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# Multiplication operators on $L(L_p)$ and $\ell_p$-strictly singular operators

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**Abstract.** A classification of weakly compact multiplication operators on $L(L_p)$, $1 < p < \infty$, is given. This answers a question raised by Saksman and Tylli in 1992. The classification involves the concept of $\ell_p$-strictly singular operators, and we also investigate the structure of general $\ell_p$-strictly singular operators on $L_p$. The main result is that if an operator $T$ on $L_p$, $1 < p < 2$, is $\ell_p$-strictly singular and $T_{|X}$ is an isomorphism for some subspace $X$ of $L_p$, then $X$ embeds into $L_r$ for all $r < 2$, but $X$ need not be isomorphic to a Hilbert space.

It is also shown that if $T$ is convolution by a biased coin on $L_p$ of the Cantor group, $1 \le p < 2$, and $T_{|X}$ is an isomorphism for some reflexive subspace $X$ of $L_p$, then $X$ is isomorphic to a Hilbert space. The case $p = 1$ answers a question asked by Rosenthal in 1976.

**Keywords.** Elementary operators, multiplication operators, strictly singular operators, $L_p$ spaces, biased coin

## 1. Introduction

Given (always bounded, linear) operators $A$, $B$ on a Banach space $X$, define $L_A$, $R_B$ on $L(X)$ (the space of bounded linear operators on $X$) by $L_A T = AT$, $R_B T = TB$. Operators of the form $L_A R_B$ on $L(X)$ are called *multiplication operators*. The beginning point of this paper is a problem raised in 1992 by E. Saksman and H.-O. Tylli [ST1] (see also [ST2, Problem 2.8]):

Characterize the multiplication operators on $L(L_p)$, $1 < p \neq 2 < \infty$, which are weakly compact.

Here $L_p$ is $L_p(0, 1)$ or, equivalently, $L_p(\mu)$ for any purely non-atomic separable probability $\mu$.

In Theorem 1 we answer the Saksman–Tylli question. The characterization is rather simple but gives rise to questions about operators on $L_p$, some of which were asked by Tylli. First we set some terminology. Given an operator $T : X \to Y$ and a Banach space $Z$, say that $T$ is $Z$-*strictly singular* provided there is no subspace $Z_0$ of $X$ which

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is isomorphic to $Z$ for which $T_{|Z_0}$ is an isomorphism. An operator $S : Z \to W$ *factors through an operator* $T : X \to Y$ provided there are operators $A : Z \to X$ and $B : Y \to W$ so that $S = BTA$. If $S$ factors through the identity operator on $X$, we say that $S$ *factors through* $X$.

If $T$ is an operator on $L_p$, $1 < p < \infty$, then $T$ is $\ell_p$-strictly singular (respectively, $\ell_2$-strictly singular) if and only if $I_{\ell_p}$ (respectively, $I_{\ell_2}$) does not factor through $T$. This is true because every subspace of $L_p$ which is isomorphic to $\ell_p$ (respectively, $\ell_2$) has a subspace which is still isomorphic to $\ell_p$ (respectively, $\ell_2$) and is complemented in $L_p$. Actually, a stronger fact is true: if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $L_p$ which is equivalent to the unit vector basis for either $\ell_p$ or $\ell_2$, then $\{x_n\}_{n=1}^{\infty}$ has a subsequence which spans a complemented subspace of $L_p$. For $p > 2$, an even stronger theorem was proved by Kadec–Pełczyński [KP]. When $1 < p < 2$ and $\{x_n\}_{n=1}^{\infty}$ is a sequence in $L_p$ which is equivalent to the unit vector basis for $\ell_2$, one takes $\{y_n\}_{n=1}^{\infty}$ in $L_{p'}$ (where $p' = p/(p-1)$ is the conjugate index to $p$) which are uniformly bounded and biorthogonal to $\{x_n\}_{n=1}^{\infty}$. By passing to a subsequence which is weakly convergent and subtracting the limit from each $y_n$, one may assume that $y_n \to 0$ weakly and hence, by the Kadec–Pełczyński dichotomy [KP], has a subsequence that is equivalent to the unit vector basis of $\ell_2$ (since it is clearly impossible that $\{y_n\}_{n=1}^{\infty}$ has a subsequence equivalent to the unit vector basis of $\ell_{p'}$). This implies that the corresponding subsequence of $\{x_n\}_{n=1}^{\infty}$ spans a complemented subspace of $L_p$. (Pełczyński showed this argument, or something similar, to one of the authors many years ago, and a closely related result was proved in [PR].) Finally, when $1 < p < 2$ and $\{x_n\}_{n=1}^{\infty}$ is a sequence in $L_p$ which is equivalent to the unit vector basis for $\ell_p$, see the comments after the statement of Lemma 1.

Notice that the comments in the preceding paragraph imply that an operator on $L_p$, $1 < p < \infty$, is $\ell_p$-strictly singular (respectively, $\ell_2$-strictly singular) if and only if $T^*$ is $\ell_{p'}$-strictly singular (respectively, $\ell_2$-strictly singular). Better known is that an operator on $L_p$, $1 < p < \infty$, is strictly singular if it is both $\ell_p$-strictly singular and $\ell_2$-strictly singular (and hence $T$ is strictly singular if and only if $T^*$ is strictly singular). For $p > 2$ this is immediate from [KP], and Lutz Weis [We] proved the case $p < 2$.

Although Saksman and Tylli did not know a complete characterization of the weakly compact multiplication operators on $L(L_p)$, they realized that a classification must involve $\ell_p$- and $\ell_2$-strictly singular operators on $L_p$. This led Tylli to ask us about possible classifications of the $\ell_p$- and $\ell_2$-strictly singular operators on $L_p$. The $\ell_2$ case is known. It is enough to consider the case $2 < p < \infty$. If $T$ is an operator on $L_p$, $2 < p < \infty$, and $T$ is $\ell_2$-strictly singular, then it is an easy consequence of the Kadec–Pełczyński dichotomy that $I_{p,2}T$ is compact, where $I_{p,r}$ is the identity mapping from $L_p$ into $L_r$. But then by [Jo], $T$ factors through $\ell_p$. Tylli then asked whether the following conjecture is true:

**Tylli Conjecture.** If $T$ is an $\ell_p$-strictly singular operator on $L_p$, $1 < p < \infty$, then $T$ is in the closure (in the operator norm) of the operators on $L_p$ that factor through $\ell_2$. (It is clear that the closure is needed because not all compact operators on $L_p$, $p \neq 2$, factor through $\ell_2$.)

We then formulated a weaker conjecture:

**Weak Tylli Conjecture.** If $T$ is an $\ell_p$-strictly singular operator on $L_p$, $1 < p < \infty$, and $J : L_p \to \ell_\infty$ is an isometric embedding, then $JT$ is in the closure of the operators from $L_p$ into $\ell_\infty$ that factor through $\ell_2$.

It is of course evident that an operator on $L_p$, $p \neq 2$, that satisfies the conclusion of the Weak Tylli Conjecture must be $\ell_p$-strictly singular. There is a slight subtlety in these conjectures: while the Tylli Conjecture for $p$ is equivalent to the Tylli Conjecture for $p'$, it is not at all clear and is even false that the Weak Tylli Conjecture for $p$ is equivalent to the Weak Tylli Conjecture for $p'$. In fact, we observe in Lemma 2 (it is simple) that for $p > 2$ the Weak Tylli Conjecture is true, while the example in Section 4 shows that the Tylli Conjecture is false for all $p \neq 2$ and the Weak Tylli Conjecture is false for $p < 2$.

There are however some interesting consequences of the Weak Tylli Conjecture that are true when $p < 2$. In Theorem 4 we prove that if $T$ is an $\ell_p$-strictly singular operator on $L_p$, $1 < p < 2$, then $T$ is $\ell_r$-strictly singular for all $p < r < 2$. In view of theorems of Aldous [Al] (see also [KM]) and Rosenthal [Ro3], this proves that if such a $T$ is an isomorphism on a subspace $Z$ of $L_p$, then $Z$ embeds into $L_r$ for all $r < 2$. The Weak Tylli Conjecture would imply that $Z$ is isomorphic to $\ell_2$, but the example in Section 4 shows that this need not be true. When we discovered Theorem 4, we thought its proof bizarre and assumed that a more straightforward argument would yield a stronger theorem. The example suggests that in fact the proof may be "natural".

In Section 5 we discuss convolution by a biased coin on $L_p$ of the Cantor group, $1 \leq p < 2$. We prove that if $T_{|X}$ is an isomorphism for some reflexive subspace $X$ of $L_p$, $1 \leq p < 2$, then $X$ is isomorphic to a Hilbert space. This answers an old question of H. P. Rosenthal [Ro4].

The standard Banach space theory terminology and background we use can be found in [LT].

## 2. Weakly compact multiplication operators on $L(L_p)$

We use freely the result [ST2, Proposition 2.5] that if $A$, $B$ are in $L(X)$ where $X$ is a reflexive Banach space with the approximation property, then the multiplication operator $L_A R_B$ on $L(X)$ is weakly compact if and only if for every $T$ in $L(X)$, the operator $ATB$ is compact. For completeness, in Section 6 we give another proof of this under the weaker assumption that $X$ is reflexive and has the compact approximation property. This theorem implies that for such an $X$, $L_A R_B$ is weakly compact on $L(X)$ if and only if $L_{B^*} R_{A^*}$ is a weakly compact operator on $L(X^*)$. Consequently, to classify weakly compact multiplication operators on $L(L_p)$, $1 < p < \infty$, it is enough to consider the case $p > 2$. For $p \leq r$ we denote the identity operator from $\ell_p$ into $\ell_r$ by $i_{p,r}$. It is immediate from [KP] that an operator $T$ on $L_p$, $2 < p < \infty$, is compact if and only if $i_{2,p}$ does not factor through $T$.

**Theorem 1.** *Let $2 < p < \infty$ and let $A$, $B$ be bounded linear operators on $L_p$. Then the multiplication operator $L_A R_B$ on $L(L_p)$ is weakly compact if and only if one of the following (mutually exclusive) conditions hold:*

(a) *$i_{2,p}$ does not factor through A (i.e., A is compact).*
(b) *$i_{2,p}$ factors through A but $i_{p,p}$ does not factor through A (i.e., A is $\ell_p$-strictly singular) and $i_{2,2}$ does not factor through B (i.e., B is $\ell_2$-strictly singular).*
(c) *$i_{p,p}$ factors through A but $i_{2,p}$ does not factor through B (i.e., B is compact).*

*Proof.* The proof is a straightforward application of the Kadec–Pełczyński dichotomy principle [KP]: if $\{x_n\}_{n=1}^{\infty}$ is a semi-normalized (i.e., bounded and bounded away from zero) weakly null sequence in $L_p$, $2 < p < \infty$, then there is a subsequence which is equivalent to either the unit vector basis of $\ell_p$ or of $\ell_2$ and spans a complemented subspace of $L_p$. Notice that this immediately implies the "i.e.'s" in the statement of the theorem so that (a) and (c) imply that $L_A R_B$ is weakly compact. Now assume that (b) holds and let $T$ be in $L(L_P)$. If $ATB$ is not compact, then there is a normalized weakly null sequence $\{x_n\}_{n=1}^{\infty}$ in $L_p$ so that $ATBx_n$ is bounded away from zero. By passing to a subsequence, we may assume that $\{x_n\}_{n=1}^{\infty}$ is equivalent to either the unit vector basis of $\ell_p$ or of $\ell_2$. If $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis of $\ell_p$, then since $TBx_n$ is bounded away from zero, we can assume by passing to another subsequence that also $TBx_n$ is equivalent to the unit vector basis of $\ell_p$, and similarly for $ATBx_n$, which contradicts the assumption that $A$ is $\ell_p$-strictly singular. On the other hand, if $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis of $\ell_2$, then since $B$ is $\ell_2$-strictly singular we can assume by passing to a subsequence that $Bx_n$ is equivalent to the unit vector basis of $\ell_p$ and continue as in the previous case to get a contradiction.

Now suppose that (a), (b), and (c) are all false. If $i_{p,p}$ factors through $A$ and $i_{2,p}$ factors through $B$ then there is sequence $\{x_n\}_{n=1}^{\infty}$ equivalent to the unit vector basis of $\ell_2$ or of $\ell_p$ so that $Bx_n$ is equivalent to the unit vector basis of $\ell_2$ or of $\ell_p$ (of course, only three of the four cases are possible) and $Bx_n$ spans a complemented subspace of $L_p$. Moreover, there is a sequence $\{y_n\}_{n=1}^{\infty}$ in $L_p$ so that both $y_n$ and $Ay_n$ are equivalent to the unit vector basis of $\ell_p$. Since $Bx_n$ spans a complemented subspace of $L_p$, the mapping $Bx_n \mapsto y_n$ extends to a bounded linear operator $T$ on $L_p$ and $ATB$ is not compact. Finally, suppose that $i_{2,p}$ factors through $A$ but $i_{p,p}$ does not factor through $A$ and $i_{2,2}$ factors through $B$. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ so that $x_n$ and $Bx_n$ are both equivalent to the unit vector basis of $\ell_2$ and $Bx_n$ spans a complemented subspace of $L_p$. There is also a sequence $\{y_n\}_{n=1}^{\infty}$ equivalent to the unit vector basis of $\ell_2$ so that $Ay_n$ is equivalent to the unit vector basis of $\ell_2$ or of $\ell_p$. The mapping $Bx_n \mapsto y_n$ extends to a bounded linear operator $T$ on $L_p$ and $ATB$ is not compact. □

It is perhaps worthwhile to restate Theorem 1 in a way that the cases where $L_A R_B$ is weakly compact are not mutually exclusive.

**Theorem 2.** *Let $2 < p < \infty$ and let A, B be bounded linear operators on $L_p$. Then the multiplication operator $L_A R_B$ on $L(L_p)$ is weakly compact if and only if one of the following conditions hold:*

(a) *A is compact.*
(b) *A is $\ell_p$-strictly singular and B is $\ell_2$-strictly singular.*
(c) *B is compact.*

### 3. $\ell_p$-strictly singular operators on $L_p$

We recall the well known

**Lemma 1.** *Let $W$ be a bounded convex symmetric subset of $L_p$, $1 \le p \ne 2 < \infty$. The following are equivalent:*

(1) *No sequence in $W$ equivalent to the unit vector basis for $\ell_p$ spans a complemented subspace of $L_p$.*
(2) *For every $C$ there exists $n$ so that no length $n$ sequence in $W$ is $C$-equivalent to the unit vector basis of $\ell_p^n$.*
(3) *For each $\varepsilon > 0$ there is $M_\varepsilon$ so that $W \subset \varepsilon B_{L_p} + M_\varepsilon B_{L_\infty}$.*
(4) *$|W|^p$ is uniformly integrable, i.e.,* $\lim_{t\downarrow 0} \sup_{x\in W} \sup_{\mu(E)<t} \|\mathbf{1}_E x\|_p = 0$.

When $p = 1$, the assumptions that $W$ is convex and symmetric are not needed, and the conditions in Lemma 1 are equivalent to the non-weak-compactness of the weak closure of $W$. This case is essentially proved in [KP] and proofs can also be found in books; see, e.g., [Wo, Theorem 3.C.12]. (Condition (3) does not appear in [Wo], but it is easy to check the equivalence of (3) and (4). Also, in the proof in [Wo, Theorem 3.C.12] that not (4) implies not (1), Wojtaszczyk only constructs a basic sequence in $W$ that is equivalent to the unit vector basis for $\ell_1$; however, it is clear that the constructed basic sequence spans a complemented subspace.)

For $p > 2$, Lemma 1 and stronger versions of condition (1) can be deduced from [KP]. For $1 < p < 2$, one needs to modify slightly the proof in [Wo] for the case $p = 1$. The only essential modification comes in the proof that not (4) implies not (1), and this is where it is needed that $W$ is convex and symmetric. Just as in [Wo], one shows that not (4) implies that there is a sequence $\{x_n\}_{n=1}^\infty$ in $W$ and a sequence $\{E_n\}_{n=1}^\infty$ of disjoint measurable sets so that $\inf \|1_{E_n} x_n\|_p > 0$. By passing to a subsequence, we can assume that $\{x_n\}_{n=1}^\infty$ converges weakly to, say, $x$. Suppose first that $x = 0$. Then by passing to a further subsequence, we may assume that $\{x_n\}_{n=1}^\infty$ is a small perturbation of a block basis of the Haar basis for $L_p$ and hence is an unconditionally basic sequence. Since $L_p$ has type $p$, this implies that there is a constant $C$ so that for all sequences $\{a_n\}_{n=1}^\infty$ of scalars, $\|\sum a_n x_n\|_p \le C(\sum |a_n|^p)^{1/p}$. Let $P$ be the norm one projection from $L_p$ onto the closed linear span $Y$ of the disjoint sequence $\{\mathbf{1}_{E_n} x_n\}_{n=1}^\infty$. Then $Px_n$ is weakly null in a space isometric to $\ell_p$, and $\|Px_n\|_p$ is bounded away from zero, so there is a subsequence $\{Px_{n(k)}\}_{k=1}^\infty$ which is equivalent to the unit vector basis for $\ell_p$ and whose closed span is the range of a projection $Q$ from $Y$. The projection $QP$ from $L_p$ onto the the closed span of $\{Px_{n(k)}\}_{k=1}^\infty$ maps $x_{n(k)}$ to $Px_{n(k)}$, and because of the upper $p$ estimate on $\{x_{n(k)}\}_{k=1}^\infty$, maps the closed span of $\{x_{n(k)}\}_{k=1}^\infty$ isomorphically onto the closed span of $\{Px_{n(k)}\}_{k=1}^\infty$. This implies that $\{x_{n(k)}\}_{k=1}^\infty$ is equivalent to the unit vector basis for $\ell_p$ and spans a complemented subspace. Suppose now that the weak limit $x$ of $\{x_n\}_{n=1}^\infty$ is not zero. Choose a subsequence $\{x_{n(k)}\}_{k=1}^\infty$ so that $\inf \|1_{E_{n(2k+1)}}(x_{n(2k)} - x_{n(2k+1)})\|_p > 0$ and replace $\{x_n\}_{n=1}^\infty$ with $\{(x_{n(2k)} - x_{n(2k+1)})/2\}_{k=1}^\infty$ in the argument above.

Notice that the argument outlined above gives that if $\{x_n\}_{n=1}^\infty$ is a sequence in $L_p$, $1 < p \ne 2 < \infty$, which is equivalent to the unit vector basis of $\ell_p$, then there is a subsequence $\{y_n\}_{n=1}^{\infty}$ whose closed linear span in $L_p$ is complemented. This is how one proves that the identity on $\ell_p$ factors through any operator on $L_p$ which is not $\ell_p$-strictly singular.

The Weak Tylli Conjecture for $p > 2$ is an easy consequence of the following lemma.

**Lemma 2.** *Let $T$ be an operator from an $\mathcal{L}_1$ space $V$ into $L_p$, $1 < p < 2$, so that $W := TB_V$ satisfies condition* (1) *in Lemma* 1. *Then for each $\varepsilon > 0$ there is an operator $S : V \to L_2$ so that $\|T - I_{2,p}S\| < \varepsilon$.*

*Proof.* Let $\varepsilon > 0$. By condition (3) in Lemma 1, for each norm one vector $x$ in $V$ there is a vector $Ux$ in $L_2$ with $\|Ux\|_2 \le \|Ux\|_\infty \le M_\varepsilon$ and $\|Tx - Ux\|_p \le \varepsilon$. By the definition of $\mathcal{L}_1$ space, we can write $V$ as a directed union $\bigcup_\alpha E_\alpha$ of finite-dimensional spaces that are uniformly isomorphic to $\ell_1^{n_\alpha}$, $n_\alpha = \dim E_\alpha$, and let $(x_i^\alpha)_{i=1}^{n_\alpha}$ be norm one vectors in $E_\alpha$ which are, say, $\lambda$-equivalent to the unit vector basis for $\ell_1^{n_\alpha}$ with $\lambda$ independent of $\alpha$. Let $U_\alpha$ be the linear extension to $E_\alpha$ of the mapping $x_i^\alpha \mapsto Ux_i^\alpha$, considered as an operator into $L_2$. Then $\|T_{|E_\alpha} - I_{2,p}U_\alpha\| \le \lambda\varepsilon$ and $\|U_\alpha\| \le \lambda M_\varepsilon$. A standard Lindenstrauss compactness argument produces an operator $S : V \to L_2$ so that $\|S\| \le \lambda M_\varepsilon$ and $\|T - I_{2,p}S\| \le \lambda\varepsilon$. Indeed, extend $U_\alpha$ to all of $V$ by letting $U_\alpha x = 0$ if $x \notin E_\alpha$. The net $T_\alpha$ has a subnet $S_\beta$ so that for each $x$ in $V$, $S_\beta x$ converges weakly in $L_2$; call the limit $Sx$. It is easy to check that $S$ has the properties claimed. □

**Theorem 3.** *Let $T$ be an $\ell_p$-strictly singular operator on $L_p$, $2 < p < \infty$, and let $J$ be an isometric embedding of $L_p$ into an injective $Z$. Then for each $\varepsilon > 0$ there is an operator $S : L_p \to Z$ so that $S$ factors through $\ell_2$ and $\|JT - S\| < \varepsilon$.*

*Proof.* Lemma 2 gives the conclusion when $J$ is the adjoint of a quotient mapping from $\ell_1$ or $L_1$ onto $L_{p'}$. The general case then follows from the injectivity of $Z$. □

The next proposition, when souped up via "abstract nonsense" and known results, gives our main result about $\ell_p$-strictly singular operators on $L_p$. Note that it shows that an $\ell_p$-strictly singular operator on $L_p$, $1 < p < 2$, cannot be the identity on the span of a sequence of $r$-stable independent random variables for any $p < r < 2$. We do not know another way of proving even this special case of our main result.

**Proposition 1.** *Let $T$ be an $\ell_p$-strictly singular operator on $L_p$, $1 < p < 2$. If $X$ is a subspace of $L_p$ and $T_{|X} = aI_X$ with $a \neq 0$, then $X$ embeds into $L_s$ for all $s < 2$.*

*Proof.* By making a change of density, we can by [JJ] assume that $T$ is also a bounded linear operator on $L_2$, so assume, without loss of generality, that $\|T\|_p \vee \|T\|_2 = 1$, so that, in particular, $a \le 1$. Lemma 1 gives for each $\epsilon > 0$ a constant $M_\epsilon$ so that

$$TB_{L_p} \subset \epsilon B_{L_p} + M_\epsilon B_{L_2}. \tag{1}$$

Indeed, otherwise condition (1) in Lemma 1 gives a bounded sequence $\{x_n\}_{n=1}^{\infty}$ in $L_p$ so that $\{Tx_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis of $\ell_p$. By passing to a subsequence of differences of $\{x_n\}_{n=1}^{\infty}$, we can assume, without loss of generality, that $\{x_n\}_{n=1}^{\infty}$

is a small perturbation of a block basis of the Haar basis for $L_p$ and hence is an unconditionally basic sequence. Since $L_p$ has type $p$, the sequence $\{x_n\}_{n=1}^{\infty}$ has an upper $p$ estimate, which means that there is a constant $C$ so that for all sequences $\{a_n\}_{n=1}^{\infty}$ of scalars, $\|\sum a_n x_n\| \leq C\|(\sum |a_n|^p)^{1/p}\|$. Since $\{Tx_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis of $\ell_p$, $\{x_n\}_{n=1}^{\infty}$ also has a lower $p$ estimate and hence $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis of $\ell_p$. This contradicts the $\ell_p$-strict singularity of $T$.

Iterating this we get, for every $n$ and $0 < \epsilon < 1/2$,

$$a^n B_X \subset T^n B_{L_p} \subset \epsilon^n B_{L_p} + 2M_\epsilon B_{L_2}$$

or, setting $A := 1/a$,

$$B_X \subset A^n \epsilon^n B_{L_p} + 2A^n M_\epsilon B_{L_2}.$$

For $f$ a unit vector in $X$ write $f = f_n + g_n$ with $\|f_n\|_2 \leq 2A^n M_\epsilon$ and $\|g_n\|_p \leq (A\epsilon)^n$. Then $f_{n+1} - f_n = g_n - g_{n+1}$, and since evidently $f_n$ can be chosen to be of the form $(f \vee -k_n) \wedge k_n$ (with appropriate interpretation when the set $[f_n = \pm k_n]$ has positive measure), the choice of $f_n$, $g_n$ can be made so that

$$\|f_{n+1} - f_n\|_2 \leq \|f_{n+1}\|_2 \leq 2M_\epsilon A^{n+1}, \qquad \|g_n - g_{n+1}\|_p \leq \|g_n\|_p \leq (A\epsilon)^n.$$

(Alternatively, to avoid thinking, just take any $f = f_n + g_n$ so that $\|f_n\|_2 \leq 2A^n M_\epsilon$ and $\|g_n\|_p \leq (A\epsilon)^n$. Each left side of the two displayed inequalities is less than twice the corresponding right side as long as $A\varepsilon \leq 1$.)

For $p < s < 2$ write $1/s = \theta/2 + (1-\theta)/p$. Then

$$\|f_{n+1} - f_n\|_s \leq \|f_{n+1} - f_n\|_2^\theta \|g_n - g_{n+1}\|_p^{1-\theta} \leq (2M_\epsilon A)^\theta (A\epsilon^{1-\theta})^n,$$

which is summable if $\epsilon^{1-\theta} < 1/A$. But $\|f - f_n\|_p \to 0$ so $f = f_1 + \sum_{n=1}^{\infty} f_{n+1} - f_n$ in $L_p$ and hence also in $L_s$ if $\epsilon^{1-\theta} < 1/A$. So for some constant $C_s$ we conclude for all $f \in X$ that $\|f\|_p \leq \|f\|_s \leq C_s \|f\|_p$. □

We can now prove our main theorem. For background on ultrapowers of Banach spaces, see [DJT, Chapter 8].

**Theorem 4.** *Let $T$ be an $\ell_p$-strictly singular operator on $L_p$, $1 < p < 2$. If $X$ is a subspace of $L_p$ and $T_{|X}$ is an isomorphism, then $X$ embeds into $L_r$ for all $r < 2$.*

*Proof.* In view of Rosenthal's theorem [Ro3], it is enough to prove that $X$ has type $s$ for all $s < 2$. By the Krivine–Maurey–Pisier theorem, [Kr] and [MP] (or, alternatively, Aldous' theorem, [Al] or [KM]), we only need to check that for $p < s < 2$, $X$ does not contain almost isometric copies of $\ell_s^n$ for all $n$. (To apply the Krivine–Maurey–Pisier theorem we use that the second condition in Lemma 1, applied to the unit ball of $X$, implies that $X$ has type $s$ for some $p < s \leq 2$.) So suppose that for some $p < s < 2$, $X$ contains almost isometric copies of $\ell_s^n$ for all $n$. By applying Krivine's theorem [Kr] we get for each $n$ a sequence $(f_i^n)_{i=1}^n$ of unit vectors in $X$ which is $1+\epsilon$-equivalent to the unit vector basis for $\ell_s^n$ and, for some constant $C$ (which we can take independently of $n$), the sequence $(CTf_i^n)_{i=1}^n$ is also $1+\epsilon$-equivalent to the unit vector basis for $\ell_s^n$. By replacing $T$ by $CT$, we might as well assume that $C = 1$. Now consider an ultrapower $T_{\mathcal{U}}$, where $\mathcal{U}$ is a free ultrafilter on the natural numbers. The domain and codomain of $T_{\mathcal{U}}$ is the (abstract) $L_p$ space $(L_p)_{\mathcal{U}}$, and $T_{\mathcal{U}}$ is defined by $T_{\mathcal{U}}(f_1, f_2, \dots) = (Tf_1, Tf_2, \dots)$ for any (equivalence class of a) bounded sequence $(f_1, f_2, \dots)$. It is evident that $T_{\mathcal{U}}$ is an isometry on the ultraproduct of span $(f_i^n)_{i=1}^n$, $n = 1, 2, \dots$, and hence $T_{\mathcal{U}}$ is an isometry on a subspace of $(L_p)_{\mathcal{U}}$ which is isometric to $\ell_s$. Since condition (2) in Lemma 1 is obviously preserved when taking an ultrapower of a set, we see that $T_{\mathcal{U}}$ is $\ell_p$-strictly singular. Finally, by restricting $T_{\mathcal{U}}$ to a suitable subspace, we get an $\ell_p$-strictly singular operator $S$ on $L_p$ and a subspace $Y$ of $L_p$ so that $Y$ is isometric to $\ell_s$ and $S_{|Y}$ is an isometry. By restricting the domain of $S$, we can assume that $Y$ has full support and the functions in $Y$ generate the Borel sets. It then follows from the Plotkin–Rudin theorem [Pl], [Ru] (see [KK, Theorem 1]) that $S_{|Y}$ extends to an isometry $W$ from $L_p$ into $L_p$. Since any isometric copy of $L_p$ in $L_p$ is norm one complemented (see [La, §17]), there is a norm one operator $V : L_p \to L_p$ so that $VW = I_{L_p}$. Then $VS_{|Y} = I_Y$ and $VS$ is $\ell_p$-strictly singular, which contradicts Proposition 1. □

**Remark 1.** The $\ell_1$-strictly singular operators on $L_1$ also form an interesting class. They are the weakly compact operators on $L_1$. In terms of factorization, they are just the closure in the operator norm of the integral operators on $L_1$ (see, e.g., the proof of Lemma 2).

## 4. The example

Rosenthal [Ro1] proved that if $\{x_n\}_{n=1}^\infty$ is a sequence of three-valued, symmetric, independent random variables, then for all $1 < p < \infty$, the closed span in $L_p$ of $\{x_n\}_{n=1}^\infty$ is complemented by means of the orthogonal projection $P$, and $\|P\|_p$ depends only on $p$, not on the specific sequence $\{x_n\}_{n=1}^\infty$. Moreover, he showed that if $p > 2$, then for any sequence $\{x_n\}_{n=1}^\infty$ of symmetric, independent random variables in $L_p$, $\|\sum x_n\|_p$ is equivalent (with constant depending only on $p$) to $(\sum \|x_n\|_p^p)^{1/p} \vee (\sum \|x_n\|_2^2)^{1/2}$. Thus if $\{x_n\}_{n=1}^\infty$ is normalized in $L_p$, $p > 2$, and $w_n := \|x_n\|_2$, then $\|\sum a_n x_n\|_p$ is equivalent to $\|\{a_n\}_{n=1}^\infty\|_{p,w} := (\sum |a_n|^p)^{1/p} \vee (\sum |a_n|^2 w_n^2)^{1/2}$. The completion of the finitely nonzero sequences of scalars under the norm $\|\cdot\|_{p,w}$ is called $X_{p,w}$. It follows that if $w = \{w_n\}_{n=1}^\infty$ is any sequence of numbers in $[0, 1]$. Then $X_{p,w}$ is isomorphic to a complemented subspace of $L_p$. Suppose now that $w = \{w_n\}_{n=1}^\infty$ and $v = \{v_n\}_{n=1}^\infty$ are two such sequences of weights and $v_n \ge w_n$. Then the diagonal operator $D$ from $X_{p,w}$ to $X_{p,v}$ that sends the $n$th unit vector basis vector $e_n$ to $(w_n/v_n)e_n$ is contractive, and it is more or less obvious that $D$ is $\ell_p$-strictly singular if $w_n/v_n \to 0$ as $n \to \infty$. Since $X_{p,w}$ and $X_{p,v}$ are isomorphic to complemented subspaces of $L_p$, the adjoint operator $D^*$ is $\ell_{p'}$-strictly singular and (identifying $X_{p,w}^*$ and $X_{p,v}^*$ with subspaces of $L_{p'}$) extends to an $\ell_{p'}$-strictly singular operator on $L_{p'}$. Our goal in this section is to produce weights $w$ and $v$ so that $D^*$ is an isomorphism on a subspace of $X_{p,v}^*$ which is not isomorphic to a Hilbert space.

For all $0 < r < 2$ there is a positive constant $c_r$ such that

$$|t|^r = c_r \int_0^\infty \frac{1 - \cos tx}{x^{r+1}}\, dx$$

for all $t \in \mathbb{R}$. It follows that for any closed interval $[a, b] \subset (0, \infty)$ and for all $\varepsilon > 0$ there are $0 < x_1 < \cdots < x_{n+1}$ such that $\max_{1 \le j \le n} \left| \frac{x_{j+1} - x_j}{x_j^{r+1}} \right| \le \varepsilon$ and

$$\left| c_r \sum_{j=1}^{n} \frac{x_{j+1} - x_j}{x_j^{r+1}} (1 - \cos t x_j) - |t|^r \right| < \varepsilon \tag{2}$$

for all $t$ with $|t| \in [a, b]$.

Let $0 < q < r < 2$ and define $v_j$ and $a_j$, $j = 1, \dots, n$, by

$$v_j^{2q/(2-q)} = c_r \frac{x_{j+1} - x_j}{x_j^{r+1}}, \qquad \frac{a_j}{v_j^{2/(2-q)}} = x_j.$$

Let $Y_j$, $j = 1, \dots, n$, be independent, symmetric, three-valued random variables such that $|Y_j| = v_j^{-2/(2-q)} \mathbf{1}_{B_j}$ with $\mathrm{Prob}(B_j) = v_j^{2q/(2-q)}$, so that in particular $\|Y_j\|_q = 1$ and $v_j = \|Y_j\|_q / \|Y_j\|_2$. Then the characteristic function of $Y_j$ is

$$\varphi_{Y_j}(t) = 1 - v_j^{2q/(2-q)} + v_j^{2q/(2-q)} \cos(t v_j^{-2/(2-q)}) = 1 - v_j^{2q/(2-q)} (1 - \cos(t v_j^{-2/(2-q)}))$$

and

$$\begin{aligned} \varphi_{\sum a_j Y_j}(t) &= \prod_{j=1}^{n} (1 - v_j^{2q/(2-q)} (1 - \cos(t a_j v_j^{-2/(2-q)}))) \\ &= \prod_{j=1}^{n} \left( 1 - c_r \frac{x_{j+1} - x_j}{x_j^{r+1}} (1 - \cos(t x_j)) \right). \end{aligned} \tag{3}$$

To evaluate this product we use the estimates on $\frac{x_{j+1} - x_j}{x_j^{r+1}}$ to deduce that, for each $j$

$$\begin{aligned} &\left| \log\left( 1 - c_r \frac{x_{j+1} - x_j}{x_j^{r+1}} (1 - \cos(t x_j)) \right) + c_r \frac{x_{j+1} - x_j}{x_j^{r+1}} (1 - \cos(t x_j)) \right| \\ &\qquad \le C \varepsilon c_r^2 \frac{x_{j+1} - x_j}{x_j^{r+1}} (1 - \cos(t x_j)) \end{aligned}$$

for some absolute $C < \infty$. Then, by (2),

$$\begin{aligned} &\left| \sum_{j=1}^{n} \log\left( 1 - c_r \frac{x_{j+1} - x_j}{x_j^{r+1}} (1 - \cos(t x_j)) \right) + c_r \sum_{j=1}^{n} \frac{x_{j+1} - x_j}{x_j^{r+1}} (1 - \cos(t x_j)) \right| \\ &\qquad \le C \varepsilon c_r (\varepsilon + b^r). \end{aligned}$$

Using (2) again we get

$$\left| \sum_{j=1}^{n} \log\left( 1 - c_r \frac{x_{j+1} - x_j}{x_j^{r+1}} (1 - \cos(t x_j)) \right) + |t|^r \right| \le (C + 1) \varepsilon (\varepsilon + b^r)$$

(assuming as we may that $b \geq 1$), and from (3) we get

$$\varphi_{\sum a_j Y_j}(t) = (1 + O(\varepsilon)) \exp(-|t|^r)$$

for all $|t| \in [a, b]$, where the function hiding under the $O$ notation depends on $r$ and $b$ but on nothing else. It follows that, given any $\eta > 0$, one can find $a, b$ and $\varepsilon$ such that for the corresponding $\{a_j, Y_j\}$ there is a symmetric $r$-stable $Y$ (with characteristic function $e^{-|t|^r}$) satisfying

$$\Big\| Y - \sum_{j=1}^{n} a_j Y_j \Big\|_q \leq \eta.$$

This follows from classical translation of various convergence notions; see e.g. [Ro2, p. 154].

Let now $0 < \delta < 1$. Put $w_j = \delta v_j$, $j = 1, \ldots, n$, and let $Z_j$, $j = 1, \ldots, n$, be independent, symmetric, three-valued random variables such that $|Z_j| = w_j^{-2/(2-q)} \mathbf{1}_{C_j}$ with $\mathrm{Prob}(C_j) = w_j^{2q/(2-q)}$, so that in particular $\|Z_j\|_q = 1$ and $w_j = \|Z_j\|_q / \|Z_j\|_2$. In a similar manner to the argument above we see that

$$\begin{aligned}
\varphi_{\sum \delta a_j Z_j}(t) &= \prod_{j=1}^{n} (1 - w_j^{2q/(2-q)} (1 - \cos(t \delta a_j w_j^{-2/(2-q)}))) \\
&= \prod_{j=1}^{n} (1 - \delta^{2q/(2-q)} v_j^{2q/(2-q)} (1 - \cos(t \delta^{-q/(2-q)} a_j v_j^{-2/(2-q)}))) \\
&= (1 + O(\varepsilon)) \exp(-\delta^{q(2-r)/(2-q)} |t|^r)
\end{aligned}$$

for all $|t| \in [\delta^{q/(2-q)} a, \delta^{q/(2-q)} b]$, where the $O$ now depends also on $\delta$.

Assuming $\delta^{q(2-r)/(2-q)} > 1/2$ and for a choice of $a, b$ and $\varepsilon$ depending on $\delta, r, q$ and $\eta$ we find that there is a symmetric $r$-stable random variable $Z$ (with characteristic function $e^{-\delta^{q(2-r)/(2-q)} |t|^r}$) such that

$$\Big\| Z - \sum_{j=1}^{n} \delta a_j Z_j \Big\|_q \leq \eta.$$

Note that the ratio between the $L_q$ norms of $Y$ and $Z$ is bounded away from zero and infinity by universal constants and each of these norms is also universally bounded away from zero. Consequently, if $\varepsilon$ is small enough the ratio between the $L_q$ norms of $\sum_{j=1}^n a_j Y_j$ and $\sum_{j=1}^n \delta a_j Z_j$ is bounded away from zero and infinity by universal constants.

Let now $\delta_i$ be any sequence decreasing to zero and $r_i$ any sequence such that $q < r_i \uparrow 2$ and $\delta_i^{q(2-r_i)/(2-q)} > 1/2$. Then for any sequence $\varepsilon_i \downarrow 0$ we can find two sequences of symmetric, independent, three-valued random variables $\{Y_i\}$ and $\{W_i\}$, all normalized in $L_q$, with the following additional properties:

- Put $v_j = \|Y_j\|_q / \|Y_j\|_2$ and $w_j = \|Z_j\|_q / \|Z_j\|_2$. Then there are disjoint finite subsets of the integers $\sigma_i$, $i = 1, 2, \ldots$, such that $w_j = \delta_i v_j$ for $j \in \sigma_i$.

- There are independent random variables $\{\bar{Y}_i\}$ and $\{\bar{Z}_i\}$ with $\bar{Y}_i$ and $\bar{Z}_i$ $r_i$-stable with bounded, from zero and infinity, ratio of $L_q$ norms and there are coefficients $\{a_j\}$ such that
$$\Big\|\bar{Y}_i - \sum_{j\in\sigma_i} a_j Y_j\Big\|_q < \varepsilon_i \quad \text{and} \quad \Big\|\bar{Z}_i - \sum_{j\in\sigma_i} \delta_i a_j Z_j\Big\|_q < \varepsilon_i.$$

From [Ro1] we know that the spans of $\{Y_j\}$ and $\{Z_j\}$ are complemented in $L_q$, $1 < q < 2$, and the dual spaces are naturally isomorphic to $X_{p,\{v_j\}}$ and $X_{p,\{w_j\}}$ respectively; both the isomorphism constants and the complementation constants depend only on $q$. Here $p = q/(q-1)$ and

$$\|\{\alpha_j\}\|_{X_{p,\{u_j\}}} = \max\Big\{\Big(\sum |\alpha_j|^p\Big)^{1/p}, \Big(\sum u_j^2\alpha_j^2\Big)^{1/2}\Big\}.$$

Under this duality the adjoint $D^*$ to the operator $D$ that sends $Y_j$ to $\delta_i Z_j$ for $j \in \sigma_i$ is formally the same diagonal operator between $X_{p,\{w_i\}}$ and $X_{p,\{v_i\}}$. The relation $w_j = \delta_i v_j$ for $j \in \sigma_i$ easily implies that this is a bounded operator; $\delta_i \to 0$ implies that this operator is $\ell_q$-strictly singular. If $\varepsilon_i \to 0$ fast enough, $D^*$ preserves a copy of span$\{\bar{Y}_i\}$. Finally, if $r_i$ tends to 2 not too fast this span is not isomorphic to a Hilbert space. Indeed, let $1 \le s_j \uparrow 2$ be arbitrary and let $\{n_j\}_{j=1}^{\infty}$ be a sequence of positive integers with $n_j^{1/s_j - 1/2} \ge j$, $j = 1, 2, \ldots$, say. For $1 \le k \le n_j$, put $r_{n_1+\cdots+n_{j-1}+k} = s_j$. Then the span of $\{Y_i\}_{i=n_1+\cdots+n_{j-1}+1}^{n_1+\cdots+n_j}$ is isomorphic, with constant independent of $j$, to $\ell_{s_j}^{n_j}$ and this last space is of distance at least $j$ from a Euclidean space.

It follows that if $J : L_q \to \ell_\infty$ is an isometric embedding, then $JD^*$ cannot be arbitrarily approximated by an operator which factors through a Hilbert space, and hence the Weak Tylli Conjecture is false in the range $1 < q < 2$.

## 5. Convolution by a biased coin

In this section we regard $L_p$ as $L_p(\Delta)$, where $\Delta = \{-1, 1\}^{\mathbb{N}}$ is the Cantor group and the measure is the Haar measure $\mu$ on $\Delta$; i.e., $\mu = \prod_{n=1}^{\infty} \mu_n$, where $\mu_n(-1) = \mu_n(1) = 1/2$. For $0 < \varepsilon < 1$, let $\nu_\varepsilon$ be the $\varepsilon$-biased coin tossing measure, i.e., $\nu_\varepsilon = \prod_{n=1}^{\infty} \nu_{\varepsilon,n}$, where $\nu_{\varepsilon,n}(1) = (1+\varepsilon)/2$ and $\nu_{\varepsilon,n}(-1) = (1-\varepsilon)/2$. Let $T_\varepsilon$ be convolution by $\nu_\varepsilon$, so that for a $\mu$-integrable function $f$ on $\Delta$, $(T_\varepsilon f)(x) = (f * \nu_\varepsilon)(x) = \int_\Delta f(xy)\, d\nu_\varepsilon(y)$. The operator $T_\varepsilon$ is a contraction on $L_p$ for all $1 \le p \le \infty$. Let us recall how $T_\varepsilon$ acts on the characters on $\Delta$. For $t = \{t_n\}_{n=1}^{\infty} \in \Delta$, let $r_n(t) = t_n$. The characters on $\Delta$ are finite products of these *Rademacher functions* $r_n$ (where the void product is the constant one function). For $A$ a finite subset of $\mathbb{N}$, set $w_A = \prod_{n\in A} r_n$ and let $W_n$ be the linear span of $\{w_A : |A| = n\}$. Then $T_\varepsilon w_A = \varepsilon^{|A|} w_A$.

We are interested in studying $T_\varepsilon$ on $L_p$, $1 \le p < 2$. The background we mention below is all contained in Bonami's paper [Bo] (or see [Ro4]). On $L_p$, $1 < p < 2$, $T_\varepsilon$ is $\ell_p$-strictly singular; in fact, $T_\varepsilon$ even maps $L_p$ into $L_r$ for some $r = r(p,\varepsilon) > p$. Indeed, by interpolation it is sufficient to check that $T_\varepsilon$ maps $L_s$ into $L_2$ for some $s = s(\varepsilon) < 2$. But there is a constant $C_s$ which tends to 1 as $s \uparrow 2$ so that for all $f \in W_n$, $\|f\|_2 \le C_s^n \|f\|_s$

and the orthogonal projection $P_n$ onto (the closure of) $W_n$ satisfies $\|P_n\|_p \leq C_s^n$. From this it is easy to check that if $\varepsilon C_s^2 < 1$, then $T_\varepsilon$ maps $L_s$ into $L_2$. We remark in passing that Bonami [Bo] found for each $p$ (including $p \geq 2$) and $\varepsilon$ the largest value of $r = r(p, \varepsilon)$ such that $T_\varepsilon$ maps $L_p$ into $L_r$.

Thus Theorem 4 shows that if $X$ is a subspace of $L_p$, $1 < p < 2$, and $T_\varepsilon$ (considered as an operator from $L_p$ to $L_p$) is an isomorphism on $X$, then $X$ embeds into $L_s$ for all $s < 2$. Since, as we mentioned above, $T_\varepsilon$ maps $L_s$ into $L_2$ for some $s < 2$, it then follows from an argument in [Ro4] that $X$ must be isomorphic to a Hilbert space. (Actually, as we show after the proof, Lemma 3 is that we can prove Theorem 5 without using Theorem 4.) Since [Ro4] is not generally available, we repeat Rosenthal's argument in Lemma 3 below.

Now $T_\varepsilon$ is not $\ell_1$-strictly singular on $L_1$. Nevertheless, we still find that if $X$ is a reflexive subspace of $L_1$, and $T_\varepsilon$ (considered as an operator from $L_1$ to $L_1$) is an isomorphism on $X$, then $X$ is isomorphic to a Hilbert space. Indeed, Rosenthal showed (see Lemma 3) that then there is another subspace $X_0$ of $L_1$ which is isomorphic to $X$ so that $X_0$ is contained in $L_p$ for some $1 < p < 2$, the $L_p$ and $L_1$ norms are equivalent on $X_0$, and $T_\varepsilon$ is an isomorphism on $X_0$. This implies that as an operator on $L_p$, $T_\varepsilon$ is an isomorphism on $X_0$ and hence $X_0$ is isomorphic to a Hilbert space. (To apply Lemma 3, use the fact [Ro3] that if $X$ is a relexive subspace of $L_1$, then $X$ embeds into $L_p$ for some $1 < p < 2$.)

We summarize this discussion in the first sentence of Theorem 5. The case $p = 1$ solves Problem B from Rosenthal's 1976 paper [Ro4].

**Theorem 5.** *Let $1 \leq p < 2$, let $0 < \varepsilon < 1$, and let $T_\varepsilon$ be considered as an operator on $L_p$. If $X$ is a reflexive subspace of $L_p$ and the restriction of $T_\varepsilon$ to $X$ is an isomorphism, then $X$ is isomorphic to a Hilbert space. Moreover, if $p > 1$, then $X$ is complemented in $L_p$.*

We now prove Rosenthal's lemma [Ro4, proof of Theorem 5] and defer the proof of the "moreover" statement in Theorem 5 until after the proof of the lemma.

**Lemma 3.** *Suppose that $T$ is an operator on $L_p$, $1 \leq p < r < s < 2$, $X$ is a subspace of $L_p$ which is isomorphic to a subspace of $L_s$, and $T_{|X}$ is an isomorphism. Then there is another subspace $X_0$ of $L_p$ which is isomorphic to $X$ so that $X_0$ is contained in $L_r$, the $L_r$ and $L_p$ norms are equivalent on $X_0$, and $T$ is an isomorphism on $X_0$.*

*Proof.* We want to find a measurable set $E$ so that

(1) $X_0 := \{\mathbf{1}_E x : x \in X\}$ is isomorphic to $X$,
(2) $X_0 \subset L_r$,
(3) $T_{|X_0}$ is an isomorphism.

(We did not say that $\|\cdot\|_p$ and $\|\cdot\|_r$ are equivalent on $X_0$ since that follows formally from the closed graph theorem. The isomorphism $X \to X_0$ guaranteed by (a) is of course the mapping $x \mapsto \mathbf{1}_E x$.)

Assume, without loss of generality, that $\|T\| = 1$. Take $a > 0$ so that $\|Tx\|_p \geq a\|x\|_p$ for all $x$ in $X$. Since $\ell_p$ does not embed into $L_s$ we see from (4) in Lemma 1 that there is

$\eta > 0$ so that if $E$ has measure larger than $1 - \eta$, then $\|\mathbf{1}_{\sim E} x\|_p \le (a/2)\|x\|_p$ for all $x$ in $X$. Obviously (1) and (3) are satisfied for any such $E$. It is proved in [Ro3] that there is a strictly positive $g$ with $\|g\|_1 = 1$ so that $x/g$ is in $L_r$ for all $x$ in $X$. Now simply choose $t < \infty$ so that $E := [g < t]$ has measure at least $1 - \eta$; then $E$ satisfies (1)–(3). □

Next we remark how to avoid using Theorem 4 in proving Theorem 5. Suppose that $T_\varepsilon$ is an isomorphism on a reflexive subspace $X$ of $L_p$, $1 \le p < 2$. Let $s$ be the supremum of those $r \le 2$ such that $X$ is isomorphic to a subspace of $L_r$, so $1 < s \le 2$. It is sufficient to show that $s = 2$. But if $s < 2$, the interpolation formula implies that if $r < s$ is sufficiently close to $s$, then $T_\varepsilon$ maps $L_r$ into $L_t$ for some $t > s$ and hence, by Lemma 3, $X$ embeds into $L_t$.

Finally, we prove the "moreover" statement in Theorem 5. We now know that $X$ is isomorphic to a Hilbert space. In the proof of Lemma 3, instead of using Rosenthal's result from [Ro3], use Grothendieck's theorem [DJT, Theorem 3.5], which implies that there is a strictly positive $g$ with $\|g\|_1 = 1$ so that $x/g$ is in $L_2$ for all $x$ in $X$. Choosing $E$ the same way as in the proof of Lemma 3 with $T := T_\varepsilon$, we see that (1)–(3) are true with $r = 2$. Now the $L_2$ and $L_p$ norms are equivalent on both $X_0$ and on $T_\varepsilon X_0$. But it is clear that the only way that $T_\varepsilon$ can be an isomorphism on a subspace $X_0$ of $L_2$ is for the orthogonal projection $P_n$ onto the closed span of $W_k$, $0 \le k \le n$, to be an isomorphism on $X_0$ for some finite $n$. But then also in the $L_p$ norm the restriction of $P_n$ to $X_0$ is an isomorphism, because the $L_p$ norm and the $L_2$ norm are equivalent on the span of $W_k$, $0 \le k \le n$, and $P_n$ is bounded on $L_p$ (since $p > 1$). It follows that the operator $S := P_n \circ \mathbf{1}_E$ on $L_p$ maps $X_0$ isomorphically onto a complemented subspace of $L_p$, which implies that $X_0$ is also complemented in $L_p$.

Here is the problem that started us thinking about $\ell_p$-strictly singular operators:

**Problem 1.** Let $1 < p < 2$ and $0 < \varepsilon < 1$. On $L_p(\Delta)$, does $T_\varepsilon$ satisfy the conclusion of the Tylli Conjecture?

After we submitted this paper, G. Pisier [Pi] answered Problem 1 in the affirmative.

Although the example in Section 4 shows that the Tylli Conjecture is false, something close to it may be true:

**Problem 2.** Let $1 < p < r < 2$. Is every $\ell_p$-strictly singular operator on $L_p$ in the closure of the operators on $L_p$ that factor through $L_r$?

## 6. Appendix

In this appendix we prove a theorem that is essentially due to Saksman and Tylli. The only novelty is that we assume the compact approximation property rather than the approximation property.

**Theorem 6.** *Let $X$ be a reflexive Banach space and let $A$, $B$ be in $L(X)$. Then*

(a) *If $ATB$ is a compact operator on $X$ for every $T$ in $L(X)$, then $L_A R_B$ is a weakly compact operator on $L(X)$.*

(b) *If $X$ has the compact approximation property and $L_A R_B$ is a weakly compact operator on $L(X)$, then $ATB$ is a compact operator on $X$ for every $T$ in $L(X)$.*

*Proof.* To prove (a), recall [Kal] that for a reflexive space $X$, on bounded subsets of $K(X)$ the weak topology is the same as the weak operator topology (the operator $T \mapsto f_T \in C((B_X, \text{weak}) \times (B_{X^*}, \text{weak}))$, where $f_T(x, x^*) := \langle x^*, Tx \rangle$, is an isometric isomorphism from $K(X)$ into a space of continuous functions on a compact Hausdorff space). Now if $(T_\alpha)$ is a bounded net in $L(X)$, then since $X$ is reflexive there is a subnet (which we still denote by $(T_\alpha)$) which converges in the weak operator topology to, say, $T \in L(X)$. Then $AT_\alpha B$ converges in the the weak operator topology to $ATB$. But since all these operators are in $K(X)$, $AT_\alpha B$ converges weakly to $ATB$ by Kalton's theorem. This shows that $L_A R_B$ is a weakly compact operator on $L(X)$.

To prove (b), suppose that we have a $T \in L(X)$ with $ATB$ not compact. Then there is a weakly null normalized sequence $\{x_n\}_{n=1}^\infty$ in $X$ and $\delta > 0$ so that for all $n$, $\|ATBx_n\| > \delta$. Since a reflexive space with the compact approximation property also has the compact metric approximation property [CJ], there are $C_n \in K(X)$ with $\|C_n\| < 1 + 1/n$ and $C_n Bx_i = Bx_i$ for $i \le n$. Since the $C_n$ are compact, for each $n$, $\|C_n Bx_m\| \to 0$ as $m \to \infty$. Thus $A(TC_n)Bx_i = ATBx_i$ for $i \le n$ and $\|A(TC_n)Bx_m\| \to 0$ as $m \to \infty$. This implies that no convex combination of $\{A(TC_n)B\}_{n=1}^\infty$ can converge in the norm of $L(X)$ and hence $\{A(TC_n)B\}_{n=1}^\infty$ has no weakly convergent subsequence. This contradicts the weak compactness of $L_A R_B$ and completes the proof. □

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