REPRESENTING COMPLETELY CONTINUOUS OPERATORS THROUGH WEAKLY ∞-COMPACT OPERATORS

WILLIAM B. JOHNSON, RAUNI LILLEMETS, AND EVE OJA

Abstract. Let $\mathcal{V}, \mathcal{W}_\infty$, and $\mathcal{W}$ be operator ideals of completely continuous, weakly ∞-compact, and weakly compact operators, respectively. We prove that $\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1}$. As an immediate application, the recent result by Dowling, Freeman, Lennard, Odell, Randrianantoanina, and Turett follows: the weak Grothendieck compactness principle holds only in Schur spaces.

1. Introduction

Let $\mathcal{L}, \mathcal{K}, \mathcal{W}$, and $\mathcal{V}$ denote the operator ideals of bounded linear, compact, weakly compact, and completely continuous operators. Let X and Y be Banach spaces. Recall that a linear map $T\colon X \to Y$ is *completely continuous*, i.e., $T \in \mathcal{V}(X, Y)$, if T takes weakly null sequences in X to null sequences in Y.

Recall that $\mathcal{K} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{W}$, but $\mathcal{V}$ and $\mathcal{W}$ are incomparable [9, 1.11.8]. The starting point for the present note was the following well-known formula [9, 3.2.3]:

$$\mathcal{V} = \mathcal{K} \circ \mathcal{W}^{-1}.$$

Recall that the *right-hand quotient* $\mathcal{A} \circ \mathcal{B}^{-1}$ of two operator ideals $\mathcal{A}$ and $\mathcal{B}$ is the operator ideal that consists of all operators $T \in \mathcal{L}(X, Y)$ such that $TS \in \mathcal{A}(Z, Y)$ whenever $S \in \mathcal{B}(Z, X)$ for some Banach space Z [9, 3.1.1].

Let $(x_n) \subset X$ be a bounded sequence. It is well known and easy to see that (x_n) defines an operator $\Phi_{(x_n)} \in \mathcal{L}(\ell_1, X)$ through the equality

$$\Phi_{(x_n)}(a_k) = \sum_{k=1}^{\infty} a_k x_k, \ (a_k) \in \ell_1.$$

The main tool in the proof of the formula $\mathcal{V} = \mathcal{K} \circ \mathcal{W}^{-1}$ in [9, 3.2.3] is the simple fact that $\Phi_{(x_n)}\colon c_0^* \to X$ *is weak***-to-weak continuous* if (x_n) is weakly null.

Let B_X denote the closed unit ball of X. A subset K of X is called *relatively weakly ∞-compact* if $K \subset \Phi_{(x_n)}(B_{\ell_1})$ for some weakly null sequence (x_n) in X. An operator $T \in \mathcal{L}(X, Y)$ is *weakly ∞-compact* if $T(B_X)$ is a relatively weakly ∞-compact subset of Y. Weakly ∞-compact (more generally, weakly p-compact) operators were considered by Castillo and Sanchez [4] in 1993 and by Sinha and Karn [10] in 2002 (for an even more general version of weakly (p, r)-compact operators, see [3]).

Denote by $\mathcal{W}_\infty$ the class of all weakly ∞-compact operators acting between arbitrary Banach spaces. An easy straightforward verification (as in [1, Propositions 2.1 and 2.2]) shows that $\mathcal{W}_\infty$ is a surjective operator ideal. The main result of this note reads as follows.

Theorem 1.1. $\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1}$.

An immediate consequence is that the weak Grothendieck compact principle holds only in Schur spaces. Recall that X has the *Schur property* (is a Schur space) if weakly null sequences in X are norm null.

Corollary 1.2. *[7, Theorem 1] Every weakly compact subset of a Banach space X is contained in the closed convex hull of a weakly null sequence if and only if X has the Schur property.*

Our method of proof relies on the Davis–Figiel–Johnson–Pełczyński factorization theorem [5], providing also an alternative proof for the Dowling–Freeman–Lennard–Odell–Randrianantoanina–Turett theorem [7], where Schauder basis theory was used.

2. Proof of Theorem 1.1

The following fact is well known; we include a proof for completeness.

Proposition 2.1. *If (x_n) is a weakly null sequence in a Banach space X, then $\Phi_{(x_n)}(B_{\ell_1})$ is weakly compact and coincides with the closed absolutely convex hull of (x_n).*

Proof. (cf. [2, proof of the "if" part of Theorem 3]). The set $\Phi_{(x_n)}(B_{\ell_1})$ is clearly absolutely convex. It is also weakly compact because $\Phi_{(x_n)}\colon c_0^* \to X$ is weak*-to-weak continuous and $B_{\ell_1} = B_{c_0^*}$ is weak* compact. Hence, $\Phi_{(x_n)}(B_{\ell_1})$ is a closed absolutely convex subset of X containing (x_n). Since $\Phi_{(x_n)}(B_{\ell_1})$ is obviously contained in the closed absolutely convex hull of (x_n), it coincides with the latter set. □

Let X be a Banach space. By the Grothendieck compactness principle, any compact subset of X is contained in $\Phi_{(x_n)}(B_{\ell_1})$ for some null sequence (x_n) in X. Therefore, the *relatively compact sets are relatively weakly ∞-compact*, and we get from Proposition 2.1 the following (known) result.

Corollary 2.2. $\mathcal{K} \subset \mathcal{W}_\infty \subset \mathcal{W}$.

From the proof of Proposition 3.1 below, it can be seen that these inclusions are strict.

Since $\mathcal{K} \subset \mathcal{W}_\infty$, we clearly have that $\mathcal{K} \circ \mathcal{W}^{-1} \subset \mathcal{W}_\infty \circ \mathcal{W}^{-1}$. Since also $\mathcal{V} \subset \mathcal{K} \circ \mathcal{W}^{-1}$ (this is the obvious "part" of the equality $\mathcal{V} = \mathcal{K} \circ \mathcal{W}^{-1}$),

$$\mathcal{V} \subset \mathcal{W}_\infty \circ \mathcal{W}^{-1}.$$

For the **proof of Theorem 1.1**, it remains to show that

$$\mathcal{W}_\infty \circ \mathcal{W}^{-1} \subset \mathcal{V}.$$

Proof. Let X and Y be Banach spaces and let $T \in \mathcal{W}_\infty \circ \mathcal{W}^{-1}(X, Y)$. Assume for contradiction that $T \notin \mathcal{V}(X, Y)$. Then there exists a weakly null sequence (x_n) in X such that (Tx_n) is not a null sequence in Y. Passing to a subsequence of (x_n), we may assume that $\|Tx_n\| \geq \delta, n \in \mathbb{N}$, for some $\delta > 0$. Hence, (Tx_n) is not relatively compact.

Since $\Phi_{(x_n)} \in \mathcal{W}(\ell_1, X)$ (see Proposition 2.1), by the Davis–Figiel–Johnson–Pełczyński factorization theorem [5], there exist a reflexive space R and weakly compact operators $\Phi\colon \ell_1 \to R$ with $\|\Phi\| = 1$ and $J\colon R \to X$ such that $\Phi_{(x_n)} = J\Phi$. From the definition of $\mathcal{W}_\infty \circ \mathcal{W}^{-1}$, we get that $TJ \in \mathcal{W}_\infty(R, Y)$ because $T \in \mathcal{W}_\infty \circ \mathcal{W}^{-1}(X, Y)$ and $J \in \mathcal{W}(R, X)$. Hence, there exists a weakly null sequence (y_n) in Y such that $TJ(B_R) \subset \Phi_{(y_n)}(B_{\ell_1})$. In particular, $Tx_n = T\Phi_{(x_n)}e_n = TJ\Phi e_n \in \Phi_{(y_n)}(B_{\ell_1})$, $n \in \mathbb{N}$, where (e_n) is the unit vector basis in ℓ_1.

Denote by $\overline{\Phi}_{(y_n)}$ the injective associate of $\Phi_{(y_n)}$, which means that $\Phi_{(y_n)} = \overline{\Phi}_{(y_n)}q$, where $q\colon \ell_1 \to Z := \ell_1/\ker\Phi_{(y_n)}$ is the quotient mapping. Since $\operatorname{ran} TJ \subset \operatorname{ran}\Phi_{(y_n)} = \operatorname{ran}\overline{\Phi}_{(y_n)}$, we can consider the linear operator $\overline{\Phi}_{(y_n)}^{-1}TJ\colon R \to Z$. This operator is bounded: if $r \in B_R$, then $TJr = \Phi_{(y_n)}\alpha$ for some $\alpha \in B_{\ell_1}$ and $\left\|\overline{\Phi}_{(y_n)}^{-1}TJr\right\| = \|q\alpha\| \le 1$.

We claim that Z has the Schur property. Indeed, by Grothendieck's result [8, Theorem 10] (see also Remark 2.3 below), the dual W^* of any closed subspace W of c_0 has the Schur property. It remains to observe that $\ker\Phi_{(y_n)}$ is weak* closed in $\ell_1 = c_0^*$ (because $\Phi_{(y_n)}$ is weak*-to-weak continuous), hence $\ker\Phi_{(y_n)} = W^\perp$ for some closed subspace W of c_0, and $Z = W^*$.

Since R is reflexive and Z has the Schur property, $\mathcal{L}(R, Z) = \mathcal{K}(R, Z)$. In particular, $\overline{\Phi}_{(y_n)}^{-1}TJ$ and therefore also $\overline{\Phi}_{(y_n)}\overline{\Phi}_{(y_n)}^{-1}TJ = TJ$ are compact operators. It follows that $(Tx_n) = (TJ\Phi e_n) \subset (TJ)(B_R)$ is relatively compact, a contradiction that completes the proof of Theorem 1.1. □

Remark 2.3. Let W be a closed subspace of c_0. To prove that the dual W^* has the Schur property, Grothendieck [8, Theorem 10] first establishes that W has the Dunford–Pettis property (DPP). Grothendieck's easy and beautiful proof can be found in Diestel's survey article [6, pages 25–26, see also Theorem 4]. Since W does not contain a copy of ℓ_1, relying on Rosenthal's ℓ_1 theorem, Diestel [6, Theorem 3] quickly concludes that W^* has the Schur property. Let us provide a version of Grothendieck's proof [8, pages 171–172], showing that the DPP of W implies that W^* has the Schur property.

The DPP of W means that every weakly compact operator with domain W is completely continuous, hence compact (because W^* is separable). It follows easily that then W^* has the Schur property: given (w_n^*) weakly null in W^*, consider the weakly compact operator $S : W \to c_0$ defined by $Sw = (\langle w_n^*, w\rangle)$, and use that $S^* = \Phi_{(w_n^*)}$ is a compact operator.

3. Applications of Theorem 1.1

It is well known that $\mathcal{K} \subset \mathcal{V}$. As we see now, $\mathcal{W}_\infty$ lies strictly between $\mathcal{K}$ and $\mathcal{V}$.

Proposition 3.1. *$\mathcal{K} \subset \mathcal{W}_\infty \subset \mathcal{V}$ and both inclusions are strict.*

Proof. It is obvious that $\mathcal{A} \subset \mathcal{A} \circ \mathcal{B}^{-1}$ for any two operator ideals $\mathcal{A}$ and $\mathcal{B}$. Hence, $\mathcal{W}_\infty \subset \mathcal{W}_\infty \circ \mathcal{W}^{-1} = \mathcal{V}$ by Theorem 1.1, and the inclusion $\mathcal{K} \subset \mathcal{W}_\infty$ was observed in Corollary 2.2.

To see that $\mathcal{K} \ne \mathcal{W}_\infty$, consider the identity embedding $j\colon \ell_1 \to c_0$ that is not compact but is weakly ∞-compact, because $j = \Phi_{(e_n)}$, where (e_n) is the unit vector basis of c_0. On the other hand, the identity operator on ℓ_1 is completely continuous (because ℓ_1 has the Schur property) but since it is not weakly compact, it is not weakly ∞-compact either (recall that $\mathcal{W}_\infty \subset \mathcal{W}$). (Another way to see that $\mathcal{W}_\infty \neq \mathcal{V}$ is to use that $\mathcal{W}_\infty = \mathcal{W}_\infty^{\mathrm{sur}}$, the surjective hull, but $\mathcal{V} \neq \mathcal{V}^{\mathrm{sur}} = \mathcal{L}$.) Now it is easy to see that the inclusion $\mathcal{W}_\infty \subset \mathcal{W}$ in Corollary 2.2 is strict: the identity operator on ℓ_2 is weakly compact but since it is not completely continuous, it is not weakly ∞-compact. □

Let X and Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. It is well known (and clear thanks to the Eberlein–Šmulian theorem) that $T \in \mathcal{V}(X,Y)$ *if and only if T takes relatively weakly compact subsets of X into relatively compact subsets of Y.*

Theorem 3.2. *Let X and Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. Then $T \in \mathcal{V}(X,Y)$ if and only if T takes relatively weakly compact subsets of X into relatively weakly ∞-compact subsets of Y.*

Proof. The "only if" part is obvious because relatively compact sets are relatively weakly ∞-compact. From the definition of $\mathcal{W}_\infty \circ \mathcal{W}^{-1}$, it is clear that if T takes relatively weakly compact sets into relatively weakly ∞-compact sets, then $T \in \mathcal{W}_\infty \circ \mathcal{W}^{-1}(X,Y)$. By Theorem 1.1, this means that $T \in \mathcal{V}(X,Y)$. □

Let $\mathcal{A}$ be an operator ideal. Recall that the *space ideal* Space($\mathcal{A}$) is defined as the class of all Banach spaces X such that the identity operator on X belongs to $\mathcal{A}(X,X)$. If $\mathcal{A}$ and $\mathcal{B}$ are operator ideals, then obviously $X \in \mathrm{Space}(\mathcal{A} \circ \mathcal{B}^{-1})$ if and only if $\mathcal{B}(Z,X) \subset \mathcal{A}(Z,X)$ for all Banach spaces Z.

From the definitions, it is clear that Space($\mathcal{V}$) is the class of all Banach spaces with the Schur property. Theorem 1.1 yields that

$$\mathrm{Space}(\mathcal{V}) = \mathrm{Space}(\mathcal{W}_\infty \circ \mathcal{W}^{-1}).$$

This can be reformulated as follows. Note that the equivalence (a) $\Leftrightarrow$ (b) below is precisely Corollary 1.2 and, as was mentioned in the Introduction, it is due to [7, Theorem 1].

Theorem 3.3. *For a Banach space X, the following statements are equivalent:*

(a) *X has the Schur property;*
(b) *the weak Grothendieck compactness principle holds in X;*
(c) *$\mathcal{W}(Z,X) \subset \mathcal{W}_\infty(Z,X)$ for all Banach spaces Z.*

Proof. We already observed that (a) $\Leftrightarrow$ (c) thanks to Theorem 1.1. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious. (But (a) $\Leftrightarrow$ (b) is also the special case of Theorem 3.2, when T is the identity operator on X.) □

Remark 3.4. By the Davis–Figiel–Johnson–Pełczyński factorization theorem, (c) is equivalent to

(d) all injective operators from reflexive Banach spaces to X are weakly ∞-compact.

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