Weakly null sequences in L_1*

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Abstract

We construct a weakly null normalized sequence $\{f_i\}_{i=1}^{\infty}$ in L_1 so that for each $\varepsilon > 0$, the Haar basis is $1+\varepsilon$-equivalent to a block basis of every subsequence of $\{f_i\}_{i=1}^{\infty}$. In particular, the sequence $\{f_i\}_{i=1}^{\infty}$ has no unconditionally basic subsequence. This answers a question raised by the second author and H. P. Rosenthal in 1977. A similar example is given in an appropriate class of rearrangement invariant function spaces.

1 Introduction

In [MR], the second author and H. P. Rosenthal constructed the first examples of weakly null normalized sequences which do not have any unconditionally basic subsequences. Much more recently, the second author and W. T. Gowers [GM] constructed infinite dimensional Banach spaces which do not contain any unconditionally basic sequences. These later examples are of a different character than the examples in [MR]. For example, in [MR], it is shown that if K is a sufficiently complex countable compact metric space, then the space $C(K)$ contains a weakly null normalized sequence which does not have an unconditionally basic subsequence, and it is known [PS] that every infinite dimensional subspace of such a $C(K)$ space contains an unconditionally basic sequence; in fact, a sequence which is equivalent to the unit vector basis of c_0. In [MR] it was asked if a similar example exists in L_1. The space L_1 is similar to $C(K)$ in that every infinite dimensional subspace contains an unconditionally basic sequence. Indeed, if the subspace is not reflexive, then ℓ_1 embeds into the subspace [KP]. If a subspace X of L_1 is reflexive, then it embeds into L_p for some $1 < p \leq 2$ [R]. Since L_p has an unconditional basis, every weakly null normalized sequence in X has an unconditionally basic subsequence.

The main result in this paper, Theorem 1, is that there is a weakly null normalized sequence $\{f_i\}_{i=1}^{\infty}$ in L_1 which has no unconditionally basic subsequence. In fact, the

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sequence $\{f_i\}_{i=1}^\infty$ has the stronger property that for every $\varepsilon > 0$, the (conditional) Haar basis is $1+\varepsilon$-equivalent to a block basis of every subsequence of $\{f_i\}_{i=1}^\infty$. This is analogous to the result in [MR] that if K is a sufficiently complex countable compact metric space, then the space $C(K)$ contains a weakly null normalized sequence $\{x_n\}_{n=1}^\infty$ so that every initial segment of the (conditional) summing basis is $1+\varepsilon$-equivalent to a block basis of every subsequence of $\{x_n\}_{n=1}^\infty$.

Theorem 1 can also be compared to the result of [MS] that for $1 < p < 2$, there is a 1-symmetric basic sequence $\{g_n\}_{n=1}^\infty$ in L_p so that the Haar basis is equivalent to a block basis of every subsequence of $\{g_n\}_{n=1}^\infty$. In this result, there is a lower bound $C_p > 1$ to the constant of equivalence to the Haar basis because every block basis of $\{g_n\}_{n=1}^\infty$ is monotonely unconditional and the Haar basis for L_p is not.

The approach to proving Theorem 1 also yields information about other rearrangement invariant function spaces. In Theorem 2 we prove that if X is a rearrangement invariant function space on $[0,1]$ such that $\inf_{t>0} \frac{\|\chi_{[0,t]}\|}{t^{1/2}} = 0$ and $\int_0^1 \|\chi_{[0,t]}\| \frac{dt}{t} < \infty$, then there is a weakly null normalized sequence $\{f_i\}_{i=1}^\infty$ in X such that for each $\varepsilon > 0$, the Haar basis is $1+\varepsilon$-equivalent to a block basis of every subsequence of $\{f_i\}_{i=1}^\infty$. The first condition says that, in some sense, X is not to the right of L_2. Some such condition is needed because the theorem is false in L_p for $p > 2$ by the results of [KP]. The second condition in Theorem 2 is technical and probably can be weakened; however, for anything resembling our construction to work, the Rademacher functions in X must be equivalent to an orthonormal sequence in a Hilbert space.

We use standard Banach space theory terminology as can be found in [LT]. By a Haar system we mean here any sequence of functions distributionally equivalent to the (L_∞ normalized, mean zero) traditional Haar functions; i.e., any sequence of functions $\{k_{i,n}\}_{n=0,i=1}^{\infty,\ 2^n}$ with $|k_{i,n}|$ being a characteristic function of a set $A_{i,n}$ of measure 2^{-n}, $k_{i,n}$ equal to 1 on $A_{2i-1,n+1}$ and to -1 on $A_{2i,n+1}$. (In particular $A_{2i,n+1}$ and $A_{2i-1,n+1}$ are disjoint and their union is $A_{i,n}$.)

2 A peculiar weakly null sequence in L_1

In proving Theorem 1 we use the well-known fact that j independent sets each having measure less than $1/j$ are essentially disjoint. Rather than hunt for a reference that states this in a form suitable for our use, we formulate what we need as a lemma. Thanks are due S. Kwapień for simplifying the proof.

Lemma 1 *Let $0 < \theta < 1$ and let j be a positive integer. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $h_1, h_2, \ldots, h_j$ be independent symmetric random variables on Ω so that for each i, $|h_i|$ is the indicator function of a set having probability θ. Then there are disjoint sets A and B in the algebra generated by $h_1, h_2, \ldots, h_j$ so that*

$$\mu(A) = \mu(B) \geq \frac{\theta j}{2}(1 - \theta(j-1)) \quad \text{and} \tag{1}$$

$$\Big\| \sum_{i=1}^{j} h_i - \chi_A + \chi_B \Big\|_1 \le \theta^2 j(j-1). \tag{2}$$

Proof: Since $\int \sum_{i=1}^{j} |h_i| = \theta j$ and

$$\mu\Big[\sum_{i=1}^{j} |h_i| = 1\Big] = \int_{\left[\sum_{i=1}^{j}|h_i|=1\right]} \sum_{i=1}^{j} |h_i| = \theta j(1-\theta)^{j-1}, \tag{3}$$

$$\begin{aligned} \textstyle\int_{\left[\sum_{i=1}^{j}|h_i|\ge 2\right]} \sum_{i=1}^{j} |h_i| &= \theta j - \textstyle\int_{\left[\sum_{i=1}^{j}|h_i|=1\right]} \sum_{i=1}^{j} |h_i| \\ &= \theta j - \theta j(1-\theta)^{j-1} \qquad (4) \\ &\le \theta^2 j(j-1). \end{aligned}$$

Set

$$A := \Big[\sum_{i=1}^{j} |h_i| = 1 \ \text{ and } \ \sum_{i=1}^{j} h_i = 1\Big]$$

$$B := \Big[\sum_{i=1}^{j} |h_i| = 1 \ \text{ and } \ \sum_{i=1}^{j} h_i = -1\Big]$$

then, by (3),

$$\mu(A) = \mu(B) \ge \frac{\theta j}{2}(1 - \theta(j-1))$$

and by (4)

$$\Big\| \sum_{i=1}^{j} h_i - \chi_A + \chi_B \Big\|_1 \le \theta^2 j(j-1).$$

■

Theorem 1 *There is a weakly null normalized sequence $\{f_i\}_{i=1}^{\infty}$ in L_1 such that for each $\varepsilon > 0$, the Haar basis is $1+\varepsilon$-equivalent to a block basis of every subsequence of $\{f_i\}_{i=1}^{\infty}$. Consequently, $\{f_i\}_{i=1}^{\infty}$ has no unconditionally basic subsequence.*

Proof: Let $\mathcal{A}$ be the algebra generated by the dyadic subintervals of $(0,1)$ and let $\{E_n\}_{n=1}^{\infty}$ be an ordering of the non empty elements of $\mathcal{A}$ so that each element of $\mathcal{A}$ appears infinitely many times in the sequence $\{E_n\}_{n=1}^{\infty}$. We will define by recursion an increasing sequence $\{a_n\}_{n=1}^{\infty}$ of positive number and a rapidly growing sequence $\{k_n\}_{n=1}^{\infty}$ of powers of two to satisfy certain conditions to be specified later. We then define, for each n, a sequence $\{h_{i,n}\}_{i=1}^{\infty}$ of functions on $(0,1)$ so that

(i) $|h_{i,n}| = \chi_{A_{i,n}}$ with $A_{i,n} \subset E_n$

(ii) $\int h_{i,n} = 0$

(iii) $\lambda(A_{i,n}) = \lambda(E_n)/k_n$ (λ is Lebesgue measure)

(iv) $h_{i,n}$ is $\mathcal{A}$-measurable

(v) $\{h_{i,n}\}_{i=1}^{\infty}$ are independent relative to E_n with normalized Lebesgue measure on E_n.

Having done this, we define the desired sequence $\{f_i\}_{i=1}^{\infty}$ by

$$f_i = \sum_{n=1}^{\infty} a_n h_{i,n}. \tag{5}$$

The a_n's are going to be chosen so that $\sum_{n=1}^{\infty} a_n \|h_{i,n}\|_1 = \sum_{n=1}^{\infty} a_n \lambda(E_n)/k_n$ converges and, since each sequence $\{h_{i,n}\}_{i=1}^{\infty}$ is clearly weakly null, also $\{f_i\}$ is. Since the proof that $\{f_i\}_{i=1}^{\infty}$ has the other desired properties is a bit technical, we first describe in words the main part of the construction. The sequence $\{f_i\}_{i=1}^{\infty}$ itself of course cannot be symmetric if it has the properties we claim. However, notice that for each n, the sequence $\{a_n h_{i,n}\}_{i=1}^{\infty}$ is a 1-symmetric basic sequence which is equivalent (with constant depending on n) to the unit vector basis of ℓ_2. It turns out that this allows all estimates we make when summing the f_i's to depend only on the number of terms in the sums.

Suppose that we want to build a Haar function whose support is a set E in $\mathcal{A}$; that is, we want a linear combination of $\{f_i\}_{i=1}^{\infty}$ (for i in a given infinite set of natural numbers) which approximates a mean zero function whose absolute value is approximately χ_E. The definitions of $\{a_n\}_{n=1}^{\infty}$ and $\{k_n\}_{n=1}^{\infty}$ guarantee that for an appropriate m_n, if we sum m_n different f_i (say, for i in B) and normalize appropriately, the resulting vector $a_n^{-1}\sum_{i\in B} f_i$ is a small perturbation of $\sum_{i\in B} h_{i,n}$, and this latter vector is (by Lemma 1) a small perturbation of an $\mathcal{A}$-measurable function, h_B, which takes on only the values $\{-1, 0, 1\}$. Moreover, the measure of the support $S(B)$ of h_B will be a percentage of the measure of the set E_n. If that percentage were 100%, we would have achieved our goal since there are arbitrarily large n with $E_n = E$. We cannot make the percentage 100%, but we can repeat the process, replacing E by $E \backslash S(B)$, and get another linear combination of $\{f_i\}_{i=1}^{\infty}$ whose absolute value is approximately χ_F with F the same percentage of $E \setminus S(B)$. By iterating this process, we obtain a linear combination of $\{f_i\}_{i=1}^{\infty}$ whose absolute value is approximately χ_E.

We turn now to the actual proof of Theorem 1. Suppose we have defined $\{a_1, a_2, \ldots, a_N\}$ and $\{k_1, k_2, \ldots, k_N\}$. Since for each n, the sequence $\{a_n h_{i,n}\}_{i=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_2, there is a constant M_N so that for all finite sets σ of natural numbers we have

$$\sum_{n=1}^{N} \Big\| \sum_{i\in\sigma} a_n h_{i,n} \Big\| < |\sigma|^{1/2} M_N. \tag{6}$$

We want to define k_{N+1} and a_{N+1} so that if we add up εk_{N+1} (with $\varepsilon > 2^{-N}$, say) of the f_i's, then the terms involving $a_{N+1} h_{i,N+1}$ are dominant. To guarantee that these terms dominate the terms involving $a_n h_{i,n}$ with $n \leq N$, it suffices, as we shall see, to have

$$2^N k_{N+1}^{1/2} M_N \leq a_{N+1}\lambda(E_{N+1}). \tag{7}$$

We shall also see below that in order to guarantee that the terms involving $a_{N+1}h_{i,N+1}$ are negligible with respect to the terms involving $a_n h_{i,n}$ with $n \leq N$ if we add up fewer than k_N of the f_i's, it suffices (in fact, is "over kill") to have

$$\frac{a_{N+1}\lambda\left(E_{N+1}\right)}{k_{N+1}} \leq \frac{\min\{\lambda\left(E_n\right) : 1 \leq n \leq N\}}{2^{N+1}k_N}. \tag{8}$$

There is of course no difficulty in achieving (7) and (8) simultaneously. For example, let $a_{N+1} = k_{N+1}^{3/4}$ with k_{N+1} a sufficiently large power of 2.

This completes the description of how to define the sequence $\{f_i\}_{i=1}^{\infty}$. To see that $\{f_i\}_{i=1}^{\infty}$ satisfies the conclusion of Theorem 1, we need the following:

Claim 1 *Let $\tau > 0$, let F_1 be a non empty set in $\mathcal{A}$, and suppose that $\mathbb{M}$ is an infinite subset of $\mathbb{N}$. Then there is an f in the linear span of $\{f_i\}_{i\in\mathbb{M}}$ and disjoint subsets $A \in \mathcal{A}$, $B \in \mathcal{A}$ of F_1 with $\lambda(A) = \lambda(B) > \lambda(F_1)/2 - \tau$ so that $\|f - \chi_A + \chi_B\|_1 < \tau$.*

Once we have the Claim it is of course easy to get the stronger conclusion with $A \cup B = F_1$ keeping the second approximation conclusion as is. Just divide $F_1 \setminus (A \cup B)$ into two disjoint sets of $\mathcal{A}$ of the same measure; add one of them to A and the other to B. It is now evident from Claim 1 that for any sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers and any infinite subset $\mathbb{M}$ of $\mathbb{N}$ we can build a Haar system $\{g_n\}_{n=1}^{\infty}$ and a block basis $\{u_n\}_{n=1}^{\infty}$ of $\{f_i\}_{i\in\mathbb{M}}$ so that for each n, $\|g_n - u_n\|_1 < \varepsilon_n$. This implies that $\{f_i\}_{i=1}^{\infty}$ satisfies the conclusion of Theorem 1.

Proof of Claim 1: Let ε^{-1} be an appropriately large power of 2; say, $\varepsilon^{-1} = 2^m$. We choose an appropriately large N_1 with $E_{N_1} = F_1$ and let σ_1 be a subset of $\mathbb{M}$ which has cardinality εk_{N_1}. By conditions (i)–(v) and Lemma 1 (applied in the probability space $(F_1, \frac{\lambda}{\lambda(F_1)})$ with $\theta = k_{N_1}^{-1}$ and $j = \varepsilon k_{N_1}$), there are disjoint sets $A_1 \subset F_1$, $B_1 \subset F_1$ in $\mathcal{A}$ so that

$$\lambda(A_1) = \lambda(B_1) \geq \frac{\varepsilon}{2}(1-\varepsilon)\lambda(F_1) \quad \text{and}$$

$$\Big\| \sum_{i\in\sigma_1} h_{i,k_{N_1}} - \chi_{A_1} + \chi_{B_1} \Big\|_1 < \varepsilon^2.$$

Set $F_2 := F_1 \setminus (A_1 \cup B_1)$ (so that $\lambda(F_2) \leq [1 - \varepsilon(1-\varepsilon)]\lambda(F_1)$) and repeat the construction, replacing F_1 by F_2. We use the same ε and choose an appropriately large $N_2 > N_1$ with $E_{N_2} = F_2$ and let σ_2 be a subset of $\mathbb{M}$ of cardinality εk_{N_2} with $\max \sigma_1 < \min \sigma_2$. This time we get disjoint sets $A_2 \subset F_2$, $B_2 \subset F_2$ in $\mathcal{A}$ so that

$$\lambda(A_2) = \lambda(B_2) \geq \frac{\varepsilon}{2}(1-\varepsilon)\lambda(F_2) \quad \text{and}$$

$$\Big\|\sum_{i\in\sigma_2} h_{i,k_{N_2}} - \chi_{A_2} + \chi_{B_2}\Big\|_1 < \varepsilon^2.$$

Next set $F_3 := F_2 \setminus (A_2 \cup A_3)$ (so that $\lambda(F_3) \le [1-\varepsilon(1-\varepsilon)]^2\lambda(F_1)$) and repeat. Continue in this way $m\varepsilon^{-1} = m2^m$ steps, thereby obtaining $k_1 < k_2 < \cdots < k_{m2^m}$ (which can grow as fast as we like), disjoint subsets $A_1, A_2, \dots, A_{m2^m}, B_1, B_2 \dots, B_{m2^m}$ of F_1 which are in $\mathcal{A}$, and subsets $\sigma_1, \sigma_2, \dots, \sigma_{m2^m}$ of $\mathbb{M}$, with $\max\sigma_{j-1} < \min\sigma_j$ and $|\sigma_j| = \varepsilon k_{N_j}$ so that $\lambda(A_j) = \lambda(B_j)$,

$$\lambda(F_1 \setminus \cup_{j=1}^n (A_j \cup B_j)) \le [1-\varepsilon(1-\varepsilon)]^n\lambda(F_1) \quad \text{and} \tag{9}$$

$$\Big\|\sum_{i\in\sigma_j} h_{i,k_{N_j}} - \chi_{A_j} + \chi_{B_j}\Big\|_1 < \varepsilon^2. \tag{10}$$

Set $A := \cup_{j=1}^{m2^m} A_j$, $B := \cup_{j=1}^{m2^m} B_j$, and $\sigma := \cup_{j=1}^{m2^m}\sigma_j$. Then from (9) we have that

$$\lambda(A) = \lambda(B) \ge \frac{1-[1-\varepsilon(1-\varepsilon)]^{m/\varepsilon}}{2}\lambda(F_1) \tag{11}$$

while (10) gives

$$\Big\|\sum_{j=1}^{m2^m}\sum_{i\in\sigma_j} h_{i,k_{N_j}} - \chi_A + \chi_B\Big\|_1 < m\varepsilon. \tag{12}$$

We need to verify that $\sum_{j=1}^{m2^m}\sum_{i\in\sigma_j} h_{i,k_{N_j}}$ is close to the linear span of $\{f_i\}_{i\in\mathbb{M}}$. From (6), (7), and (8) we get

$$\begin{aligned}\|\textstyle\sum_{i\in\sigma_j}(a_{N_j}^{-1}f_i - h_{i,k_{N_j}})\|_1 &\le a_{N_j}^{-1}\Big((\varepsilon k_{N_j})^{1/2}M_{N_j-1} + \varepsilon k_{N_j}\textstyle\sum_{n=N_j+1}^{\infty}\frac{a_n\lambda(E_n)}{k_n}\Big)\\ &\le 2^{-N_j+1}\varepsilon^{1/2}\lambda(E_{N_j}) + \varepsilon 2^{-N_j-1}\lambda(E_{N_j}) := (*),\end{aligned}$$

and we can assume that $(*) \le 2^{-j-1}\varepsilon$ since ε is specified before N_1. From this and (12) we get that

$$\|\sum_{j=1}^{m2^m} a_{N_j}^{-1}\sum_{i\in\sigma_j} f_i - \chi_A + \chi_B\|_1 < (m+1)\varepsilon = \varepsilon(1-\log_2\varepsilon). \tag{13}$$

From (11) and (13) it is clear that if ε is sufficiently small, the claim is satisfied if we set $f := \sum_{j=1}^{m2^m} a_{N_j}^{-1}\sum_{i\in\sigma_j} f_i$. ∎

3 Rearrangement invariant function spaces

Here we generalize the first statement of Theorem 1 to the case of rearrangement invariant function spaces. It is clear that this statement does not hold for every rearrangement invariant function spaces; even not for nice ones. Indeed, for $2 \le p < \infty$, every weakly null normalized sequence in L_p contains a subsequence equivalent to the unit vector basis in either ℓ_p or ℓ_2. It is natural to conjecture that the theorem may still hold for spaces which are strictly "to the left" of L_2, in some sense. Actually a lot more is true:

Theorem 2 *Let $(X, \|\cdot\|)$ be a rearrangement invariant function space on $[0,1]$ such that*

$$\inf_{t>0} \frac{\|\chi_{[0,t]}\|}{t^{1/2}} = 0 \tag{14}$$

and

$$\int_0^1 \|\chi_{[0,t]}\| \frac{dt}{t} < \infty. \tag{15}$$

Then there is a weakly null normalized sequence $\{f_i\}_{i=1}^\infty$ in X such that for each $\varepsilon > 0$, the Haar basis is $1+\varepsilon$-equivalent to a block basis of every subsequence of $\{f_i\}_{i=1}^\infty$.

We begin with a lemma which replaces Lemma 1. The main difference is that we give a lattice, rather than norm, estimate for the error.

Lemma 2 *Let $0 < \theta < 1$ and let j be a positive integer. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $h_1, h_2, \dots, h_j$ be independent symmetric random variables on Ω so that for each i, $|h_i|$ is the indicator function of a set having probability θ. Denote by S the support of $\sum_{i=1}^j |h_i|$. Then there are disjoint subsets A and B of S in the algebra generated by $h_1, h_2, \dots, h_j$ and, for each $s = 2, 3, \dots,$ a set $C_s \subset S$ of measure at most*

$$\frac{(\theta j)^{s-1} \exp(\frac{2\theta j}{1-\theta})}{s!} \mu(S)$$

so that

$$\mu(A) = \mu(B) \ge \frac{\theta j}{2} \exp\Big(\frac{-\theta(j-1)}{1-\theta}\Big) \ge \frac{\mu(S)}{2} \exp\Big(\frac{-\theta(j-1)}{1-\theta}\Big) \tag{16}$$

and

$$\Big|\sum_{i=1}^j h_i - \chi_A + \chi_B\Big| \le 2\chi_{C_2} + \sum_{s=3}^\infty \chi_{C_s}. \tag{17}$$

Remark 1 In particular, if $(\Omega, \mathcal{F}, \mu)$ is a subinterval of $[0,1]$ with normalized Lebesgue measure and $\|\cdot\|$ is a rearrangement invariant norm on $[0,1]$ then, putting $\sigma = \mu(S)\exp\big(\frac{2\theta j}{1-\theta}\big)$,

$$\Big\|\sum_{i=1}^j h_i - \chi_A + \chi_B\Big\| \le 2\|\chi_{[0,\theta j\sigma/2]}\| + \sum_{t=2}^{j-1} \|\chi_{[0,(\theta j)^t\sigma/(t+1)!]}\|. \tag{18}$$

However, this norm estimate of the error does not seem to be enough to deduce Theorem 2 and we will have to use the lattice estimate given in the Lemma.

Proof: As in the proof of Lemma 1 set

$$A := \Big[\sum_{i=1}^{j} |h_i| = 1 \ \text{ and } \ \sum_{i=1}^{j} h_i = 1\Big]$$

and

$$B := \Big[\sum_{i=1}^{j} |h_i| = 1 \ \text{ and } \ \sum_{i=1}^{j} h_i = -1\Big].$$

For $s = 2, 3, \ldots$ set

$$C_s := \Big[\sum_{i=1}^{j} |h_i| \geq s\Big].$$

Then clearly

$$\Big|\sum_{i=1}^{j} h_i - \chi_A + \chi_B\Big| \leq \sum_{r=2}^{j} \Big(\sum_{i=1}^{j} |h_i|\Big)\chi_{[\sum_{i=1}^{j}|h_i|=r]} \leq 2\chi_{C_2} + \sum_{s=3}^{\infty} \chi_{C_s}.$$

For $2 \leq s \leq j$ the measure of C_s is

$$\sum_{r=s}^{j} \binom{j}{r} \theta^r (1-\theta)^{j-r} \leq \sum_{r=s}^{j} \frac{(\theta j)^r}{r!} \leq \frac{(\theta j)^s}{s!} e^{\theta j}.$$

The measure of S is

$$1 - (1-\theta)^j \leq \theta j,$$

while A and B both have measure

$$\frac{\theta j (1-\theta)^{j-1}}{2} \geq \frac{\theta j}{2} \exp\Big(\frac{-\theta(j-1)}{1-\theta}\Big).$$

Since $\mu(S) \geq \mu(A) + \mu(B) \geq \theta j \exp\big(\frac{-\theta(j-1)}{1-\theta}\big)$ we also get that

$$\mu(C_s) \leq \frac{(\theta j)^{s-1} \exp(\theta j + \frac{\theta(j-1)}{1-\theta})}{s!} \mu(S) \leq \frac{(\theta j)^{s-1} \exp(\frac{2\theta j}{1-\theta})}{s!} \mu(S).$$

■

The proof of Theorem 2 is very similar to that of Theorem 1 and we shall only sketch it.

Proof of Theorem 2. We are going to define inductively an increasing sequence of positive numbers $\{a_n\}$, a sequence $\{k_n\}_{n=1}^{\infty}$ of powers of two and a double sequence $\{h_{i,n}\}$ of functions on $[0, 1]$. Given k_n, $\{h_{i,n}\}_{i=1}^{\infty}$ is defined in exactly the same way as in the beginning of the proof of Theorem 1 to satisfy conditions (i)-(v) there. We now describe how, given $a_1, \ldots, a_N$ and $k_1, \ldots, k_N$, to define a_{N+1} and k_{N+1}.

We note first that (15) implies in particular that for each n, $\{h_{i,n}\}_{i=1}^{\infty}$ is equivalent to an orthonormal sequence. We delay the proof of that to Lemma 3 below. Once we know this we deduce, as in the L_1 case, that there is a constant M_N so that for all finite sets σ of natural numbers we have

$$\sum_{n=1}^{N}\Big\|\sum_{i\in\sigma} a_n h_{i,n}\Big\| < |\sigma|^{1/2} M_N. \tag{19}$$

We want to define $k_{N+1} > k_N$ (and a power of two) and $a_{N+1} > a_N$ so as to satisfy

$$2^N k_{N+1}^{1/2} M_N \leq a_{N+1}\|\chi_{E_{N+1}}\|. \tag{20}$$

and

$$a_{N+1}\|\chi_{[0,\frac{\lambda(E_{N+1})}{k_{N+1}}]}\| \leq \frac{\min\{\|\chi_{E_n}\| : 1\leq n\leq N\}}{2^{N+1}k_N}. \tag{21}$$

We can choose k_{N+1} and a_{N+1} to satisfy (20) and (21) simultaneously since, by condition (14) on the space, we clearly can build increasing positive sequences $\{a_n\}$ and $\{k_n\}$ tending to infinity with $a_n/k_n^{1/2}\to\infty$ and $a_n\|\chi_{[0,k_n^{-1}]}\|\to 0$. By passing to a subsequence the convergence to these limits can be made arbitrarily fast.

Condition (21) implies in particular that $\sum_{n=1}^{\infty} a_n\|h_{i,n}\| < \infty$. So we can define $f_i = \sum_{n=1}^{\infty} a_n h_{i,n}$ as before and Lemma 3 guarantees that $\{f_i\}$ is weakly null.

The main part of the proof is contained in the following claim, replacing Claim 1, which is clearly enough to finish the proof of the theorem. Note that since the conditions on the space ensure that it is not L_∞ the measure condition on the sets A and B in the claim imply that $\chi_A - \chi_B$ approximates a Haar function supported on F_1 to an arbitrary degree of approximation (see the comment after the statement of Claim 1).

Claim 2 *Let $\tau > 0$, let F_1 be a non empty set in $\mathcal{A}$, and suppose that $\mathbb{M}$ is an infinite subset of $\mathbb{N}$. Then there are f in the linear span of $\{f_i\}_{i\in\mathbb{M}}$ and disjoint subsets $A, B\in\mathcal{A}$ of F_1 with $\lambda(A)=\lambda(B)>\lambda(F_1)/2-\tau$ so that $\|f-\chi_A+\chi_B\|<\tau$.*

Proof of Claim 2: Let ε^{-1} be an appropriately large power of 2; say, $\varepsilon^{-1}=2^m$. We choose an appropriately large N_1 with $E_{N_1}=F_1$ and let σ_1 be a subset of $\mathbb{M}$ which has cardinality εk_{N_1}. By conditions (i)–(v) and Lemma 2 (applied in the probability space $(F_1, \frac{\lambda}{\lambda(F_1)})$ with $\theta = k_{N_1}^{-1}$ and $j=\varepsilon k_{N_1}$), there are disjoint sets A_1, B_1 in $\mathcal{A}$ and another set S_1 in $\mathcal{A}$ such that $A_1, B_1\subset S_1\subset F_1$ and, for each $s=2,3,\dots$, a set $C_{1,s}\subset S_1$ of measure at most

$$\frac{\varepsilon^{s-1}e^{3\varepsilon}}{s!}\lambda(S_1)$$

so that

$$\lambda(A_1)=\lambda(B_1)\geq\frac{\varepsilon e^{-2\varepsilon}}{2}\lambda(F_1)\geq\frac{e^{-2\varepsilon}}{2}\lambda(S_1)\quad\text{and}$$

$$\Big|\sum_{i\in\sigma_1} h_{i,k_{N_1}} - \chi_{A_1} + \chi_{B_1}\Big| \le 2\chi_{C_{1,2}} + \sum_{s=3}^{\infty} \chi_{C_{1,s}}.$$

Set $F_2 := F_1 \setminus S_1$ and repeat the construction, replacing F_1 by F_2. We use the same ε and choose an appropriately large $N_2 > N_1$ with $E_{N_2} = F_2$ and let σ_2 be a subset of $\mathbb{M}$ of cardinality εk_{N_2} with $\max\sigma_1 < \min\sigma_2$. This time we get sets $A_2, B_2 \subset S_2 \subset F_2$ in $\mathcal{A}$ with A_2, B_2 disjoint and, for each $s = 2, 3, \dots$, a set $C_{2,s} \subset S_2$ of measure at most

$$\frac{\varepsilon^{s-1}e^{3\varepsilon}}{s!}\lambda(S_2)$$

so that

$$\lambda(A_2) = \lambda(B_2) \ge \frac{\varepsilon e^{-2\varepsilon}}{2}\lambda(F_2) \ge \frac{e^{-2\varepsilon}}{2}\lambda(S_2) \quad \text{and}$$

$$\Big|\sum_{i\in\sigma_2} h_{i,k_{N_2}} - \chi_{A_2} + \chi_{B_2}\Big| \le 2\chi_{C_{2,2}} + \sum_{s=3}^{\infty} \chi_{C_{2,s}}.$$

We continue in an obvious way, setting $F_3 = F_1 \setminus (S_1 \cup S_2)$ ..., getting subsets $\sigma_1, \dots, \sigma_l$ of $\mathbb{M}$ with $\max\sigma_i < \min\sigma_{i+1}$, disjoint subsets $S_1, S_2, \dots, S_l$ of of F_1, a disjoint couple of sets $A_j, B_j \subset S_j$, $j = 1, 2, \dots, l$, all in $\mathcal{A}$, and a double sequence of sets $C_{j,s}$, $j = 1, 2, \dots, l$, $s = 1, 2, \dots$, satisfying for all $j = 1, 2, \dots, l$,

$$\lambda(C_{j,s}) \le \frac{\varepsilon^{s-1}e^{3\varepsilon}}{s!}\lambda(S_j),$$

$$\lambda(A_j) = \lambda(B_j) \ge \frac{\varepsilon e^{-2\varepsilon}}{2}\lambda(F_1 \setminus \cup_{r=1}^{j-1} S_r) \ge \frac{e^{-2\varepsilon}}{2}\lambda(S_j) \quad \text{and}$$

$$\Big|\sum_{i\in\sigma_j} h_{i,k_{N_j}} - \chi_{A_j} + \chi_{B_j}\Big| \le 2\chi_{C_{j,2}} + \sum_{s=3}^{\infty} \chi_{C_{2j,s}}.$$

We choose l so that $\lambda(F_1 \setminus \cup_{r=1}^{l} S_r) < \varepsilon\lambda(F_1)$. This clearly can be done since S_j "eats" at least a $\varepsilon e^{-2\varepsilon}$ portion of $F_1 \setminus \cup_{r=1}^{j-1} S_r$. Set

$$A = \cup_{j=1}^{l} A_j, \;\; B = \cup_{j=1}^{l} B_j$$

and for all $s = 2, 3, \dots$

$$C_s = \cup_{j=1}^{l} C_{j,s}.$$

Then

$$\lambda(C_s) \le \frac{\varepsilon^{s-1}e^{3\varepsilon}}{s!}\lambda(F_1),$$

$$\lambda(A) = \lambda(B) \ge \frac{e^{-2\varepsilon}}{2}\lambda(\cup_{r=1}^{l} S_r) \ge \frac{e^{-2\varepsilon} - \varepsilon}{2}\lambda(F_1), \tag{22}$$

and

$$\Big|\sum_{j=1}^{l}\sum_{i\in\sigma_j} h_{i,k_{N_j}} - \chi_A + \chi_B\Big| \le 2\chi_{C_2} + \sum_{s=3}^{\infty}\chi_{C_s}.$$

Consequently,

$$\Big\|\sum_{j=1}^{l}\sum_{i\in\sigma_j} h_{i,k_{N_j}} - \chi_A + \chi_B\Big\| \le 2\|\chi_{[0,\varepsilon e^{3\varepsilon}/2]}\| + \sum_{t=2}^{\infty}\|\chi_{[0,\varepsilon^t e^{3\varepsilon}/(t+1)!]}\|. \tag{23}$$

For ε small enough the last expression is smaller than

$$10\int_0^{\varepsilon}\|\chi_{[0,t]}\|\frac{dt}{t}$$

and condition (15) implies that the last quantity is smaller than $\tau/2$ if ε is small enough. Also, (22) implies that, if ε is small enough, then

$$\lambda(A), \lambda(B) > \lambda(F_1)/2 - \tau.$$

The rest of the proof is very similar to that of the L_1 case. We need to verify that $\sum_{j=1}^{l}\sum_{i\in\sigma_j} h_{i,k_{N_j}}$ is close to the linear span of $\{f_i\}_{i\in\mathbb{M}}$. From (19), (20), and (21) we get

$$\begin{aligned}\|\textstyle\sum_{i\in\sigma_j}(a_{N_j}^{-1}f_i - h_{i,k_{N_j}})\| &\le a_{N_j}^{-1}\Big((\varepsilon k_{N_j})^{1/2}M_{N_j-1} + \varepsilon k_{N_j}\textstyle\sum_{n=N_j+1}^{\infty} a_n\|\chi_{[0,\frac{\lambda(E_n)}{k_n}]}\|\Big)\\ &\le 2^{-N_j+1}\varepsilon^{1/2}\|\chi_{E_{N_j}}\| + \varepsilon 2^{-N_j}\|\chi_{E_{N_j}}\| := (*),\end{aligned}$$

and we can assume that $(*) \le 2^{-j-1}\varepsilon$ since ε is specified before N_1. From this and (23) we get that

$$\|\sum_{j=1}^{l} a_{N_j}^{-1}\sum_{i\in\sigma_j} f_i - \chi_A + \chi_B\|_1 < \varepsilon + \tau/2 < \tau \tag{24}$$

for small enough ε. Thus the claim is satisfied for $f := \sum_{j=1}^{l} a_{N_j}^{-1}\sum_{i\in\sigma_j} f_i$. ■

Lemma 3 *Assume $(X, \|\cdot\|)$ is a rearrangement invariant function space on $[0,1]$ and $\int_0^1\|\chi_{[0,t]}\|\frac{dt}{t} < \infty$. Let $\{r_i\}$ be a sequence of three valued symmetric random variables whose absolute values are characteristic functions of sets $\{A_i\}$ of equal measure, all contained in one set A, and assume the r_i's are independent as random variables in the probability space A with the normalized Lebesgue measure. Then, $\{r_i\}$ is equivalent to an orthonormal sequence.*

Proof: By a simple and classical computation in the space $L_4(A, \frac{dt}{\lambda(A)})$ we get

$$\|\sum a_j r_j\|_4^4 = m \sum a_j^4 + 6m^2 \sum_{j<k} a_j^2 a_k^2 \leq 3m\Big(\sum a_j^2\Big)^2$$

where $m = \lambda(A_j)/\lambda(A)$, hence the L_2 and L_4 norms are equivalent, and this yields that the L_1 and L_2 norms are also equivalent. By the fact that the norm in X dominates the L_1 norm we get that $\{r_i\}$, as a sequence of elements of X, dominates the same sequence considered as a sequence in $L_1(A, \frac{dt}{\lambda(A)})$. The later is equivalent to an orthogonal sequence by the preceding argument.

For the other inequality we first assume as me may that $A = [0, 1]$. For each sequence of coefficients $\{a_i\}$ with $\sum a_i^2 = 1$ and each $t > 0$ we have the well known inequality

$$\lambda(|\sum a_i r_i| > t) \leq 2e^{-t^2/2}.$$

Since

$$|\sum a_i r_i| \leq \sum_{j=1}^{\infty} j\chi_{(j-1<|\sum a_i r_i| \leq j)} = \sum_{t=0}^{\infty} \chi_{(|\sum a_i r_i|>t)},$$

we get that $\|\sum a_i r_i\|$ is dominated by

$$\sum_{t=0}^{\infty} \|\chi_{(|\sum a_i r_i|>t)}\| \leq 2\sum_{t=0}^{\infty} \|\chi_{[0,e^{-t^2/2}]}\|.$$

The last sum is comparable to

$$\int_0^{\infty} \|\chi_{[0,e^{-t^2/2}]}\| dt = \int_0^1 \|\chi_{[0,s]}\| \frac{ds}{s\sqrt{\log \frac{1}{2s}}},$$

which, by (15), is finite. ■

Remark 2 The proofs of Theorems 1 and Theorem 2 actually show that the weakly null sequence obtained has the stronger property that every subsequence has a block basis which is an arbitrarily small perturbation of a Haar system. Under some mild extra condition on the rearrangement invariant space (separability, for example, is sufficient), the closed linear span of a Haar system is norm one complemented (via a conditional expectation), which implies that the corresponding block basis of the weakly null sequence spans a complemented subspace.

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