by an appropriate simple function, one sees that $N_p(X,\epsilon) < \infty$ for every k-dimensional X and ϵ. Moreover, it depends on X only through its dimension k so that $N_p(k,\epsilon) < \infty$. However, one gets that way a (larger than) exponential in k bound on $N_p(k,\epsilon)$. In this section we shall review results which give much better bounds, close to the best possible ones.

The case $p = 2$ is of course trivial and one can take $n = k$ even for $\epsilon = 0$. The case $p = 1$ has a nice geometrical interpretation: The unit ball of the dual to a k-dimensional subspace of ℓ_1^n is easily seen to be (isometric to) the Minkowski sum of n segments in $\mathbb{R}^k$ (centered at 0) and visa versa. Consequently, the n sought after is the smallest n such that every (centered at zero) body K in $\mathbb{R}^k$ which is the Minkowski sum of arbitrarily many segments (or the limit of such bodies - these are called zonoids) can be ϵ-approximated by a body Z which is the sum of n such segments in the sense that $Z \subset K \subset (1+\epsilon)Z$.

The history of this problem can be traced to the offsprings of Dvoretzky's theorem as discussed in [24]. There the case of X being a k-dimensional Hilbert space (which embeds isometrically in all the L_p spaces) is treated and solved quite satisfactory: For some absolute constant C, $N_p(\ell_2^k,\epsilon)$ is at most $C\epsilon^{-2}k$ for $p < 2$ and $C\epsilon^{-2}pk^{p/2}$ for $p > 2$. This is best possible except that it is unknown whether the factor ϵ^{-2} can be replaced by a smaller function of ϵ. Notice the following nonintuitive special case: The k-dimensional Euclidean ball can be approximated by a body which the sum of a constant (depending on the degree of approximation) times k segments in $\mathbb{R}^k$.

The first result in this direction which did not involve Euclidean spaces was proved in [31]. There it was shown that, for $0 < p \le 1$ and $p < q < 2$, ℓ_q^k (which is known to embed isometrically into L_p whenever $0 < p < q \le 2$) $(1+\epsilon)$-embeds into ℓ_p^n for some $n \le C(p,q,\epsilon)k$. Later the second named author found some initial results indicating in particular that the dependence of n on k in the general problem stated at the beginning of this section is polynomial rather than exponential as one is first tempted to believe. The first proofs were quite complicated and worked only for $p < 2$ as they used fine properties of q-stable random variables. Later a much simpler method was introduced in [53]. Assuming, as we may, that X is already a subspace of ℓ_p^N for some finite N, pick randomly a "few" coordinates and hope that the natural projection onto these coordinates, restricted to X, is a good isomorphism. If we do it with no additional preparation this cannot work. Indeed, X may contain a vector with small support (say one of the unit vector basis elements of ℓ_p^N), in which case the chance that a coordinate in its support is picked is small; of course, if no such coordinate is picked, the said projection cannot be an isomorphism on X. The point is that one wants to change X first to another isometric copy of X in which each element of X is "spread out". This can be done by a change of density. The method of [53] was used with other tools in [8], [57], and some other papers to produce the best known results. In these results it is not known what is the right dependence of n on ϵ and we shall not try to emphasize what are the exact estimates one gets from the proofs. However, the dependence of n on k is best possible except for log factors in some places; we shall pay more attention to this in the sequel.

We now state the best known results.

Theorem 12 *(i) For $p > 2$, $N_p(k,\epsilon) \le C(p,\epsilon)k^{p/2}\log k$.*
(ii) For $1 < p < 2$, $N_p(k,\epsilon) \le C(\epsilon)k\log k(\log\log k)^2$.
(iii) For $p = 1$, $N_1(k,\epsilon) \le C(\epsilon)k\log k$.
(iv) For $0 < p < 1$, $N_p(k,\epsilon) \le C(p,\epsilon)k(\log k)^3$.

Under some conditions ensuring that X does not contain good copies of ℓ_p^m spaces, one gets better results for $p < 2$. Recall that a quasi-normed space X is of type p with constant C for some $0 < p \le 2$ provided

$$\left(\mathbb{E}\|\sum_{i=1}^n \epsilon_i x_i\|^2\right)^{1/2} \le C\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}$$

for all finite sequences $x_1,\ldots,x_n$ of elements of X. The best C is denoted $T_p(X)$. The space L_p, $0 < p \le 2$, is of type p. Recall also that $K(X)$ denotes the K-convexity constant of X, i.e., the norm of the Rademacher projection in $L_2(X)$. See [28] for a brief discussion of these notions (although it is restricted to the normed spaces, which always have type $p \ge 1$) and [45] for a more comprehensive discussion.

Theorem 13

(i) *Let $0 < p < q < 2$ and let $0 < \epsilon, C < \infty$. Then for some constant $C' = C'(p, q, \epsilon, C)$ and all k-dimensional subspace X of L_p with $T_q(X) \le C$, $N_p(X, \epsilon) \le C'k$.*
For $p = 1$ we have a quantitatively better estimate:
(ii) *For all k-dimensional subspaces X of L_1, $N_1(X, \epsilon) \le C(\epsilon)K(X)^2 k$.*

Theorem 13(i) was proved for $p \ge 1$ in [8]. [32] contains the full statement with a different proof. [8] also contains Theorem 12(i) and somewhat weaker versions of Theorem 12$(ii), (iii)$, and Theorem 13(ii). The exact Theorem 12(ii) is contained in [58] while Theorem 13(ii) is the main result of [57]. Theorem 12(iii) follows from it since it is known ([48]) that $K(X) \le C\sqrt{\log k}$ for every k-dimensional subspace X of L_1; see Lemma 17. Finally, Theorem 12(iv) was proved only recently [55] after noticing its omission while writing this survey.

Before describing the proofs, we mention that there are several unsettled problem related to Theorems 12 and 13. The most important one (or at least the one that attracted the most attention) is whether the various log factors and the dependence on the type and K-convexity are really needed. It is strange that the constants in the proofs blow up when X contains ℓ_p^m spaces. Actually, as we shall see below, in at least some of the proofs the worst case occurs when X is isometrically ℓ_p^k. Another problem is the determination of the dependence of $N(\cdot)$ on ϵ. Scant attention has been given to that in the published work. A problem we find particularly interesting is whether there is an "isomorphic" (as oppose to "almost isometric") version to some of the results here. Here is an instance of this problem: is it true that for all $1 < p < 2$ and all $\lambda > 1$ there is a positive constant $C = C(p, \lambda)$ such that whenever $n < \lambda k$, ℓ_p^k C-embeds into ℓ_1^n?

Next we would like to **sketch some of the ideas involved in the proofs of some of the statements of Theorems 12 and 13**. As we already indicated above, a common feature of all the proofs we shall sketch is that, using a change of density, we first find an isometric copy of X with some additional good properties. We delay stating theses properties and the actual change of density that ensure them until later and denote the new space by the same notation X. We may also assume without loss of generality that X lies in a finite (but large) dimensional ℓ_p^m space, say $L_p^m(\mu)$, where μ is a probability measure on $\{1, \dots, m\}$. We denote $\mu_i = \mu(\{i\})$. We would like to show that the restriction operator to a set of relatively few of the m coordinates is a good isomorphism on X. We prefer to do it iteratively by first showing how to find a subset of cardinality at most $m/2$ such that the restriction operator is a (very) good isomorphism on X, provided m is much larger than k. We shall then show how to iterate this procedure. The choice of the subset will be random; for example, it is enough to show that, for an appropriate $\epsilon(k,m)$,

$$\begin{aligned}\text{Ave}_A \sup_{x\in X,\|x\|\le 1} &\left|2\sum_{i\in A}\mu_i|x_i|^p - \sum_{i=1}^m \mu_i|x_i|^p\right| \\ &= \mathbb{E}\sup_{x\in X,\|x\|\le 1}\left|\sum_{i=1}^m \epsilon_i\mu_i|x_i|^p\right| \le \epsilon(k,m). \qquad (4)\end{aligned}$$

Here Ave_A denotes the average over all subsets of $\{1,\ldots,m\}$ while $\mathbb{E}$ is the expectation with respect to the natural product measure on $\{-1,1\}^n$. Indeed, if this is the case then for some A the restriction operators to both A and the complement of A are $\left(\frac{1+\epsilon(k,m)}{1-\epsilon(k,m)}\right)^{1/p}$-isomorphisms and of course either A or its complement A^c is of cardinality at most $m/2$.

Alternatively, in order to find such an A, it is enough to show that

$$P\left(\sup_{x\in X,\|x\|\le 1}\left|\sum_{i=1}^m \epsilon_i\mu_i|x_i|^p\right| > \epsilon(k,m)\right) < 1. \qquad (5)$$

In both cases after iterating we get that as long as

$$\prod_{i=1}^l \left(\frac{1+\epsilon(k,m2^{-i})}{1-\epsilon(k,m2^{-i})}\right)^{1/p} \le 1+\epsilon, \qquad (6)$$

X must $(1+\epsilon)$-embed into ℓ_p^n for some $n \le m2^{-l-1}$.

So this approach reduces the problem to finding good bounds on the quantity in (4) or (5). For technical reasons involving splitting of atoms, as explained in Lemma 14 below, we may need to enlarge m to at most $3m/2$ before making the random choice in each step. This does not effect the process significantly; after the random choice we end up in $\ell_p^{3m/4}$ and this just means that instead of (6) we shall need to ensure that

$$\prod_{i=1}^l \left(\frac{1+\epsilon(k,m(\frac{3}{4})^i)}{1-\epsilon(k,m(\frac{3}{4})^i)}\right)^{1/p} \le 1+\epsilon. \qquad (7)$$

Before proceeding further, we would like to point out why there is very little hope of eliminating the log factors altogether using this approach. Consider, in $\ell_1^{cn\log n}$, n vectors of the form $x_i = (c\log n)^{-1}\sum_{j\in\sigma_i} e_j$ with σ_i, $i=1,\ldots,c\log n$, disjoint sets of cardinality $c \log n$ (assuming it is an integer). It is easy to calculate (and appears in many probability books, sometimes as the coupon collector's problem) that, if c is small, a random choice of $\frac{c}{2} n \log n$ of the coordinates will most likely miss at least one of the σ_i's. Thus the restriction to a "random" subset of cardinality $\frac{c}{2} n \log n$ will not be an isomorphism on $X = \mathrm{span}\{x_i\}$ (which incidentally is isometric to ℓ_1^n). We did not change the density here, but it is not hard to see that a change of density is not going to help to reduce the minimal m for which this procedure works below $cn \log n$ for some absolute $c > 0$.

We now continue sketching the idea of the proofs. We first **sketch a version of the simple argument of [53]** which only gives

$$\begin{aligned} N_p(k,\epsilon) &\le C(\epsilon)k^2 \text{ for } 0 < p < 2 \quad \text{and} \\ N_p(k,\epsilon) &\le C(\epsilon)k^{(p+2)/2} \text{ for } p > 2. \end{aligned} \tag{8}$$

By Lemma 4 there is a change of density on L_p^m so that for all $x \in X$,

$$\begin{aligned} \|x\|_\infty &\le k^{1/p}\|x\|_p, \text{ for } 0 < p < 2 \quad \text{and} \\ \|x\|_\infty &\le k^{1/2}\|x\|_p, \text{ for } p > 2. \end{aligned} \tag{9}$$

Splitting the atoms of this change of density we may assume in addition that $\mu_i \le 4/m$ (paying by enlarging the original m to $3m/2$). This follows from the following simple lemma.

Lemma 14 *Let μ be a probability measure on $\{1, 2, \ldots, m\}$. Then there is an $m \le M \le 3m/2$, a probability measure ν on $\{1, 2, \ldots, M\}$ and a partition $\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ of $\{1, 2, \ldots, M\}$ satisfying*

(i) $\nu(\{i\}) \le 4/m$, $i = 1, \ldots, M$, *and*
(ii) $\sum_{i \in \sigma_j} \nu(\{i\}) = \mu(\{j\})$.

The proof of the lemma is very simple. Split the atoms of μ which have mass larger than $4/m$ into pieces each of size larger than $2/m$. This does not add more than $m/2$ atoms to the original ones, thus ending the proof. Of course $L_p(\mu)$, and thus also X, naturally embeds into $L_p(\nu)$ in particular the estimates in 9 still hold for the image of X in $L_p(\nu)$. This justifies the statement in the paragraph preceding the statement of the lemma.

Fixing an $x \in X$ of norm one, we get by classical deviation inequalities (see for example Proposition 5(iii) in [54]) that for $0 < p < 2$,

$$\begin{aligned} P\left(|\textstyle\sum_{i=1}^m \epsilon_i \mu_i |x_i|^p| > t\right) &\le K \exp(-\delta t^2 / \textstyle\sum_{i=1}^m \mu_i |x_i|^p) \\ &\le K \exp(-\delta t^2 / \max_{1\le i\le m} \mu_i |x_i|^p) \\ &\le K \exp(-\delta' t^2 m/k). \end{aligned}$$

Let $0 < t < 1$ and pick a t-net $\mathcal{N}$ in the unit sphere of X which has at most $(3/t)^k$ elements (see e.g. [47] p. 7 for an easy proof of the existence of such a net). Then as long as $k^2 < ct^2 \log^{-1}(1/t)m$ (for some absolute constant $c > 0$),

$$P\left(\sup_{x\in\mathcal{N}} \left|\sum_{i=1}^m \epsilon_i \mu_i |x_i|^p\right| > t\right) \le K \exp(-\delta' t^2 m/k),$$

from which it is easy to get that, as long as $k \le cm^{1/2}$,

$$P\left(\sup_{x\in X, \|x\|\le 1} \left|\sum_{i=1}^m \epsilon_i \mu_i |x_i|^p\right| > Ck/m^{1/2}\right) < 1. \tag{10}$$

This and (7) implies the desired result (8) for $0 < p < 2$. The treatment for $p > 2$ is similar. ■

This sketch of the argument of [53] was given only to illustrate the basic method involved. We now continue to **sketch the arguments which give the stronger statements of theorems 12 and 13**. The point is to find estimates for the quantities in (4) or (5) which are better than (10). We first relate the second quantity in (4) to a similar one involving Gaussian variables. Let $g_1, \ldots, g_m$ be independent standard Gaussian variables. Then

$$\mathbb{E} \sup_{x\in X, \|x\|\le 1} \left|\sum_{i=1}^m \epsilon_i \mu_i |x_i|^p\right| \le \sqrt{\frac{\pi}{2}} \mathbb{E} \sup_{x\in X, \|x\|\le 1} \left|\sum_{i=1}^m g_i \mu_i |x_i|^p\right|.$$

This version of the "contraction principle" is easy to prove: Replace each of the g_i on the right by $\epsilon_i |g_i|$, where $\{\epsilon_i\}$ is independent of $\{g_i\}$, and replace $\mathbb{E}$ with the successive application of the two expectations $\mathbb{E}_\epsilon \mathbb{E}_g$. Now push the expectation $\mathbb{E}_g$ inside the outer $|\cdot|$ and use the fact that $\mathbb{E}|g_1| = \sqrt{\frac{2}{\pi}}$.

The problem now reduces to evaluating the quantity

$$\mathbb{E} \sup_{x\in X, \|x\|\le 1} \left|\sum_{i=1}^m g_i \mu_i |x_i|^p\right|. \tag{11}$$

Set $G_x = \sum_{i=1}^m g_i \mu_i |x_i|^p$. Then $\{G_x\}_{x \in X, \|x\| \le 1}$ is a Gaussian process indexed by elements of $\mathrm{B}(X)$ and we are required to estimate the expectation of its supremum. This is a well studied area in probability theory (see e.g. [41]), related to the continuity of Gaussian processes. This quantity (or the similar one involving the Rademacher functions in (4)) is evaluated by different means in the proofs of the different parts of Theorems 12 and 13. We shall first present a

Sketch of proof of Theorem 13(ii)**.** Consider the process $\{H_x\}_{x \in X, \|x\| \le 1}$ where $H_x = \sum_{i=1}^m g_i \mu_i x_i$. Then

$$\mathbb{E}G_x^2 = \mathbb{E}H_x^2 \text{ and } \mathbb{E}G_x G_y \ge \mathbb{E}H_x H_y$$

for all $x, y \in X$ of norm at most 1. Slepian's lemma (see e.g. [41, p. 75]) implies now that

$$\mathbb{E} \sup_{x \in X, \|x\| \le 1} \left| \sum_{i=1}^m g_i \mu_i |x_i| \right| \le \mathbb{E} \sup_{x \in X, \|x\| \le 1} \left| \sum_{i=1}^m g_i \mu_i x_i \right|.$$

We shall show that after a change of density (and possibly enlarging m to $3m/2$),

$$\mathbb{E} \sup_{x \in X, \|x\| \le 1} \left| \sum_{i=1}^m g_i \mu_i x_i \right| \le C K(X) \left(\frac{k}{m} \right)^{1/2}. \tag{12}$$

Then one concludes the proof of Theorem 13(ii) by applying (7). To prove (12) we shall use the following two propositions.

Proposition 15 *Let X be a k-dimensional subspace of $L_1^m(\mu)$ with μ a probability measure and let $f_1, \dots, f_k$ be an orthonormal basis of X (considered as a subspace of $L_2^m(\mu)$). Then, one can split some of the atoms of μ to a total of at most $M = 3m/2$ atoms getting a new probability measure ν and a natural embedding $I : L_1^m(\mu) \to L_1^M(\nu)$ satisfying*

$$\mathbb{E} \sup_{y \in Y, \|y\| \le 1} \left| \sum_{i=1}^M g_i \nu_i y_i \right| \le 2m^{-1/2} \mathbb{E} \| \sum_{i=1}^k g_i f_i \|^*.$$

Here $Y = IX$ and $\| \cdot \|^*$ is the dual norm to that of X where duality is given by $\langle \sum a_i f_i, \sum b_i f_i \rangle = \sum a_i b_i$.

Proposition 16 *With the notation of the previous proposition,*

$$\mathbb{E}\Big\|\sum_{i=1}^{k} g_i f_i\Big\|^* \leq \sqrt{2}K(X)\Big\|\Big(\sum_{i=1}^{k} f_i^2\Big)^{1/2}\Big\|_\infty.$$

We now conclude the proof of 13(ii) by applying Lewis' change of density, Theorem 2 to get an orthonormal basis for (an isometric copy of X) satisfying $\Big\|\Big(\sum_{i=1}^{k} f_i^2\Big)^{1/2}\Big\|_\infty \leq n^{1/2}$. ■

Before sketching the proofs of Propositions 15 and 16 we would like to deduce Theorem 12(iii). This follows from the following Lemma (first observed in [48]).

Lemma 17 *Let X be a k-dimensional subspace of L_1. Then $K(X) \leq C\sqrt{\log k}$ for some absolute C.*

PROOF. Using Lewis' change of density we may assume $\|x\|_\infty \leq k\|x\|_1$ for all $x \in X$. Then easily $\|x\|_1 \leq \|x\|_p \leq k^{(p-1)/p}\|x\|_1$ for all $x \in X$. Letting X_p denote X with the L_p norm we get (see [45]???) that

$$K(X) \leq k^{(p-1)/p}K(X_p) \leq k^{(p-1)/p}K(L_p) \leq Ck^{(p-1)/p}\sqrt{p/(p-1)}.$$

Picking p with $p/(p-1) = \log k$ we get the result. ■

We now turn to the

Sketch of proof of Proposition 15. Using the contraction principle in the first inequality, we get

$$\begin{aligned}
\mathbb{E}\sup_{x\in X, \|x\|\leq 1}\Big|\sum_{i=1}^{m} g_i\mu_i x_i\Big| &\leq \max_{1\leq i\leq m}\mu_i^{1/2}\mathbb{E}\sup_{x\in X, \|x\|\leq 1}\Big|\sum_{i=1}^{m} g_i\mu_i^{1/2}x_i\Big| \\
&= \max_{1\leq i\leq m}\mu_i^{1/2}\mathbb{E}\sup_{x\in X, \|x\|\leq 1}\Big|\Big\langle\sum_{i=1}^{m} g_i\mu_i^{-1/2}e_i, x\Big\rangle\Big| \\
&= \max_{1\leq i\leq m}\mu_i^{1/2}\mathbb{E}\sup_{\|\sum a_j f_j\|\leq 1}\Big|\sum_{j=1}^{k} a_j\Big\langle\sum_{i=1}^{m} g_i\mu_i^{-1/2}e_i, f_j\Big\rangle\Big|.
\end{aligned}$$

Put $h_j = \Big\langle\sum_{i=1}^{m} g_i\mu_i^{-1/2}e_i, f_j\Big\rangle$. Then $h_1, \ldots, h_k$ are standard Gaussian variables which are easily seen to be independent (check that $\mathbb{E}h_j h_l = \delta_{jl}$). Thus

$$\mathbb{E}\sup_{\|\sum a_jf_j\|\le 1}\Big|\sum_{j=1}^k a_j\Big\langle\sum_{i=1}^m g_i\mu_i^{-1/2}e_i,f_j\Big\rangle\Big| = \mathbb{E}\sup_{\|\sum a_jf_j\|\le 1}\Big|\sum_{j=1}^k a_jg_j\Big|$$
$$= \mathbb{E}\Big\|\sum_{i=1}^k g_if_i\Big\|^*. \tag{13}$$

It remains to see that splitting the atoms we may also assume that $\mu_i \le 4/m$, but this follows from the splitting of atoms lemma 14. ■

Proof of Proposition 16. The proposition follows easily from [17, Lemma 1] but we present a different proof. Let $\{\epsilon_{ij}\}_{i=1,j=1}^{n,k}$ denote nk independent Rademacher functions (i.e., the ϵ_{ij} are the coordinate functions in the product probability space $\{-1,1\}^{nk}$). In the first inequality below we use the central limit theorem and in the last Khintchine's inequality (with the best constant) [28].

$$\mathbb{E}\,\Big\|\sum_{i=1}^k g_if_i\Big\|^* \le \Big(\lim_{n\to\infty}\mathbb{E}\Big\|n^{-1/2}\sum_{j=1}^k\sum_{i=1}^n\epsilon_{ij}f_j\Big\|^{*2}\Big)^{1/2}$$
$$= \lim_{n\to\infty} n^{-1/2}\sup\Big\{\Big\langle\sum_{j=1}^k\sum_{i=1}^n\epsilon_{ij}f_j,f\Big\rangle\ ;\ \|f\|_{L_2(X)}\le 1\Big\}$$
$$\le K(X)\lim_{n\to\infty} n^{-1/2}\sup\Big\{\sum_{j=1}^k\sum_{i=1}^n\langle\epsilon_{ij}f_j,x_{ij}\rangle\ ;\ x_{ij}\in X,\ \mathbb{E}\Big\|\sum_{i,j}\epsilon_{ij}x_{ij}\Big\|^2\le 1\Big\}$$
$$\le K(X)\lim_{n\to\infty}\sup\Big\{\Big\|(\sum_{i,j}x_{i,j}^2)^{1/2}\Big\|_{L_1}\Big\|\Big(\sum_{j=1}^k\sum_{i=1}^n n^{-1}f_j^2\Big)^{1/2}\Big\|_{L_\infty}\ ;\ \mathbb{E}\Big\|\sum_{i,j}\epsilon_{ij}x_{ij}\Big\|^2\le 1\Big\}$$
$$= K(X)\Big\|\Big(\sum_{j=1}^k f_j^2\Big)^{1/2}\Big\|_{L_\infty}\sup\Big\{\Big\|(\sum_{i,j}x_{i,j}^2)^{1/2}\Big\|_{L_1}\ ;\ \mathbb{E}\Big\|\sum_{i,j}\epsilon_{ij}x_{ij}\Big\|^2\le 1\Big\}$$
$$\le \sqrt{2}K(X)\Big\|\Big(\sum_{j=1}^k f_j^2\Big)^{1/2}\Big\|_{L_\infty}.$$

■

We now sketch a proof of Theorem 13(i). We have chosen a version of the proof of [32] since it is the shortest one. However, this proof does not give a good dependence on ϵ.

Sketch of proof of Theorem 13(i). We assume first as we may that $X\subset \ell_p^m$. The condition of Theorem 11(iii) is easily seen to be satisfied for $S_0 =$ (any finite subset of) the unit ball of X and $C_3 = T_q(X)$. So we can deduce from that theorem that, without loss of generality, $X \subset L_p^m(\mu)$ for some probability measure μ and $\|x\|_{q,\infty} \le C'\|x\|_p$ for all $x \in X$ and some C' which depends only on p, q and $T_q(X)$. By Theorem 11(i) it is also easy to see that we may assume that $\mu_i \ge 1/2m$ (replace g with $\frac{g+1}{2}$). Splitting the large atoms of μ as in Lemma 14 and changing m to $3m/2$ we may assume in addition that $\mu_i \le 4/m$ for all $1 \le i \le 3m/2$.

Put $r = q/p$ and note that for $x \in X$ with $\|x\|_p \le 1$,

$$\begin{aligned} \|\{\mu_i|x_i|^p\}_{i=1}^{3m/2}\|_{r,\infty} &= \max_{1\le j\le 3m/2} j^{1/r}(\mu_j|x_j|^p)^* \\ &= \max_{t>0} t(\#\{i;\ \mu_i|x_i|^p \ge t\})^{1/r} \end{aligned} \tag{14}$$

with $\{(\mu_j|x_j|^p)^*\}$ denoting the decreasing rearrangement of $\{(\mu_j|x_j|^p)\}$.

Using the fact that μ_i is of order $1/m$ and relating the quantity in (14) to

$$\begin{aligned} \|x\|_{q,\infty} &= \max_{t>0} t(\mu(\{i;\ |x_i| \ge t\}))^{1/q} \\ &\approx \max_{t>0} t(m^{-1}\#\{i;\ |x_i| \ge t\})^{1/q}, \end{aligned} \tag{15}$$

we get that the quantity in (14) is at most $Cm^{\frac{p}{q}-1}$ for some C depending only on p, q and $T_q(X)$. We now use the inequality

$$P\Big(\Big|\sum_{i=1}^n \epsilon_i a_i\Big| > t\Big) \le 2\exp(-\delta(t/\|\{a_i\}\|_{r,\infty})^s) \tag{16}$$

which holds for all $t > 0$, $1 < r < 2$ and all sequences of scalars $\{a_i\}$. Here $s = \frac{r}{r-1}$ and δ is a positive constant depending only on r. The inequality is a special case of a martingale inequality of Pisier (see [47], p. 45 for a proof or [54], Proposition 5 for a discussion of this and other similar inequalities). Note that we may assume that $r = q/p < 2$. Using (16) we get from (14) that for all $x \in X$ with $\|x\|_p = 1$,

$$P\Big(\Big|\sum_{i=1}^{3m/2} \epsilon_i\mu_i|x_i|^p\Big| > t\Big) \le 2\exp(-\delta' t^s m)$$

for some δ' depending only on p, q and $T_q(X)$. From this we get, as in the standard argument leading to (10),

$$P\Big(\sup_{x\in X, \|x\|\le 1}\Big|\sum_{i=1}^{3m/2} \epsilon_i\mu_i|x_i|^p\Big| > t\Big) \le 2\exp(k\log(3/t) - \delta'' t^s m). \tag{17}$$

It follows that, as long as $\delta' t^s m \log^{-1}(3/t) > 2k$, We can find a set of at most $3m/4$ coordinates for which the restriction operator is an $(1+2t)$-isomorphism. Choosing $t \approx (\frac{k}{m} \log \frac{m}{k})^{1/s}$ we get that there is a set of at most $3m/4$ coordinates for which the restriction operator is an $(1 + C(\frac{k}{m} \log \frac{m}{k})^{1/s})^{1/p}$- isomorphism where C depends only on p, q and $T_q(X)$. Iterating, we get that as long as

$$\prod_{i=1}^{l} \Big(1 + C\Big(\frac{k}{m(3/4)^i} \log \frac{m(3/4)^i}{k}\Big)^{1/s}\Big)^{1/p} \leq 1 + \epsilon$$

X must $(1+\epsilon)$-embed into ℓ_p^n for some $n \leq m(3/4)^{l+1}$. Note that in each step of the iteration we get a space which is at most 2-isomorphic to X. Since the new space has type q constant at most twice that of the original space we can continue the iteration.

Now it is easy to get the conclusion. ■

2.2 Natural embeddings of ℓ_r^k into ℓ_p^n

The methods described in Section 2.1 as well as all other methods for producing "tight embeddings" are probabilistic and as such are not constructive and do not produce an explicit good embedding. The most basic question concerning explicit embeddings may be to produce a specific good embedding of ℓ_2^n into ℓ_1^m with m proportional to n ($m \leq 2n$, say). We remark in passing that for p an even integer there are specific embeddings (even isometric ones) of ℓ_2^n into ℓ_p^m with the relation between m and n close to the optimal one (in particular, for $p = 4$, $m \approx n^2$). See [38] for that.

A natural approach to get an explicit embedding of ℓ_2^n into ℓ_1^m is to fix a natural subspace X_n of L_1 which is well isomorphic to ℓ_2^n, for example the span of n independent standard Gaussian variables or n independent Rademacher functions, and find a subspace Y_m with $X_n \subset Y_m \subset L_1$, Y_m well isomorphic to ℓ_1^m, and m small. However, this fails in a very strong sense: under the requirements above, the smallest m can be is C^n for some $C > 1$ depending only on the distance of Y_m to ℓ_1^m. Here is a somewhat stronger theorem from [22]. Recall that for an operator $T : X \to Y$, $\gamma_1(T) = \inf \|u\|\|v\|$, where the inf is taken over all $L_1(\nu)$ spaces and all operators $v : X \to L_1(\nu)$, $u : L_1(\nu) \to Y$, satisfying $T = uv$.

Theorem 18 *For every $0 < K < \infty$ there exists a $\delta = \delta(K) > 0$ such that if X is the span, in L_1, of n independent Gaussian variables or n independent Rademacher functions, $X \subset Y \subset L_1$, and the inclusion $J : X \to Y$ satisfies*

$\gamma_1(J) \le K$ then, for some $m \ge e^{\delta n}$, ℓ_1^m is 2-isomorphic to a subspace of Y. In particular dim $Y \ge e^{\delta n}$.

The proof of this is rather technical and we shall not reproduce it here. [22] and [23] contain many refinements and variations of this theorem. Also, [59, p. 201] contains an exposition of the proof of the simplest instance of this class of results; namely, the statement in the "in particular" part of Theorem 18 for X being the span of n independent Rademacher functions.

2.3 ℓ_r^k subspaces of m-dimensional subspaces and quotients of ℓ_p^n

This short section deals with the question of what is the largest k such that ℓ_p^k well embeds into any m-dimensional subspace X of ℓ_p^n, as well as some related questions. We shall not present any proofs but only summarize what is known on this subject.

Note that, since $X = \ell_2^m$ well embeds into ℓ_p^n, $p < 2$, for m proportional to n, the answer to the question above for $p < 2$ is not very interesting (i.e., k must be bounded) unless $n - m = o(n)$. We shall say something about this case latter. For $2 < p < \infty$ the following theorem of Bourgain and Tzafriri [12] basically solves the problem. Let $k = k_p(X, K)$ be the maximal dimension of a subspace Y of X which is K-isomorphic to ℓ_p^k.

Theorem 19 *Let $2 < p < \infty$ and $\epsilon > 0$. Then there are positive constants $c = c(p, \epsilon), C = C(p, \epsilon)$ such that for all $m < n$ and every m-dimensional subspace X of ℓ_p^n,*

$$k_p(X, 1 + \epsilon) \ge c \min\{m^{p^*/2}, (m/n^{2/p})^{p/(p-2)}\}.$$

The result is best possible in the sense that for each $m < n$ there exists a subspace X with $k_p(X, 1+\epsilon) \le C \min\{m^{p^*/2}, (m/n^{2/p})^{p/(p-2)}\}$. ($p^* = p/(p-1)$.)

As can be suspected from the statement, the proof of this result is quite involved and very technical. It uses ideas from the work of Bourgain and Tzafriri concerning restricted invertibility (some of which is surveyed Section 4.1 below) as well as from Bourgain's work on Λ_p sets [6]. We think it worthwhile to find a simpler proof.

As we said above, there can be no similar theorem in the range $p < 2$. However, one can prove a similar theorem for quotients of ℓ_p^n, $1 \le p < 2$. This was done by Bourgain, Kalton and Tzafriri in [7].

Theorem 20 *For each $1 \le p < 2$ there is a constant $C_p > 0$ such that if X*

is an m-dimensional quotient space of ℓ_p^n then

$$k_p(X, C_p) \geq C_p^{-1}(m^p/n^{2(p-1)})^{1/(2-p)},$$

for $p > 1$, while

$$k_1(X, C_1) \geq C_1^{-1}m(/1 + \log n/m).$$

Except for the constants involved the results are best possible.

Note that for m proportional to n the resulting dimension of the contained ℓ_p^k space is also proportional to n. For $p = 1$ this case was observed earlier in [15]. Note also that the conclusion of this theorem is "isomorphic" rather than "almost isometric". We do not know if one can replace the constant C_p in the left hand side of the inequalities by $1 + \epsilon$ (of course paying by replacing the constant in the right hand side by one depending on ϵ).

There is also a version of Theorem 19 for $p = \infty$: If $m \geq n^\delta$, with $\delta > 0$, then every m-dimensional subspace X of ℓ_∞^n contains a well isomorphic copy of ℓ_∞^k with $k \geq c(\delta)m^{1/2}$. This was proved in [21] for m proportional to n, and in [5] in general.

When m is very large there is also a version of Theorem 19 for $p = 1$. It was proved in [25] that for every m-dimensional subspace X of ℓ_1^n,

$$k_1(X, K) \geq c \min\{(n/(n-m)) \log(n/(n-m)), n\}.$$

K and c are universal constants.

3 Finite dimensional subspaces of L_p with special structure

3.1 Subspaces with symmetric basis

In this section we treat the classification of the finite symmetric basic sequences in L_p, $1 \leq p < \infty$ and to some extent also the classification of the finite unconditional basic sequences in L_p, $1 \leq p < 2$.

Recall that a sequence $x_1, \ldots, x_n$ in a quasi normed space X (over $\mathbb{R}$) is said to be *K-symmetric* if for all scalars $\{a_i\}$, all sequences of signs $\{\epsilon_i\}$ and all permutations π of $\{1, \dots, n\}$

$$\|\sum_{i=1}^{n} a_i x_i\| \le K \|\sum_{i=1}^{n} \epsilon_i a_{\pi(i)} x_i\|.$$

If we require the inequality only for the identity permutation the sequence is called *K-unconditional.*

The article [60] treats the classification of symmetric basic sequences in L_p, $p > 2$ so we only state the result (from [29]; see [60], Theorem 4.4).

Theorem 21 *For every $2 < p < \infty$ and every constant K there is a constant D such that any normalized K-symmetric basic sequence in L_p is D equivalent to the unit vector basis of $\mathbb{R}^n$ with the norm*

$$\|\{a_i\}\| = \max\{(\sum |a_i|^p)^{1/p}, w(\sum |a_i|^2)^{1/2}\} \tag{18}$$

for some $w \in (0, 1)$.

Of course, since ℓ_2^n isometrically embeds in L_p, any norm of the form (18) embeds, with constant 2, into L_p.

For $1 \le p < 2$ the structure of the symmetric sequences in L_p is more involved. Let M be a Orlicz function (see [28, Section 5]) and ℓ_M the associated Orlicz sequence space. It turns out that the space ℓ_M embeds isomorphically into L_p if and only if the unit vector basis of ℓ_M is p-convex and 2-concave and this happens if and only if $M(|t|^{1/p})$ is equivalent to a convex function and $M(t^{1/2})$ is equivalent to a concave function on $[0, \infty)$ (see [14]). Recall that two functions $M_1, M_2 : \mathbb{R} \to [0, \infty)$ are equivalent (at 0) if there exist constants K_1, K_2, λ, μ and $x_0 > 0$ such that for all $|x| < x_0$ $K_1 M_2(\lambda x) \le M_1(x) \le K_2 M_2(\mu x)$.

With the right quantifiers, a similar statement holds also for finite dimensional Orlicz spaces, ℓ_M^n. The embedding, when it exists, is as a span of independent, identically distributed symmetric random variables. It follows that if $\{M_j\}_{j=1}^m$ is a collection of Orlicz functions such that $\ell_{M_j}^n$ K-embed in L_p for all j and $\lambda_j > 0$ for all j then also $\mathbb{R}^n$ with the norm $\|\cdot\| = (\sum_{j=1}^m \lambda_j \|\cdot\|_{\ell_{M_j}}^p)^{1/p}$ K-embeds in L_p. The converse is also true.

Theorem 22 *For every constant K and for every $0 < p < 2$ there is a constant D such that given any normalized K-symmetric basic sequence $\{f_i\}_{i=1}^n$ in L_p there is an m and there are m symmetric functions $M_j : \mathbb{R} \to [0, \infty)$, $j = 1, \dots, m$ for which $M_j(0) = 0$, $M_j(|t|^{1/p})$ are convex and $M_j(t^{1/2})$ are concave on $[0, \infty)$ and for some weights λ_j, $\{f_i\}_{i=1}^n$ is D equivalent to the unit* *vector basis of* $\mathbb{R}^n$ *with the norm*

$$\|\cdot\| = \Big(\sum_{j=1}^{m} \lambda_j \|\cdot\|_{\ell_{M_j}}^p\Big)^{1/p}. \tag{19}$$

This theorem is a consequence of the following very nice inequality of Kwapień and Schütt [39].

Theorem 23 *Let* $\{a_{i,j}\}_{i,j=1}^{n} \in \mathbb{R}^{n^2}$ *and denote by* $\{a_i^*\}_{i=1}^{n^2}$ *the decreasing rearrangement of* $\{a_{i,j}\}$*. Then, for any* $1 \le p < \infty$,

$$\begin{aligned} \mathrm{Ave}_\pi\Big(\textstyle\sum_{i=1}^n |a_{i,\pi(i)}|^p\Big)^{1/p} &\le \tfrac{1}{n}\textstyle\sum_{i=1}^n a_i^* + \Big(\tfrac{1}{n}\textstyle\sum_{i=n+1}^{n^2} a_i^{*p}\Big)^{1/p} \\ &\le \quad 5\mathrm{Ave}_\pi\Big(\textstyle\sum_{i=1}^n |a_{i,\pi(i)}|^p\Big)^{1/p}. \end{aligned} \tag{20}$$

Here Ave_π *denotes the average over all permutations of* $1,\dots,n$.

The case $p=1$ of Theorem 22 appears in [39]. The proof for the other values of p is quite similar and we shall sketch it below. The starting point is a lemma which gives another equivalent expression to the ones in (20). For $1 \le p < \infty$ put

$$M(t) = M_p(t) = \begin{cases} (p - \frac{p-1}{n})^{-1} n^{p-1}|t|^p & \text{if } |t| \le 1/n \\ (p - \frac{p-1}{n})^{-1}(p|t| - \frac{p-1}{n}) & \text{if } |t| > 1/n. \end{cases} \tag{21}$$

Note that M_p is an Orlicz function and that $M_p(t^{1/p})$ is a concave function on $[0,\infty)$.

Lemma 24 *Let* $1 \le p < \infty$ *and let* $a_1 \ge a_2 \ge \ldots \ge a_{n^2} \ge 0$*. Then*

$$\frac{1}{4n}\Big\|\{a_i\}_{i=1}^{n^2}\Big\|_{\ell_M} \le \frac{1}{n}\sum_{i=1}^{n} a_i + \Big(\frac{1}{n}\sum_{i=n+1}^{n^2} a_i\Big)^{1/p} \le \frac{p}{3n}\Big\|\{a_i\}_{i=1}^{n^2}\Big\|_{\ell_M} \tag{22}$$

PROOF. Note that $M(pt) \ge pM(t)$ for all t. Consequently, if $\|\{a_i\}_{i=1}^{n^2}\|_{\ell_M} \le \frac{1}{p}$ then,

$$\Big(p - \frac{p-1}{n}\Big)^{-1}\Big(\sum_{a_i > 1/n}\Big(pa_i - \frac{p-1}{n}\Big) + n^{p-1}\sum_{a_i \le 1/n} a_i^p\Big) \le \frac{1}{p}. \tag{23}$$

It follows that $\sum_{a_i>1/n} a_i \leq \sum_{a_i>1/n}\Big(pa_i - \frac{p-1}{n}\Big) < 1$ so that

$$\sum_{i=1}^{n} a_i < 2 \quad \text{and} \quad \#\{i\ ;\ a_i > 1/n\} \leq n.$$

It now follows from (23) that $\sum_{i=1}^{n} a_i + n\Big(n^{-1}\sum_{i=n+1}^{n^2} a_i^p\Big)^{1/p} \leq 3.$

To prove the other side inequality assume $\sum_{i=1}^{n} a_i + n\Big(n^{-1}\sum_{i=n+1}^{n^2} a_i^p\Big)^{1/p} \leq 1.$ Then

$$n^{p-1}\sum_{a_i\leq 1/n} a_i^p \leq n^{p-1}\Big(\sum_{i=n+1}^{n^2} a_i^p + \sum_{a_i\leq 1/n, i\leq n} a_i^p\Big) \leq 2$$

and, since $a_{n+1} \leq \frac{1}{n}\sum_{i=1}^{n} a_i \leq \frac{1}{n}$, $\sum_{a_i>1/n} a_i \leq \sum_{i=1}^{n} a_i \leq 1.$

It follows that (for $n \geq 2$)

$$\begin{aligned}\textstyle\sum_{i=1}^{n^2} M(a_i) &= (p - \tfrac{p-1}{n})^{-1}\Big(\textstyle\sum_{a_i>1/n}(pa_i - \frac{p-1}{n}) + n^{p-1}\sum_{a_i\leq 1/n} a_i^p\Big)\\ &\leq (p - \tfrac{p-1}{n})^{-1}(p+2) \leq 4.\end{aligned}$$

Since $M(4t) \geq 4M(t)$ this concludes the proof. ■

We now turn to a

sketch of the proof of Theorem 22 . Let $\{f_i\}_{i=1}^n$ be a 1-symmetric basic sequence in l_p^m. Then up to a universal constant ,

$$\begin{aligned}\|\textstyle\sum_{i=1}^{n} a_i f_i\| \approx &\quad \Big(\textstyle\sum_{k=1}^{m} \mathrm{Ave}_\pi(\sum_{i=1}^{n}(a_{\pi(i)} f_i(k))^2)^{p/2}\Big)^{1/p}\\ &= \Big(\textstyle\sum_{k=1}^{m} \lambda_k^p \mathrm{Ave}_\pi(\sum_{i=1}^{n}(a_{\pi(i)} f_i(k)/\lambda_k)^2)^{p/2}\Big)^{1/p}\end{aligned}$$

where $\lambda_k^p = \|\{f_i^p(k)\}_{i=1}^n\|_{\ell_{M_{2/p}}}$.

By Theorem 23 and Lemma 24 the last expression is equivalent, with constants depending on p only, to

$$\Big(\frac{1}{n}\sum_{k=1}^{m} \lambda_k^p \|\{a_i^p |f_j(k)|^p/\lambda_k^p\}_{i,j=1}^n\|_{\ell_{M_{2/p}}}\Big)^{1/p}. \qquad (24)$$

Put, for $k = 1, \ldots, m$,

$$N_k(t) = \sum_{i=1}^{n} M_{2/p}(t|f_i(k)|^p/\lambda_k^p).$$

It is easy to check that the N_k are Orlicz functions with $N_k(t^{p/2})$ concave and that

$$\|\{a_i^p|f_j(k)|^p/\lambda_k^p\}_{i,j=1}^n\|_{\ell_{M_{2/p}}} = \|\{a_i^p\}_{i=1}^n\|_{\ell_{N_k}} = \|\{a_i\}_{i=1}^n\|_{\ell_{M_k}}^p$$

Where $M_k(t) = N_k(|t|^p)$. From this and (24) it is easy to conclude the proof. ■

The main result of [46] states that every unconditional basic sequence in L_p, $1 \le p \le 2$, is equivalent to a block basis of a symmetric basic sequence in L_p. (The block basis can be chosen to be with equal coefficients and consequently the embedded space is also complemented. This will not be used here.) Of course if the unconditional basic sequence is finite also the containing symmetric sequence can be taken finite and the constants (of the symmetricity and of the equivalence) can be controlled by the unconditional constant. Recall that given a sequence of n Orlicz functions $\tilde{M} = \{M_i\}_{i=1}^n$ the modular space $\ell_{\tilde{M}}$ is $\mathbb{R}^n$ with the norm $\|x\|_{\tilde{M}} = \inf\{t > 0\ ;\ \sum_{i=1}^n M_i(x_i/t) \le 1\}$. Using [46] and Theorem 22 one can now easily prove the following theorem.

Theorem 25 *For every constant K and for every $1 \le p < 2$ there is a constant D such that given any normalized K-unconditional basic sequence $\{f_i\}_{i=1}^n$ in L_p there is an m, m Orlicz function sequences $\tilde{M}_j = \{M_{ji}\}_{i=1}^n$, $j = 1, \ldots, m$, and positive constants λ_j and μ_{ji} such that $M_{ji}(|t|^{1/p})$ are convex and $M_{ji}(t^{1/2})$ are concave and $\{f_i\}_{i=1}^n$ is D equivalent to the unit vector basis of $\mathbb{R}^n$ with the norm*

$$\|x\| = \Big(\sum_{j=1}^{m} \lambda_j \|\{x_i\mu_{ji}\}_{i=1}^n\|_{\tilde{M}_j}^p\Big)^{1/p}. \tag{25}$$

Of course if $M_{ji}(t^{1/p})$ are K-equivalent to convex functions and $M_{ji}(t^{1/2})$ are K-equivalent to concave functions then any norm as in (25) embeds into L_p with constant depending on K only.

Recall first that the Gordon-Lewis constant, $gl(X)$, of a Banach space X is defined to be

$$gl(X) = \sup\{\gamma_1(T)\ ;\ T : X \to \ell_2,\ \pi_1(T) \le 1\}$$

where π_1 denotes the 1-summing norm (see [28, section 10]) and $\gamma_1(T : X \to Y) = \inf\{\|A\|\|B\|\}$. Here the inf is taken over all L_1 spaces L and over all decompositions $T = AB$ with $B : X \to L$, $A : L \to Y$.

An easy but very useful theorem of Gordon and Lewis ([26] or [59, p. 260]) says that the unconditional constant of every basis of X is at least $gl(X)$.

When $2 < p \le \infty$ there is an abundance of "bad" subspaces of ℓ_p^n. By "bad" we mean here lacking good unconditional bases or even the weaker property of small Gordon-Lewis constant. Recall that L_p^n is L_p over the measure space consisting of n points and endowed with the uniform probability measure.

Theorem 26 *There are positive constants c, c' such that if X is any subspace of L_p^n, $2 < p \le \infty$, satisfying $\dim X \ge cn$ and $\|x\|_{L_2^n} \le 2\|x\|_{L_1^n}$ for all $x \in X$ then $gl(X) \ge c' n^{1/2-1/p}$.*

Theorem 26 was proved by Figiel and Johnson in [21]. A somewhat weaker theorem, still ensuring the abundance of subspaces with large $gl(X)$ was proved earlier by Figiel, Kwapień and Pełczyński [20]. We refer to [59, p. 261] for the proof of Theorem 26 for $p = \infty$. The case $2 < p < \infty$ follows easily since the Banach-Mazur distance between L_p^n and L_∞^n is $n^{1/p}$.

Of course, for some $c > 0$, a random subspace X of L_∞^n of dimension cn satisfies the second assumption of Theorem 26, i.e., $\|x\|_{L_2^n} \le 2\|x\|_{L_1^n}$ for all $x \in X$ (see [47] or [24]). This is why we claim that there is an abundance of subspaces of L_p^n satisfying the conclusion of Theorem 26.

For $1 \le p \le 2$, $gl(X)$ is uniformly bounded for any subspace X of L_p. Nevertheless, there are finite dimensional subspaces of L_p which have only bad unconditional bases (see [37]). This implies that $\sup\{\mathrm{ubc}(X)\ ;\ X \subset \ell_p^n\} \to \infty$ as $n \to \infty$ but no estimates are known.

4 Restricted invertibility and finite dimensional subspaces of L_p of maximal distance to Euclidean spaces

4.1 Restricted invertibility of operators on ℓ_p^n

Motivated by some problems about the structure of finite dimensional subspaces of L_p that will be discussed in section (4.2), in [10] Bourgain and Tzafriri [10] proved Theorem 27 about the restricted invertibility of operators on ℓ_p^n. Qualitatively, this result says (or rather implies) that a bounded operator on ℓ_p^n which has ones on the diagonal must be invertible on a coordinate subspace of proportional dimension (even after projecting back into the coordinate subspace). In order to state Theorem 27 we introduce the following notation. Given a subset σ of $\{1, 2, \ldots, n\}$, let ℓ_p^σ be the span in ℓ_p^n of the unit vector basis vectors $\{e_i : i \in \sigma\}$ and let R_σ be the natural coordinate projection from ℓ_p^n onto ℓ_p^σ.

Theorem 27 *Let $1 \leq p \leq \infty$. For each $\epsilon > 0$ there is $\delta_p(\epsilon) > 0$ so that if T is an n by n matrix, considered as an operator on ℓ_p^n, with zero diagonal, then for each $\epsilon > 0$ there is a subset σ of $\{1, 2, \ldots, n\}$ of cardinality at least $\delta_p(\epsilon) n$ so that $\|R_\sigma T R_\sigma\|_p \leq \epsilon \|T\|_p$. Consequently, if $\epsilon\|T\|_p < 1$, then $\|(R_\sigma(I + T)R_\sigma)^{-1}\|_p \leq \frac{1}{1-\epsilon\|T\|_p}$.*

The case $p = 1$, as well as the case $p = \infty$, which follows by duality, of Theorem 27 was proved earlier by the second author [30] and, independently, by Bourgain [4, p.113]. Bourgain's argument gives more than what is stated in Theorem 27; namely, that there exists a splitting $\sigma_1, \sigma_2, \ldots \sigma_k$ of $\{1, 2, \ldots, n\}$ into $k = k(\epsilon)$ disjoint sets so that for each $1 \leq i \leq k$, $\|R_{\sigma_i} T R_{\sigma_i}\|_1 \leq \epsilon \|T\|_1$. Whether this strengthening of Theorem 27 remains valid for other values of p is open. For $p = 2$ this matrix splitting question is particularly interesting because it is equivalent to the Kadison-Singer problem [35] whether every pure state on ℓ_∞ has a unique extension to a pure state on $B(\ell_2)$. For a discussion of the matrix splitting problem on ℓ_2 see [16].

For a proof (due to K. Ball) of the matrix splitting result for ℓ_1^n which gives the estimate $k(\epsilon) \leq 2/\epsilon$ see [11]. Notice that in the case of ℓ_1^n, there is no loss of generality in treating only matrices with nonnegative entries because as operators on ℓ_1^n, T and $|T|$ have the same norm. Berman, Halpern, Kaftal, and Weiss [3] independently used ideas similar to those used by Ball to prove an ℓ_2^n splitting result for matrices with nonnegative entries (which of course does not give a splitting result on ℓ_2^n for general matrices). It is amusing and instructive to note that the matrix splitting result for ℓ_1^n *formally* implies the matrix splitting result for nonnegative matrices on ℓ_p^n for all $1 \leq p < \infty$ via a change of density argument of L. Weis [61]. Here is the idea. First, after a change of density, a positive operator T on $L_p(\mu)$ (μ a probability) is nicely bounded on $L_\infty(\mu)$. This follows from the fact that T maps the order interval $[-f, f]$ for some $f > 0$ into the order interval $[-(1+\epsilon)\|T\|f, (1+\epsilon)\|T\|f]$, which in turn follows via iteration and summing (as in the proof of 7) from the inclusion $T[-g, g] \subset [-Tg, Tg]$, which is valid for all $g \geq 0$ because T is a positive operator. Secondly, by working both with T and T^*, one can get that, after a suitable change of density, a positive operator T on $L_p(\mu)$ (μ a probability) is nicely bounded on both on $L_1(\mu)$ and on $L_\infty(\mu)$. Now specialize this to a positive operator on ℓ_p^n. The above discussion shows that this operator can be modeled as a (positive) operator T on $L_p^n(\mu)$ for some probability on $\{1, 2, \ldots, n\}$ in such a way that $\|T : L_1^n(\mu) \to: L_1^n(\mu)\|$ and $\|T : L_\infty^n(\mu) \to: L_\infty^n(\mu)\|$ do not exceed $(2+\epsilon)\|T : L_p^n(\mu) \to: L_p^n(\mu)\|$. Since we know the splitting result for operators on ℓ_1^n and ℓ_∞^n, the result for ℓ_p^n follows by interpolation.

The proof of Theorem 27 for ℓ_p^n uses ℓ_1^n in a more serious way. By duality, it is enough to prove the case $1 < p < 2$, so we restrict to this range of p. The "natural" approach to prove Theorem 27 is to show that if the set σ is chosen at random from among the subsets of $\{1, 2, \ldots, n\}$ having cardinality δn for small enough $\delta = \delta(\epsilon)$, then with big probability $\|R_\sigma T R_\sigma\|_p \leq \epsilon \|T\|_p$. This is, unfortunately, obviously wrong (consider the right shift operator). However, regarding ℓ_p^n as L_p^n (so that the injection $I_{p,1} : L_p^n \to L_1^n$ has norm one), it *is* true that for most such choices of σ the operator $W := I_{p,1} R_\sigma T R_\sigma$ has norm not exceeding $\epsilon \delta^{1/p^*} \|T\|_p$ (the factor δ^{1/p^*} is natural; it goes away when we regard W as an operator from L_p^σ into L_1^σ). For a proof, which is nice, and not particularly difficult, see [10, Proposition 1.10]. What is remarkable is that the $p = 2$ case in Theorem 27 then follows immediately by an application of Grothendieck's inequality via Proposition 1! Indeed, W^* maps L_∞^n into a Hilbert space and thus has 2-summing norm at most $K_G\|W\|$, where K_G is Grothendieck's constant (see [28, section 10]. Applying Proposition 1, we conclude that $W = DU$ for some operator U on L_2^n of norm at most $\epsilon K_G \|T\|_2$ and some norm one diagonal operator $D : L_2^n \to L_1^n$. The operator D is multiplication by some function g which has norm one in L_2^n, hence U is defined by the formula $Uf = \frac{R_\sigma T R_\sigma f}{g}$, whence $\|\frac{R_\sigma T R_\sigma f}{g}\|_2 \leq \epsilon K_G \|T\|_2 \|f\|_2$ for all f in L_2^n. Since $\|g\|_2 = 1$, $\mathbb{P}[|g| \geq \sqrt{2/\delta}] \leq \delta/2$; that is, $|g(j)| \geq \sqrt{2/\delta}$ for at most $(\delta n)/2$ coordinates j. Throwing away any of these which are in the set σ and calling the resulting set σ', we have that σ' has cardinality at least $(\delta n)/2$ and $\|R_{\sigma'} T R_{\sigma' f}\|_2 \leq \sqrt{2} K_G \epsilon \|T\|_2$.

That completes the outline of the proof of Theorem 27 in the Hilbertian case $p = 2$, which by itself has many applications (see [10] and [11]). It is, however, the other cases of Theorem 27 that have applications to the structure theory of finite dimensional L_p spaces. The proof we sketched for $p = 2$ does not carry over because operators from L_∞ into L_{p^*} need not be p^*-summing when

$1 < p < 2$ (they are s-summing for all $s > p^*$, but this is not sufficient). Lemma 28 is used to get around this problem.

Lemma 28 *Suppose* $1 < r < p < 2$, μ, ν, τ *are probabilities,* X *is a subspace of* $L_p(\mu)$, $T : X \to L_1(\nu)$ *and* $U : X \to L_p(\tau)$ *are operators so that*

$$\|Tx\|_1 \leq \|Ux\|_r, \quad x \in X. \tag{26}$$

Then $\pi_{p^*}(T^*) \leq \gamma(r,p)\|U\|$, *where* $\gamma(r,p)$ *is the* L_r*-norm of a* p*-stable random variable which is normalized in the* L_1*-norm.*

In view of Corollary 6, to prove Lemma 28 it suffices to verify the estimate $M^{(p)}(T) \leq \gamma(r,p)\|U\|$, which obviously follows from the the following inequality, valid for all finite sets of vectors x_i in X:

$$\|(\sum_i |Tx_i|^p)^{1/p}\|_1 \leq \gamma(r,p)\|(\sum_i |Ux_i|^p)^{1/p}\|_p. \tag{27}$$

To prove (27), write $(\sum\limits_i |Tx_i|^p)^{1/p} = \mathbb{E}|\sum\limits_i f_i T x_i|$ where the f_i are independent p-stable random variables with $\mathbb{E}|f_i| = 1$ and interchange the expectation and the L_1 norm to see that this is estimated from above by

$$\|(\mathbb{E}|\sum_i f_i U x_i|^r)^{1/r}\|_r. \tag{28}$$

But $(\mathbb{E}|\sum\limits_i f_i U x_i|^r)^{1/r} = \gamma(r,p)(\sum\limits_i |Ux_i|^p)^{1/p}$, so (28) is dominated by the right side of (27).

The main tool for proving Theorem 27 is:

Proposition 29 *For each* $0 < \epsilon < 10^{-2}$ *there is* $\rho = \rho(\epsilon) > 0$ *so that if* $1 < r < 2$ *and* S *is an operator on* L_2^n $(n \geq n(\epsilon, r))$ *with* $\|S\|_2 \leq \rho$, *then there is a subset* σ *of* $\{1, 2, \ldots, n\}$ *of cardinality at least* ϵn *so that for all* x,

$$\|R_\sigma S x\|_1 \leq C\epsilon(\|x\|_r + \|Sx\|_r), \tag{29}$$

where C *is a numerical constant.*

Here we are using as usual the L_p^n normalization. The important thing is that the factor $C\epsilon$ in (29) is a gain over the trivial factor ϵ^{1/r^*}.

The $p < 2$ case of Theorem 27 follows easily from Lemma 28, Proposition 29, and the $p = 2$ case of Theorem 27. Indeed, Corollary 8 says that to prove Theorem 27 it is enough to consider norm one operators on L_p^n which have norm at most $2J_p$ as operators on L_2^n. The $p=2$ case in Theorem 27 then says that it is enough to consider norm one operators on L_p^n which have norm at most $\rho = \rho(\epsilon)$ as operators on L_2^n. Now if S is an operator on L_p^n with $\|S\|_p = 1$ and $\|S\|_2 \le \rho$, apply Proposition 29 to get σ. Define $T = R_\sigma S R_\sigma$, considered as an operator from L_p^n into L_p^σ (so that both the domain and range are L_p spaces of a probability), and set $U = 2C(R_\sigma \oplus S R_\sigma)$, considered as an operator from L_p^n into $L_p^{2n} \overset{2}{\approx} L_p^n \oplus_p L_p^n$. Then for, say, $r = \frac{1+p}{2}$, we have from Proposition 29 that for all x in $\mathbb{R}^n$, $\|Tx\|_{L_1^\sigma} \le \|Ux\|_{L_r^{2n}}$. Lemma 28 then gives that $\pi_{p^*}(T^*) \le \gamma(r,p)\|U\|_p \le 4C\gamma(r,p)$. Changing back to the L_p^n normalization (that is, regarding $R_\sigma S R_\sigma$ as an operator on L_p^n), we have that $\pi_{p^*}(R_\sigma S^* R_\sigma) \le \epsilon^{1/p} 4C\gamma(r,p)$. The completion of the proof of Theorem 27 is now just as in the $p=2$ case.

For a proof of Proposition 29 see [11, Section 5]. Here is the idea: By the same kind of reasoning that works in the first part of the proof of Theorem 27, most choices of a subset σ of $\{1,\dots,n\}$ of cardinality ϵn make it true that $\|R_\sigma Sx\|_1 \le C\epsilon\|Sx\|_r$ for all vectors x whose support has cardinality at most τn (an estimate for $\tau = \tau(\epsilon)$ comes out of the proof). For a general vector x, apply this to $1_A x$, where A is the set of the τn largest coordinates of $|x|$, and use the smallness of $\|S\|_2$ to take care of $x - 1_A x$.

Corollary 30 gives a method, to be exploited in the next section, for building complemented copies of ℓ_p^n in subspaces of L_p.

Corollary 30 *Let $1 \le p \le \infty$ and $1 > \epsilon > 0$. If T is a norm one operator from ℓ_p^n into $L_p(\mu)$ and there are disjoint μ-measurable sets A_i so that $\|1_{A_i} T e_i\|_p \ge \epsilon$ for each $1 \le i \le n$, then there is an operator S from $L_p(\mu)$ into ℓ_p^n with $\|S\| \le 2/\epsilon$ and a subset σ of $\{1,\dots,n\}$ of cardinality at least $\delta_p(\epsilon/2)$ so that $ST_{|\ell_p^\sigma}$ is the identity of ℓ_p^σ. Consequently, $TR_\sigma S$ is a projection of $L_p(\mu)$ onto span$\{Te_i\}_{i\in\sigma}$.*

Here the function $\delta_p(\cdot)$ is taken from Theorem 27. For the proof of Corollary 30, take norm one vectors h_i, supported on A_i, so that $\langle Te_i, h_i\rangle = \|Te_i\|_p$. Define $\tilde{S}$ from $L_p(\mu)$ to ℓ_p^n by $\tilde{S}f = \sum_i^n \|1_{A_i} T e_i\|_p{}^{-1}\langle f, h_i\rangle e_i$. Then $\|\tilde{S}\| \le 1/\epsilon$ and $\tilde{S}T$ has ones on the diagonal when represented as a matrix with respect to the unit vector basis for ℓ_p^n. Therefore, by Theorem 27 there is a subset σ of $\{1,\dots,n\}$ of cardinality at least $\delta_p(\epsilon/2)$ so that $R_\sigma \tilde{S} T R_\sigma$ is a 1/2-perturbation of the identity operator on ℓ_p^σ. Set $S = (R_\sigma \tilde{S} T R_\sigma)^{-1} R_\sigma S$.

4.2 Subspaces of L_p with maximal distance to Hilbert spaces

In this section we give some applications to the isomorphic structure theory of finite dimensional subspaces of L_p, $1 \le p < \infty$, of the restricted invertibility theorem from the previous section. First, however, we mention what is known in the isometric and almost isometric theories. It is classical (see [1]) that a subspace of $L_p(\mu)$ which is isometric to an $L_p(\nu)$ space must be 1-complemented in $L_p(\mu)$. This was extended in [19] (for $p = 1$) and [52] (for $1 < p < \infty$) to the almost isometric setting:

Theorem 31 *For each $1 \le p < \infty$ there is $\delta_p > 0$ so that if $0 < \epsilon < p$ and X is a subspace of $L_p(\mu)$ which is $1+\epsilon$-isomorphic to an $L_p(\nu)$ space, then X is $C_p(\epsilon)$-complemented in $L_p(\mu)$. Moreover, $C_p(\epsilon) \to 1$ as $\epsilon \to 0$.*

Although Theorem 31 is stated for general subspaces X, it reduces easily to the case where X is finite dimensional, which is the case treated in [19] and [52]. A simpler proof of the $p > 1$ case is given in [11, Section 3].

Conversely, it is classical that a 1-complemented subspace of $L_p(\mu)$ is isometric to a $L_p(\nu)$ space (see [40]), but the almost isometric version of this result is open even for finite dimensional subspaces. (Here a result for finite dimensional spaces does not seem to imply formally a result for infinite dimensional spaces.)

It was mentioned already in section 1.2 that an n-dimensional subspace of $L_p(\mu)$ has Banach-Mazur distance at most $n^{|1/p-1/2|}$ to ℓ_2^n. The converse to this is also true for $1 \le p < 2$; see [10, Section 4] (as noted in [10], the $p = 1$ case is essentially done in [21]). For $2 < p < \infty$ it is open whether there is an n-dimensional subspace of $L_p(\mu)$, not isometric to ℓ_p^n, which has Banach-Mazur distance $n^{|1/p-1/2|}$ to ℓ_2^n.

We turn now to the isomorphic theory. Here and in the sequel we shall speak qualitatively about finite dimensional spaces, similar to the way one speaks about infinite dimensional spaces. In order for our statements to have content, statements should be quantified so as not to depend on dimension (but there may be dependence on other parameters). It takes only one example to illustrate why this convention is followed by Banach space local theorists. One of the main results we shall discuss is the following:

Theorem 32 *If X is an n dimensional complemented subspace of $L_p(\mu)$, $1 < p < \infty$, whose distance to ℓ_2^n is of maximal order, then X contains a subspace Y of proportional dimension (say, k) which is isomorphic to ℓ_p^k and complemented in $L_p(\mu)$.*

The meaning of this statement is that there are functions $f(p, \delta, C) > 0$ and $g(p, \delta, C) < \infty$ (defined for $1 < p < \infty$, $\delta > 0$, and $C < \infty$) so that if X is an n-dimensional C-complemented subspace of $L_p(\mu)$ for some n and p, and $d(X,\ell_2^n) \geq \delta n^{|1/p-1/2|}$, then for some $k \geq f(p,\delta,C)n$ there is a k-dimensional subspace Y of X with $d(Y,\ell_p^k) \leq g(p,\delta,C)$ and Y is $g(p,\delta,C)$-complemented in $L_p(\mu)$.

Theorem 32 is also true for $p=1$ ([30]), but in a stronger form – the complementation assumption is not needed. Given the method of [30], the main step in the proof of Theorem 32 which we sketch below is Theorem 27.

For $p>2$ the complementation assumption in Theorem 32 is essential; this follows easily from Theorem 26. Whether the complementation assumption is needed when $1<p<2$ is open.

A criterion for building a complemented copy of ℓ_p^n in a subspace X of $L_p(\mu)$ was given in Corollary 30. The idea for getting an operator T taking values in X which satisfies the hypothesis of the corollary is first to find vectors f_i, $1 \leq i \leq n$, in the unit sphere of X for which the norm of the square function $S_2(\{f_i\}_{i=1}^n)$ (where $S_r(\{f_i\}_{i=1}^n) := (\sum\limits_{i=1}^n |f_i|^r)^{1/r}$ for $r<\infty$ and $S_\infty(\{f_i\}_{i=1}^n) := \max\limits_{1\leq i\leq n} |f_i|$) is nearly extremal; that is, the norm of $S_2(\{f_i\})$ is proportional to $n^{1/p}$. Once one has such a sequence, one uses an extrapolation argument. Indeed, for $1 \leq p < 2$, we shall outline an argument which deals with a somewhat more general situation. Suppose $\bar{f}_i$, $1 \leq i \leq n$, are in $L_p(\mu)$; and $S_2(\{\bar{f}_i\}_{i=1}^n) \geq \delta n^{1/p-1/2}(\sum_{i=1}^n \|\bar{f}_i\|^2)^{1/2}$. Set $\theta = p/2$ and $f_i = \bar{f}_i/\|\bar{f}_i\|$. Then, using the factorization $S_2^2(\{\bar{f}_i\}_{i=1}^n) = \sum_{i=1}^n |\bar{f}_i|^p \|\bar{f}_i\|^{2-p}|f_i|^{2-p}$,

$$\begin{aligned}
\delta n^{1/p-1/2}(\textstyle\sum_{i=1}^n \|\bar{f}_i\|^2)^{1/2} &\leq \|S_2(\{\bar{f}_i\}_{i=1}^n)\|_p \\
&\leq \|S_p(\{\bar{f}_i\|\bar{f}_i\|^{(2-p)/p}\}_{i=1}^n)^\theta \, S_\infty(\{f_i\}_{i=1}^n)^{(1-\theta)}\|_p \\
&\leq \|S_p(\{\bar{f}_i\|\bar{f}_i\|^{(2-p)/p}\}_{i=1}^n)\|_p^\theta \, \|S_\infty(\{f_i\}_{i=1}^n)\|_p^{(1-\theta)} \\
&= (\textstyle\sum_{i=1}^n \|\bar{f}_i\|^2)^{1/2} \, \|S_\infty(\{f_i\}_{i=1}^n)\|_p^{(1-\theta)}
\end{aligned}$$

Letting $\tilde{A}_j = [|f_j| = S_\infty(\{f_i\}_{i=1}^n)]$ and setting $A_j = \tilde{A}_j \sim \cup_{i<j}\tilde{A}_i$, we get disjoint sets A_i so that $\delta^{2/(2-p)}n^{1/p} \leq \|\sum\limits_{i=1}^n 1_{A_i}f_i\|_p$. Thus $\|1_{A_i}f_i\|_p$ is larger than $\delta^{2/(2-p)}/2$ for at least $(\delta^{2p/(2-p)}/2)n$ values of i.

Next suppose that $2<p<\infty$ and that f_i, $1 \leq i \leq n$, are norm one vectors in $L_p(\mu)$ for which $\|S_2(\{f_i\}_{i=1}^n)\| \leq Cn^{1/p}$. Set $\theta = 2/p$. Then

$$\begin{aligned}
n^{1/p} = \|S_p(\{f_i\}_{i=1}^n)\| &\leq \|S_2(\{f_i\}_{i=1}^n)^\theta S_\infty(\{f_i\}_{i=1}^n)^{1-\theta}\| \\
&\leq \|S_2(\{f_i\}_{i=1}^n)\|^\theta \|S_\infty(\{f_i\}_{i=1}^n)\|^{1-\theta} \leq C^\theta n^{\theta/p}\|S_\infty(\{f_i\}_{i=1}^n)\|^{1-\theta}.
\end{aligned}$$

Thus there are disjoint sets A_i so that $\|1_{A_i} f_i\|_p$ is larger than $1/(2C^{2/(p-2)})$ for at least $\left(2C^{\frac{2p}{p-2}}\right)^{-1} n$ values of i.

To derive Theorem 32 from Corollary 30 we need the sequence $\{f_i\}$ for which $\|1_{A_i} f_i\|_p$ is bounded away from zero also to be dominated by the ℓ_p^k basis. This can be accomplished in the following way.

Let $2 < p < \infty$ and let $P : L_p(\mu) \to L_p(\mu)$ be the projection with range X. Assume that, for some k proportional to n, $\{f_i\}_{i=1}^k$ is a sequence in the ball of the range of P^* in $L_{p^*}(\mu)$ satisfying $\|1_{A_i} f_i\|_p > \delta > 0$ for some disjoint sets $\{A_i\}_{i=1}^k$. Put $g_i = P(|f_i|^{p^*-1}\mathrm{sgn}(f_i)1_{A_i})$. Note that $\{g_i\}_{i=1}^k$ is dominated by the ℓ_p^k basis and in particular the norm of $S_2(\{g_i\})$ is dominated by $k^{1/p}$ and that the norms of the g_i-s are bounded away from zero since $\|g_i\| \geq \int g_i f_i \geq \delta^p$. It follows that for a proportion of the g_i-s there are disjoint B_i-s for which $\|g_i 1_{B_i}\|$ are bounded away from zero. Now Corollary 30 applies to produce a subspace of X of proportional dimension isomorphic to ℓ_p of its dimension and complemented in $L_p(\mu)$. The case $1 < p < 2$ follows by duality. We remark that this part of the argument, which is what we had in mind when we wrote [30], is considerably simpler than the one presented in [10].

Finally, we briefly indicate how to find in any n-dimensional subspace X of $L_p(\mu)$ of maximal order distance from ℓ_2^n a sequence $\{f_i\}_{i=1}^n$ with $\|S_2(\{f_i\})\|$ of order $n^{1/p-1/2}(\sum_{i=1}^n \|\bar{f}_i\|^2)^{1/2}$. Assume for example that $1 < p < 2$. By a theorem of Kwapien, [59, Th. 13.15] the type 2 constant of X is of order $n^{1/p-1/2}$, and by a theorem of Tomczak [59, Th. 25.6], this type 2 constant is attained, up to a universal constant, on n vectors; i.e., there are vectors $\{f_i\}_{i=1}^n$ in X satisfying $\|S_2(\{f_i\})\| \geq \delta n^{1/p-1/2}(\sum_{i=1}^n \|f_i\|^2)^{1/2}$.

5 Complemented subspaces

5.1 Fine embeddings of complemented subspaces of L_p in ℓ_p^n

Let X be a k-dimensional subspace of L_p, $1 \leq p < \infty$, and assume there is a projection of norm K from L_p onto X. Let $\epsilon > 0$ and denote by $P_p(X, K, \epsilon)$ the smallest n such that X $(1+\epsilon)$-embeds in ℓ_p^n as a $(1+\epsilon)K$-complemented subspace; i.e., the smallest n such that there is a k-dimensional subspace Y of ℓ_p^n, an isomorphism $T : X \to Y$ with $\|T\|\|T^{-1}\| \leq 1 + \epsilon$, and a projection of norm at most $(1+\epsilon)K$ from ℓ_p^n onto Y. Also, denote by $P_p(k, K, \epsilon)$ the maximal $P_p(X, K, \epsilon)$ when X ranges over all k-dimensional subspaces of L_p. If $1 < p < \infty$, looking at the case of $X = \ell_2^k$ shows that, at least for some K depending on p, $P_p(k, K, \epsilon) \geq c(p) \max\{k^{p/2}, k^{p^*/2}\}$, where $p^* = p/(p-1)$. It turns out that, except for logarithmic terms, one can achieve this bound.

Theorem 33 *([8])* $P_p(k,K,\epsilon) \le C(p,K,\epsilon)\max\{k^p/2, k^{p^*}\}(\log k)^{C(p)}$

For $p = 1$ one gets a better result

Theorem 34 $P_1(k,K,\epsilon) \le C(K,\epsilon)k\log k$

The proofs follow the method of Section 2.1. Given a k-dimensional $X \subset \ell_p^m$, $1 < p < \infty$, and a projection P on ℓ_p^m with range X, we let $Y = P^*\ell_{p^*}^m$ and following the scheme of Section 2.1 we find a subset $\Omega \subset \{1,2,\dots,m\}$ of the right cardinality such that

$$\Big|m|\Omega|^{-1}\sum_{i\in\Omega}|x_i|^p - \sum_{i=1}^m |x_i|^p\Big| < \epsilon, \qquad x \in X,\ \|x\|_p = 1 \tag{30}$$

$$\Big|m|\Omega|^{-1}\sum_{i\in\Omega}|y_i|^{p^*)} - \sum_{i=1}^m |y_i|^{p^*}\Big| < \epsilon, \quad y \in Y,\ \|y\|_{p^*} = 1 \tag{31}$$

$$\Big|m|\Omega|^{-1}\sum_{i\in\Omega}x_iy_i - \sum_{i=1}^m x_iy_i\Big| < \epsilon, \quad x \in X, y \in Y,\ \|x\|_p = \|y\|_{p^*} = 1. \tag{32}$$

For $p = 1$, (31) causes of course a problem. Fortunately enough, in this case the relevant inequality (stating that the restriction to Ω is a good isomorphism when restricted to $Y \subset \ell_\infty^m$) follows immediately from (30), (32) and the fact that a restriction to a subset is a norm one operator on ℓ_∞^m. Finally, it is easy to see that (30), (31) and (32) imply the desired result.

5.2 Finite decomposition and uniqueness of complements

The class of finite dimensional well complemented subspaces of L_p is a rich class, at least in the range $1 < p \ne 2 < \infty$. It follows from the previous section that the same is true for the well complemented k-dimensional subspaces of ℓ_p^n as long as k is smaller than a specific power of n. The situation for larger k is not known. In particular, it is not known whether a well complemented subspace of ℓ_p^n of proportional dimension is well isomorphic to a ℓ_p^k space. It is also not known whether ℓ_p^n is "primary"; i.e., whether $\ell_p = X \oplus Y$ implies that either X or Y is well isomorphic to some ℓ_p^k (here we mean of course that the isomorphism constant should depend on the norm of the projection on X with kernel Y (and p) only). It is true however that if Y is "small" enough then X is well isomorphic to an ℓ_p^k space. There are two known instances of this statement with two different notions of smallness. These were proved in [33] following somewhat weaker results in [9] for the first theorem and [8] for the second.

Theorem 35 *Assume* $1 < p < \infty$, $\ell_p^n = X + Y$, $d(Y, \ell_2^k) \le K$, *and there is a projection* P *from* ℓ_p^n *onto* X *with kernel* Y *and* $\|P\| \le L$. *Then* $d(X, \ell_p^{n-k}) \le C = C(K, L, p)$.

Theorem 36 *Let* $1 < p < \infty$ *and put* $\bar{p} = \max\{p, p^*\}$. *Assume* $\ell_p^n = X + Y$ *with dim*$Y = k < n^{2/\bar{p}}/(\log n)^{C(p)}$ *and that there is a projection* P *from* ℓ_p^n *onto* X *with kernel* Y *and* $\|P\| \le L$. *Then* $d(X, \ell_p^{n-k}) \le C = C(L, p)$.

The proofs relay on a simple but powerful "decomposition method" applied to a "path" of complemented subspaces of L_p together with a proposition concerning "uniqueness of complements".

We begin with the decomposition method from [33], which improved on the original finite dimensional decomposition method introduced in [2].

Lemma 37 *Let* $1 \le p \le \infty$. *Let* Z_i, $i = 1, \dots, m$, *be Banach spaces and, for* $i = 1, \dots, m$, *let* Y_i *be a* K*-complemented subspace of* Z_i. *Assume also* $d(Y_i, Y_{i-1}) \le L$, $i = 1, \dots, m$. *Then* $Y_0 \oplus \sum_{i=1}^m \oplus_p Z_i$ *is* CKL*-isomorphic to* $Y_m \oplus \sum_{i=1}^m \oplus_p Z_i$. *In particular, if* $Z_i = \ell_p^{s_i}$, $i = 1, \dots, m$, *and* $s = \sum_{i=1}^m s_i$ *then* $Y_0 \oplus \ell_p^s$ *is* CKL*-isomorphic to* $Y_m \oplus \ell_p^s$.

The proof is very simple. Let X_i be the complement of Y_i in Z_i, $i = 1, \dots, m$. Then,

$$Y_0 \oplus \sum_{i=1}^m \oplus_p Z_i \approx \begin{pmatrix} Y_0 \\ \oplus_p \\ 0 \end{pmatrix} \oplus_p \begin{pmatrix} Y_1 \\ \oplus_p \\ X_1 \end{pmatrix} \oplus_p \begin{pmatrix} Y_2 \\ \oplus_p \\ X_2 \end{pmatrix} \oplus_p \dots \oplus_p \begin{pmatrix} Y_m \\ \oplus_p \\ X_m \end{pmatrix}. \tag{33}$$

We now shift the top row to the right

$$\approx \begin{pmatrix} Y_0 \\ \oplus_p \\ X_1 \end{pmatrix} \oplus_p \begin{pmatrix} Y_1 \\ \oplus_p \\ X_2 \end{pmatrix} \oplus_p \dots \oplus_p \begin{pmatrix} Y_{m-1} \\ \oplus_p \\ X_m \end{pmatrix} \oplus_p \begin{pmatrix} Y_m \\ \oplus_p \\ 0 \end{pmatrix} \tag{34}$$

to get

$$Y_0 \oplus \sum_{i=1}^m \oplus_p Z_i \approx Y_m \oplus \sum_{i=1}^m \oplus_p Z_i. \tag{35}$$

This picturesque proof can easily be justified.

To prove Theorem 35 we assume that $p > 2$ and $k = 2^m$. We build a path $Y_0, \dots, Y_m$ of spaces connecting $Y_0 = \ell_2^k$ with $Y_m = \ell_p^k$ with $d(Y_i, Y_{i-1}) \leq 2$. This can be done in several ways but, since we also want Y_i to be well isomorphic to a well complemented subspace of $\ell_p^{s_i}$ with s_i as small as possible, we take Y_i to be the ℓ_p sum of 2^i copies of $\ell_2^{2^{m-i}}$.

Given any $\delta > 0$, $\ell_2^{2^{m-i}}$ embeds as a complemented subspace, with constants depending only on p and δ, into $\ell_p^{r_i}$ with $r_i = [\delta 2^{(m-i)p/2}]$. The fact that this holds for some $\delta < \infty$ is exposed for example in [24]. To get it for all $\delta > 0$ represent ℓ_2^{uv} as the ℓ_p sum of u copies of ℓ_2^v (introducing of course a constant depending on u and p), embed each summand complementably in an appropriate ℓ_p space, and take the ℓ_p sum of these u spaces as the containing ℓ_p space.

It follows that Y_i well embeds complementably into $\ell_p^{s_i}$ for $s_i = [\delta 2^i 2^{(m-i)p/2}]$. Lemma 37 implies then that, for $s = \sum_{i=1}^m s_i$, $\ell_2^k \oplus \ell_p^s \approx \ell_p^{s+k}$.

The assumption that ℓ_2^k is well complemented in ℓ_p^n implies that $n = \gamma k^{p/2}$ for some γ bounded away from zero (this is a result of [2]). It is now easy to see that, with the right choice of δ, $s = n - k$ and $\ell_2^k \oplus \ell_p^{n-k} \approx \ell_p^n$.

It remains to show that if also $\ell_2^k \oplus X \approx \ell_p^n$ then $X \approx \ell_p^{n-k}$. This follows from the following simple "uniqueness of complement" result of [9] in the form proved in [33]. In the statement "+" denotes a direct sum of subspaces and the isomorphism constants implicit in the notation " $\approx$ " of the conclusions depend only on the constants for the " $\approx$ " and the projections in the hypotheses.

Proposition 38 *Assume $Z = Y + X = H + G$ with $H \subset X$, and assume $H \approx Y \oplus W$. Then $X \approx G \oplus W$.*
In particular, if for $i = 1, 2$, $Z = Y_i + X_i = H_i + G_i$ with $Y \approx Y_i$, $G \approx G_i$, $H \approx H_i \subset X_i$ and $H \approx Y \oplus W$. Then $X_1 \approx X_2$.

To end the proof of Theorem 35 we only have to show that X, the complement of $Y \approx \ell_2^k$ in ℓ_p^n, contains a subspace well isomorphic to ℓ_2^k and well complemented. Since n is of order at least $k^{p/2}$, the general theory of Euclidean sections as exposed in [24] implies that, for some $\delta > 0$, X contains a subspace U_1 well isomorphic to $\ell_2^{\delta k}$. U_1 is automatically well complemented (see [44] or [18, p. 46]). Now find another copy of $\ell_2^{\delta' k}$ in the complement of U_1 in X and iterate.

The proof of Proposition 38 ([33]) is very easy: Put $F = X \cap G$. Then $Z = Y + X = Y + F + H$ and consequently $G \approx Y \oplus F$. Thus $X = F + H \approx F \oplus Y \oplus W \approx G \oplus W$. ■

The proof of Theorem 36 follows the same outline as that of the proof of Theorem 35. Given a k-dimensional well complemented subspace Y_0 of L_p, $2 < p < \infty$ we find a path $Y_0, Y_1, \ldots, Y_m$ of well complemented subspaces of L_p with $d(Y_i, Y_{i-1}) < 4$, Y_m well isomorphic to ℓ_2^k, and $m = [\log_2 k]$. We then use Theorem 33 to embed each of Y_i in a well complemented fashion in a low dimensional ℓ_p space and continue in a way similar to the proof of Theorem 35.

To build the path of spaces Y_i, apply first the change of density of Theorem 2 and then that of Theorem 7 to get that without loss of generality $\|y\|_p \leq 2n^{1/2-1/p}\|y\|_2$ for each $y \in Y_0$ and the projection P from L_p onto Y_0 is also well bounded with respect to the L_2 norm. Put

$$Y_i = \{(y, 2^i y);\ y \in Y_0\} \subset L_p \oplus_p L_2, \quad i = 1, 2, \ldots, m.$$

Then clearly $d(Y_i, Y_{i-1}) < 4$, $i = 1, 2, \ldots, m$, Y_m is well isomorphic to ℓ_2^k and it only remains to show that each of Y_i is well isomorphic to a well complemented subspace of L_p.

Since P is bounded in both L_p and L_2, Y_i is well complemented in $Z_i = \{(z, 2^i z);\ y \in L_p\} \subset L_p \oplus_p L_2$. Finally, by a theorem of Rosenthal [51] (see also [1]) Z_i is well isomorphic to a well complemented subspace of L_p. ■

References

[1] Alspach, D. E.; Odell, E. W. This handbook.

[2] Bennett, G.; Dor, L. E.; Goodman, V.; Johnson, W. B.; Newman, C. M. On uncomplemented subspaces of L_p, $1 < p < 2$. Israel J. Math. 26 (1977), no. 2, 178–187.

[3] Berman, Kenneth; Halpern, Herbert; Kaftal, Victor; Weiss, Gary Matrix norm inequalities and the relative Dixmier property. (English) Integral Equations Operator Theory 11 (1988), no. 1, 28–48.

[4] Bourgain, J. New classes of $\mathcal{L}^p$-spaces. Lecture Notes in Mathematics, 889. Springer-Verlag, Berlin-New York, 1981.

[5] Bourgain, J. Subspaces of l_N^∞, arithmetical diameter and Sidon sets. Probability in Banach spaces, V (Medford, Mass., 1984), 96–127, Lecture Notes in Math., 1153, Springer, Berlin-New York, 1985.

[6] Bourgain, J. Bounded orthogonal systems and the $\Lambda(p)$-set problem. Acta Math. 162 (1989), no. 3-4, 227–245.

[7] Bourgain, J.; Kalton, N. J.; Tzafriri, L. Geometry of finite-dimensional subspaces and quotients of L_p. Geometric aspects of functional analysis (1987–88), 138–175, Lecture Notes in Math., 1376, Springer, Berlin-New York, 1989.

[8] Bourgain, J.; Lindenstrauss, J.; Milman, V. M. Approximation of zonoids by zonotopes Acta Math. series 162 (1989), 73–141.

[9] Bourgain, J.; Tzafriri, L. Complements of subspaces of l_p^n, $p \geq 1$, which are uniquely determined. Geometrical aspects of functional analysis (1985/86), 39–52, Lecture Notes in Math., 1267, Springer, Berlin-New York, 1987.

[10] Bourgain, J.; Tzafriri, L. Invertibility of "large" submatrices with applications to the geometry of Banach spaces and harmonic analysis. Israel J. Math. 57 (1987), no. 2, 137–224.

[11] Bourgain, J.; Tzafriri, L. Restricted invertibility of matrices and applications. Analysis at Urbana, Vol. II (Urbana, IL, 1986–1987), 61–107, London Math. Soc. Lecture Note Ser., 138, Cambridge Univ. Press, Cambridge-New York, 1989.

[12] Bourgain, J.; Tzafriri, L. Embedding l_p^k in subspaces of L_p for $p > 2$. Israel J. Math. 72 (1990), no. 3, 321–340 (1991).

[13] Bourgain, J.; Tzafriri, L. On a problem of Kadison and Singer. J. Reine Angew. Math. 420 (1991), 1–43.

[14] Bretagnolle, Jean; Dacunha-Castelle, Didier. Application de l'étude de certaines formes linéaires aléatoires au plongement d'espaces de Banach dans des espaces L^p. (French) Ann. Sci. École Norm. Sup. (4) 2 1969 437–480.

[15] Carl, Bernd; Pajor, Alain. Gelfand numbers of operators with values in a Hilbert space. Invent. Math. 94 (1988), no. 3, 479–504.

[16] Davidson, K. R.; Szarek, S. J. Banach space theory and "local" operator theory. This handbook.

[17] Davis, W. J.; Milman, V. D.; Tomczak-Jaegermann, N. The distance between certain n-dimensional Banach spaces. Israel J. Math. 39 (1981), no. 1-2, 1–15.

[18] Diestel, Joe; Jarchow, Hans; Tonge, Andrew. Absolutely summing operators. Cambridge studies in advanced mathematics 43, Cambridge University Press, Cambridge-New York-Port Chester-Melbourne-Sydney, 1995.

[19] Dor, Leonard E. On projections in L_1. Ann. of Math. (2) 102 (1975), no. 3, 463–474.

[20] Figiel, Tadeusz; Kwapień, Stanisław; Pełczyński, Aleksander. Sharp estimates for the constants of local unconditional structure of Minkowski spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25 (1977), no. 12, 1221–1226.

[21] Figiel, T.; Johnson, W. B. Large subspaces of l_∞^n and estimates of the Gordon-Lewis constant. Israel J. Math. 37 (1980), no. 1-2, 92–112.

[22] Figiel, T.; Johnson, W. B.; Schechtman, G. Factorizations of natural embeddings of l_p^n into L_r. I. Studia Math. 89 (1988), no. 1, 79–103.

[23] Figiel, T.; Johnson, W. B.; Schechtman, G. Factorizations of natural embeddings of l_p^n into L_r. II. Pacific J. Math. 150 (1991), no. 2, 261–277.

[24] Giannopoulos, A. A.; Milman, V. M. Euclidean structure in finite dimensional normed spaces. This handbook.

[25] Gluskin, E. D.; Tomczak-Jaegermann, N.; Tzafriri, L. Subspaces of l_p^N of small codimension. Israel J. Math. 79 (1992), no. 2-3, 173–192.

[26] Gordon, Y.; Lewis, D. R. Absolutely summing operators and local unconditional structures. Acta Math. 133 (1974), 27–48.

[27] Johnson, W. B.; Jones, L. Every L_p operator is an L_2 operator. Proc. Amer. Math. Soc. 72 (1978), no. 2, 309–312.

[28] Johnson, W. B.; Lindenstrauss, J. Basic concepts in the geometry of Banach spaces. This handbook.

[29] Johnson, W. B.; Maurey, B.; Schechtman, G.; Tzafriri, L. Symmetric structures in Banach spaces. Mem. Amer. Math. Soc. 19 (1979), no. 217

[30] Johnson, W. B.; Schechtman, G. On subspaces of L_1 with maximal distances to Euclidean space. Proceedings of research workshop on Banach space theory (Iowa City, Iowa, 1981), 83–96, Univ. Iowa, Iowa City, Iowa, 1982.

[31] Johnson, W. B.; Schechtman, G. Embedding l_p^m into l_1^n. Acta Math. series 149 (1982), 71–85.

[32] Johnson, W. B.; Schechtman, G. Remarks on Talagrand's deviation inequality for Rademacher functions, in: Functional analysis (Austin, TX, 1987/1989) (Lecture Notes in Math., 1470, Springer, Berlin, 1991) 72–77.

[33] Johnson, W. B.; Schechtman, G. On the distance of subspaces of l_p^n to l_p^k. Trans. Amer. Math. Soc. 324 (1991), no. 1, 319–329.

[34] Johnson, W. B.; Schechtman, G. Computing p-summing norms with few vectors. Israel J. Math. 87 (1994), no. 1-3, 19–31.

[35] Kadison, Richard V.; Singer, I. M. Extensions of pure states. Amer. J. Math. 81 1959 383–400.

[36] Kalton, N. J. The endomorphisms of $L_p(0 \leq p \leq i)$. Indiana Univ. Math. J. 27 (1978), no. 3, 353–381.

[37] Ketonen, Timo On unconditionality in L_p spaces. Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes No. 35 (1981).

[38] Koldobsky, A.; König, H.. Aspects of the isometric theory of Banach spaces. This handbook.

[39] Kwapień, Stanisław; Schütt, Carsten Some combinatorial and probabilistic inequalities and their application to Banach space theory. Studia Math. 82 (1985), no. 1, 91–106.

[40] Lacey, H. Elton. The isometric theory of classical Banach spaces. Die Grundlehren der mathematischen Wissenschaften, Band 208. Springer-Verlag, New York-Heidelberg, 1974. x+270 pp.

[41] Ledoux, M.; Talagrand, M.. Probability in Banach spaces, Isoperimetry and processes. Springer-Verlag, Berlin, (1991).

[42] Lewis, D. R. Finite dimensional subspaces of L_p. Studia Math. 63 (1978), no. 2, 207–212.

[43] Lewis, D. R. Ellipsoids defined by Banach ideal norms. Mathematika 26 (1979), no. 1, 18–29.

[44] Maurey, Bernard Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p. (French) With an English summary. Astérisque, No. 11. Société Mathématique de France, Paris, 1974.

[45] Maurey, B. This handbook.

[46] Maurey, B.; Schechtman, G. Some remarks on symmetric basic sequences in L_1. Compositio Math. 38 (1979), no. 1, 67–76.

[47] Milman, V. D.; Schechtman, G. Asymptotic theory of finite–dimensional normed spaces. Lecture Notes in Mathematics, 1200. Springer-Verlag, Berlin-New York (1986).

[48] Pisier, G. Un thèorème sur les opérateurs linéaires entre espaces de Banach qui se factorisent par un espace de Hilbert. (French) Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 1, 23–43.

[49] Pisier, Gilles On the dimension of the l_p^n-subspaces of Banach spaces, for $1 \leq p < 2$. Trans. Amer. Math. Soc. 276 (1983), no. 1, 201–211.

[50] Pisier, Gilles Factorization of operators through $L_{p\infty}$ or L_{p1} and noncommutative generalizations. Math. Ann. 276 (1986), no. 1, 105–136.

[51] Rosenthal, Haskell P. On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables. Israel J. Math. 8 1970 273–303.

[52] Schechtman, G. Almost isometric L_p subspaces of $L_p(0, 1)$. J. London Math. Soc. (2) 20 (1979), no. 3, 516–528.

[53] Schechtman, G. . More on embedding subspaces of L_p in l_r^n. Compositio Math. series 61 (1987), 159–169.

[54] Schechtman, G. This handbook.

[55] Schechtman, G.; Zvavitch, A. Embedding subspaces of L_p into ℓ_p^N, $0 < p < 1$. Math. Nachr., to appear.

[56] Szarek, S. J. Computing summing norms and type constants on few vectors. Studia Math. 98 (1991), no. 2, 147–156.

[57] Talagrand, M. Embedding subspaces of L_1 into l_1^N. Proc. Amer. Math. Soc. series 108 (1990), 363–369.

[58] Talagrand, M. Embedding subspaces of L_p in l_p^N. Geometric aspects of functional analysis (Israel, 1992–1994), 311–325, Oper. Theory Adv. Appl., 77, Birkhäuser, Basel, 1995.

[59] Tomczak-Jaegermann, Nicole Banach-Mazur distances and finite-dimensional operator ideals. Pitman Monographs and Surveys in Pure and Applied Mathematics, 38. Longman Scientific & Technical, Harlow, 1989.

[60] Tzafriri, L. Uniqueness of structure in Banach spaces. This handbook.

[61] Weis, L. Integral operators and changes of density. Indiana Univ. Math. J. 31 (1982), no. 1, 83–96.

[62] Wojtaszczyk, P. Banach spaces for analysts. Cambridge Studies in Advanced Mathematics, 25. Cambridge University Press, Cambridge, 1991