EMBEDDING BANACH SPACES INTO THE SPACE OF BOUNDED FUNCTIONS WITH COUNTABLE SUPPORT

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Abstract. We prove that a WLD subspace of the space $\ell_\infty^c(\Gamma)$ consisting of all bounded, countably supported functions on a set Γ embeds isomorphically into ℓ_∞ if and only if it does not contain isometric copies of $c_0(\omega_1)$. Moreover, a subspace of $\ell_\infty^c(\omega_1)$ is constructed that has an unconditional basis, does not embed into ℓ_∞, and whose every weakly compact subset is separable (in particular, it cannot contain any isomorphic copies of $c_0(\omega_1)$).

1. Introduction

It is classical that every separable Banach space is isometrically isomorphic to a subspace of ℓ_∞, the space of bounded sequences with the supremum norm. Since every weakly compact subset of ℓ_∞ is separable, any weakly compactly generated space; in particular, any reflexive space; that admits an injective bounded linear operator into ℓ_∞ must be separable. (A Banach space is *weakly compactly generated*, WCG for short, when it contains a weakly compact subset whose linear span is dense.) For this reason, there is no bounded linear injection from $c_0(\Gamma)$ into ℓ_∞ when the set Γ is uncountable. Nevertheless, $c_0(\omega_1)$ sits naturally as a subspace of ℓ_∞'s close cousin, the space $\ell_\infty^c(\Gamma)$, which consists of all bounded scalar-valued functions on Γ that are non-zero on at most countably many points in Γ.

The aim of this note is to study Banach spaces that embed into $\ell_\infty^c(\omega_1)$ but do not embed into ℓ_∞ and their relation to containment of isomorphic or even isometric copies of $c_0(\omega_1)$. In particular, we prove that a non-separable weakly Lindelöf determined (WLD) subspace of $\ell_\infty^c(\Gamma)$ contains an isometric copy of $c_0(\omega_1)$. (A Banach space X is WLD provided for some set Γ there exists an injective linear operator $T\colon X^* \to \ell_\infty^c(\Gamma)$ that is continuous as a map from X^* with the weak* topology to $\ell_\infty^c(\Gamma)$ with the topology of pointwise convergence.)

The notation is standard. We just mention that all operators are assumed to be bounded and linear, and an isomorphism is a bounded linear operator that is bounded below on the unit sphere of its domain. We consider cardinal numbers as initial ordinal numbers. A cardinal number λ is *regular* whenever a set of cardinality λ cannot be expressed as a union of fewer than λ sets that have cardinality less than λ.

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2. The results

For a WLD space X, the density character of X^* endowed with the weak* topology is the same as of X with the norm topology ([3, Proposition 5.40]), hence the existence of a bounded linear injection $T\colon X \to \ell_\infty$ implies that X is separable. Indeed, by Goldstine's theorem, $\ell_\infty^* = \ell_1^{**}$ is weak*-separable. By injectivity of T, the adjoint map $T^*\colon \ell_\infty^* \to X^*$ has dense range, and so X^* has a weak*-separable dense subspace, therefore it must be weak*-separable. As X is WLD, also X must be separable. We shall employ this fact to construct copies of $c_0(\omega_1)$ in non-separable WLD subspaces of $\ell_\infty^c(\Gamma)$. (In the case of $X = c_0(\Gamma)$ the result was already recorded in [7] and [4, Lemma 6]).

Theorem 2.1. *Let Γ be a set and let X be a WLD subspace of $\ell_\infty^c(\Gamma)$. Then the following are equivalent:*

(i) *X is separable,*
(ii) *X embeds into ℓ_∞,*
(iii) *there exists a bounded linear injection from X into ℓ_∞,*
(iv) *X does not contain a subspace that is isomorphic to $c_0(\omega_1)$,*
(v) *X does not contain a subspace that is isometrically isomorphic to $c_0(\omega_1)$.*

In particular, every reflexive subspace of $\ell_\infty^c(\Gamma)$ is separable.

First we introduce some notation. Let Γ be a set and let $\Lambda \subseteq \Gamma$. Consider the contractive projection $P_\Lambda\colon \ell_\infty^c(\Gamma) \to \ell_\infty^c(\Gamma)$ given by

$$(P_\Lambda f)(\gamma) = \begin{cases} f(\gamma) & \gamma \in \Gamma, \\ 0, & \gamma \in \Lambda \setminus \Gamma \end{cases} \qquad \big(f \in \ell_\infty^c(\Gamma)\big).$$

We identify the range of P_Γ with the space $\ell_\infty^c(\Gamma)$. Certainly, when Γ is countably infinite, the range of P_Γ is isometrically isomorphic to ℓ_∞.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are clear. We have already observed in the introduction that for WLD spaces (iii) $\Rightarrow$ (i). We shall prove now that (v) $\Rightarrow$ (iii) by contraposition.

Suppose that there is no bounded linear injection from X into ℓ_∞. In particular, for every countable set $\Lambda \subset \Gamma$, the restriction operator $P_\Gamma|_X$ is not injective, which means that $P_\Lambda f_\Lambda = 0$ for some unit vector f_Λ in the range of P_Λ. Consequently, it is possible to choose by transfinite recursion an uncountable family of pairwise disjoint countable subsets $(\Lambda_\alpha)_{\alpha<\omega_1}$ and unit vectors $f_\alpha \in \ell_\infty(\Lambda_\alpha) \cap X$. Then, the closed linear span of $\{f_\alpha\colon \alpha < \omega_1\}$ is isometric to $c_0(\omega_1)$. □

Remark 2.2. Theorem A implies that the unit sphere of a non-separable WLD subspace of $\ell_\infty^c(\Gamma)$ contains an uncountable symmetrically (1+)-separated subset, that is, a set A such that $\|x \pm y\| > 1$ for distinct $x, y \in A$; this is because $c_0(\omega_1)$ has this property. This observation complements [2, Corollary 3.6], where it was proved that WLD spaces of density greater than the continuum contain such sets. It should be noted however that not every renorming of $c_0(\omega_1)$ embeds isometrically into $\ell_\infty^c(\Gamma)$, as at least under the Continuum

Hypothesis, there exists a renorming of $c_0(\omega_1)$ that does not contain isometric copies of itself ([2, Theorem 5.9]).

The hypothesis of being WLD cannot be removed completely from the statement of Theorem A. Before we give a relevant example, we prove a simple lemma.

Lemma 2.3. *Let X be a subspace of $\ell_\infty^c(\omega_1)$. If X embeds into ℓ_∞, then there is a $\alpha < \omega_1$ such that the operator $P_{[0,\alpha)}|_X$ is bounded below; that is, bounded below on the unit sphere of X.*

Proof. Since ℓ_∞ is injective, there is an operator $J\colon \ell_\infty^c(\omega_1) \to \ell_\infty$ so that the restriction of J to X is bounded below (indeed, J is any extension of an embedding of X into ℓ_∞ to $\ell_\infty^c(\omega_1)$). It is therefore enough to observe that for any operator $J\colon \ell_\infty^c(\omega_1) \to \ell_\infty$ there is a countable ordinal α such that J vanishes on $\ell_\infty^c\big([\alpha,\omega_1)\big)$ (because then J factors through the quotient $\ell_\infty^c(\omega_1)/\ell_\infty^c\big([\alpha,\omega_1)\big)$, which is isomorphic to $\ell_\infty^c\big([0,\alpha)\big) \cong \ell_\infty$). $\square$

Theorem 2.4. *There exists a subspace Z of $\ell_\infty^c(\omega_1)$ with an unconditional basis such that Z does not embed into ℓ_∞ and $c_0(\omega_1)$ does not embed into Z. Moreover, Z contains an isomorphic copy of $\ell_1(\omega_1)$ and there is an injective operator from Z into ℓ_∞.*

Proof. Since ℓ_∞ contains an isometric copy of $\ell_1(\mathfrak{c})$, we may fix a countable set Γ_0 in ω_1 and a family of unit vectors $(f_\alpha)_{\alpha<\omega_1}$ in the range of P_{Γ_0} that is isometrically equivalent to the unit vector basis of $\ell_1(\omega_1)$. Let $\alpha = \gamma(\alpha) + n(\alpha)$ be a countable ordinal, where $\gamma(\alpha)$ is a (possibly zero) limit ordinal and $n(\alpha)$ is a finite ordinal. We set

$$z_\alpha = e_\alpha + \tfrac{1}{n(\alpha)+1} z_\alpha \qquad (\alpha < \omega_1),$$

where $(e_\alpha)_{\alpha<\omega_1}$ is the standard unit vector basis of $c_0(\omega_1) \subset \ell_\infty^c(\omega_1)$. Let Z be the closed linear span of z_α ($\alpha < \omega_1$) in $\ell_\infty^c(\omega_1)$. Then $(z_\alpha)_{\alpha<\omega_1}$ is a 1-unconditional basis for Z. Moreover the operator $P_{\Gamma_0}|_Z$ is injective, so Z cannot contain non-separable weakly compact sets as every weakly compact subset of ℓ_∞ is separable. In particular, X does not contain any isomorphic copies of $c_0(\omega_1)$.

By Lemma 2.3, Z does not embed into ℓ_∞ because the operator $P_{[0,\alpha)}|_X$ is not bounded below for any countable ordinal α.

Finally, we remark that P_{Γ_0} is an isomorphism when restricted to the copy of $\ell_1(\omega_1)$ spanned by the family $\{z_\alpha\colon \alpha < \omega_1, n(\alpha) = 0\}$. $\square$

Remark 2.5. As was noted in the proof of Theorem 2.4, the example Z does not embed into ℓ_∞ but there is an injective operator from Z into ℓ_∞. The first space having these properties was constructed in [6], but that space does not have an unconditional basis.

2.1. **An extension to higher densities.** For every cardinal λ, there is a natural generalisation of the space $\ell_\infty^c(\Gamma)$; namely, $\ell_\infty^\lambda(\Gamma)$, the subspace of $\ell_\infty(\Gamma)$ that comprises functions whose supports have cardinality strictly less than λ. In this notation, $\ell_\infty^c(\Gamma) = \ell_\infty^{\omega_1}(\Gamma)$. We note that Theorem 2.1 has a natural counterpart for spaces $\ell_\infty^\lambda(\Gamma)$, whenever λ is a regular cardinal.

Theorem 2.6. *Let Γ be a set, λ a regular cardinal number, and let X be a subspace of $\ell_\infty^\lambda(\Gamma)$. Then the following are equivalent:*

(i) $w^*\mathrm{-dens}\, X^* < \lambda$,
(ii) *X embeds into $\ell_\infty(\kappa)$ for some $\kappa < \lambda$,*
(iii) *there exists a bounded linear injection from X into $\ell_\infty(\kappa)$ for some $\kappa < \lambda$,*
(iv) *X does not contain a subspace that is isomorphic to $c_0(\lambda)$,*
(v) *X does not contain a subspace that is isometrically isomorphic to $c_0(\lambda)$.*

Proof. Note that (iii) $\Rightarrow$ (i). Indeed, if there is a bounded linear injection T from X into $\ell_\infty(\kappa)$, then T^* has weak*-dense range. As (by Goldstine's theorem) the weak* density of $\ell_\infty(\kappa)^*$ is κ, the conclusion follows. The implication (iii) $\Rightarrow$ (iv) follows from the fact that the weak* density of $c_0(\lambda)^*$ is λ and thus there is no bounded linear injection from $c_0(\lambda)$ to $\ell_\infty(\kappa)$ for $\kappa < \lambda$ (see, *e.g.*, [3, Fact 4.10]). (We remark in passing that the implication (ii) $\Rightarrow$ (iv) was proved directly in [5, Proposition 3.4].) As previously, it is enough to prove that (v) $\Rightarrow$ (iii).

Assume contrapositively that for all $\kappa < \lambda$ there is no bounded linear injection from X into $\ell_\infty(\kappa)$. In particular, $|\Gamma| \geqslant \lambda$ as otherwise $\ell_\infty^\lambda(\Gamma) = \ell_\infty(\Gamma)$ but X is a subspace of $\ell_\infty^\lambda(\Gamma)$. Without loss of generality we may assume that $|\Gamma| = \lambda$.

Let $\mathcal{A}$ be a family of non-zero vectors in X that is maximal with respect to the property that the vectors have pairwise disjoint supports. If $\mathcal{A}$ has cardinality λ, the conclusion follows as $\mathcal{A}$ spans an isometric copy of $c_0(\lambda)$. So assume that $|\mathcal{A}| < \lambda$. Let

$$\Lambda = \bigcup_{f\in\mathcal{A}} \operatorname{supp} f.$$

As λ is regular (and $|\Gamma| \geqslant \lambda$), $|\Lambda| < \lambda$. Consequently, by maximality of $\mathcal{A}$, the contractive projection $P_\Lambda\colon \ell_\infty^\lambda(\Gamma) \to \ell_\infty^\lambda(\Lambda)$ maps X injectively into $\ell_\infty(\Lambda)$; a contradiction $\square$

Theorem 2.6 fails for singular cardinal numbers. Indeed, let $\lambda = \omega_\omega = \lim_{n\to\infty} \omega_n$. The space $\ell_\infty(\omega_n)$ contains an isometric copy of $\ell_2(\omega_n)$. In particular, the c_0-direct sum of $\ell_2(\omega_n)$ ($n \in \mathbb{N}$) embeds isometrically into ℓ_∞^λ, has density λ, and is WCG (and even Asplund). On the other hand, it does not contain $c_0(\lambda)$.

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