Multiplication operators on $L(L_p)$ and ℓ_p-strictly singular operators*

William B. Johnson[‡] Gideon Schechtman[§]

September 18, 2007

Abstract

A classification of weakly compact multiplication operators on $L(L_p)$, $1 < p < \infty$, is given. This answers a question raised by Saksman and Tylli in 1992. The classification involves the concept of ℓ_p-strictly singular operators, and we also investigate the structure of general ℓ_p-strictly singular operators on L_p. The main result is that if an operator T on L_p, $1 < p < 2$, is ℓ_p-strictly singular and $T_{|X}$ is an isomorphism for some subspace X of L_p, then X embeds into L_r for all $r < 2$, but X need not be isomorphic to a Hilbert space.

It is also shown that if T is convolution by a biased coin on L_p of the Cantor group, $1 \le p < 2$, and $T_{|X}$ is an isomorphism for some reflexive subspace X of L_p, then X is isomorphic to a Hilbert space. The case $p = 1$ answers a question asked by Rosenthal in 1976.

1 Introduction

Given (always bounded, linear) operators A, B on a Banach space X, define L_A, R_B on $L(X)$ (the space of bounded linear operators on X) by $L_AT = AT$, $R_BT = TB$. Operators of the form L_AR_B on $L(X)$ are called *multiplication operators*. The beginning point of this paper is a problem raised in 1992 by E. Saksman and H.-O. Tylli [ST1] (see also [ST2, Problem 2.8]):

Characterize the multiplication operators on $L(L_p)$, $1 < p \neq 2 < \infty$, which are weakly compact.

---

*AMS subject classification: 46B20,46E30. Key words: Elementary operators, multiplication operators, strictly singular operators, L_p spaces, biased coin

†Supported in part by NSF DMS-0503688 and U.S.-Israel Binational Science Foundation.

‡Corresponding author.

§Supported in part by Israel Science Foundation and U.S.-Israel Binational Science Foundation; participant, NSF Workshop in Analysis and Probability, Texas A&M University.

Here L_p is $L_p(0,1)$ or, equivalently, $L_p(\mu)$ for any purely non-atomic separable probability μ.

In Theorem 1 we answer the Saksman-Tylli question. The characterization is rather simple but gives rise to questions about operators on L_p, some of which were asked by Tylli. First we set some terminology. Given an operator $T : X \to Y$ and a Banach space Z, say that T is Z-strictly singular provided there is no subspace Z_0 of X which is isomorphic to Z for which $T_{|Z_0}$ is an isomorphism. An operator $S : Z \to W$ factors through an operator $T : X \to Y$ provided there are operators $A : Z \to X$ and $B : Y \to W$ so that $S = BTA$. If S factors through the identity operator on X, we say that S factors through X.

If T is an operator on L_p, $1 < p < \infty$, then T is ℓ_p-strictly singular (respectively, ℓ_2-strictly singular) if and only if I_{ℓ_p} (respectively, I_{ℓ_2}) does not factor through T. This is true because every subspace of L_p which is isomorphic to ℓ_p (respectively, ℓ_2) has a subspace which is still isomorphic to ℓ_p (respectively, ℓ_2) and is complemented in L_p. Actually, a stronger fact is true: if $\{x_n\}_{n=1}^\infty$ is a sequence in L_p which is equivalent to the unit vector basis for either ℓ_p or ℓ_2, then $\{x_n\}_{n=1}^\infty$ has a subsequence which spans a complemented subspace of L_p. For $p > 2$, an even stronger theorem was proved by Kadec-Pełczyński [KP]. When $1 < p < 2$ and $\{x_n\}_{n=1}^\infty$ is a sequence in L_p which is equivalent to the unit vector basis for ℓ_2, one takes $\{y_n\}_{n=1}^\infty$ in $L_{p'}$ (where $p' = \frac{p}{p-1}$ is the conjugate index to p) which are uniformly bounded and biorthogonal to $\{x_n\}_{n=1}^\infty$. By passing to a subsequence which is weakly convergent and subtracting the limit from each y_n, one may assume that $y_n \to 0$ weakly and hence, by the Kadec–Pełczyński dichotomy [KP], has a subsequence that is equivalent to the unit vector basis of ℓ_2 (since it is clearly impossible that $\{y_n\}_{n=1}^\infty$ has a subsequence equivalent to the unit vector basis of $\ell_{p'}$) . This implies that that the corresponding subsequence of $\{x_n\}_{n=1}^\infty$ spans a complemented subspace of L_p. (Pełczyński showed this argument, or something similar, to one of the authors many years ago, and a closely related result was proved in [PR].) Finally, when $1 < p < 2$ and $\{x_n\}_{n=1}^\infty$ is a sequence in L_p which is equivalent to the unit vector basis for ℓ_p, see the comments after the statement of Lemma 1.

Notice that the comments in the preceding paragraph yield that an operator on L_p, $1 < p < \infty$, is ℓ_p-strictly singular (respectively, ℓ_2-strictly singular) if and only if T^* is $\ell_{p'}$-strictly singular (respectively, ℓ_2-strictly singular). Better known is that an operator on L_p, $1 < p < \infty$, is strictly singular if it is both ℓ_p-strictly singular and ℓ_2-strictly singular (and hence T is strictly singular if and only if T^* is strictly singular). For $p > 2$ this is immediate from [KP], and Lutz Weis [We] proved the case $p < 2$.

Although Saksman and Tylli did not know a complete characterization of the weakly compact multiplication operators on $L(L_p)$, they realized that a classification must involve ℓ_p and ℓ_2-strictly singular operators on L_p. This led Tylli to ask us about possible classifications of the ℓ_p and ℓ_2-strictly singular operators on L_p. The ℓ_2 case is known. It is enough to consider the case $2 < p < \infty$. If T is an operator on L_p, $2 < p < \infty$, and T is ℓ_2-strictly singular, then it is an easy consequence of the Kadec-Pełczyński dichotomy that $I_{p,2}T$ is compact, where $I_{p,r}$ is the identity mapping from L_p into L_r. But then by

[Jo], T factors through ℓ_p. Tylli then asked whether the following conjecture is true:

Tylli Conjecture. If T is an ℓ_p-strictly singular operator on L_p, $1 < p < \infty$, then T is in the closure (in the operator norm) of the operators on L_p that factor through ℓ_2. (It is clear that the closure is needed because not all compact operators on L_p, $p \neq 2$, factor through ℓ_2.)

We then formulated a weaker conjecture:

Weak Tylli Conjecture. If T is an ℓ_p-strictly singular operator on L_p, $1 < p < \infty$, and $J : L_p \to \ell_\infty$ is an isometric embedding, then JT is in the closure of the operators from L_p into ℓ_∞ that factor through ℓ_2.

It is of course evident that an operator on L_p, $p \neq 2$, that satisfies the conclusion of the Weak Tylli Conjecture must be ℓ_p-strictly singular. There is a slight subtlety in these conjectures: while the Tylli Conjecture for p is equivalent to the Tylli Conjecture for p', it is not at all clear and is even false that the Weak Tylli Conjecture for p is equivalent to the Weak Tylli Conjecture for p'. In fact, we observe in Lemma 2 (it is simple) that for $p > 2$ the Weak Tylli Conjecture is true, while the example in Section 4 yields that the Tylli Conjecture is false for all $p \neq 2$ and the Weak Tylli Conjecture is false for $p < 2$.

There are however some interesting consequences of the Weak Tylli Conjecture that are true when $p < 2$. In Theorem 4 we prove that if T is an ℓ_p-strictly singular operator on L_p, $1 < p < 2$, then T is ℓ_r-strictly singular for all $p < r < 2$. In view of theorems of Aldous [Al] (see also [KM]) and Rosenthal [Ro3], this proves that if such a T is an isomorphism on a subspace Z of L_p, then Z embeds into L_r for all $r < 2$. The Weak Tylli Conjecture would imply that Z is isomorphic to ℓ_2, but the example in Section 4 shows that this need not be true. When we discovered Theorem 4, we thought its proof bizarre and assumed that a more straightforward argument would yield a stronger theorem. The example suggests that in fact the proof may be "natural".

In Section 5 we discuss convolution by a biased coin on L_p of the Cantor group, $1 \leq p < 2$. We do not know whether such an operator T on L_p, $1 < p < 2$, must satisfy the Tylli Conjecture or the weak Tylli conjecture. We do prove, however, that if $T_{|X}$ is an isomorphism for some reflexive subspace X of L_p, $1 \leq p < 2$, then X is isomorphic to a Hilbert space. This answers an old question of H. P. Rosenthal [Ro4].

The standard Banach space theory terminology and background we use can be found in [LT].

2 Weakly compact multiplication operators on $L(L_p)$

We use freely the result [ST2, Proposition 2.5] that if A, B are in $L(X)$ where X is a reflexive Banach space with the approximation property, then the multiplication operator $L_A R_B$ on $L(X)$ is weakly compact if and only if for every T in $L(X)$, the operator ATB is compact. For completeness, in section 6 we give another proof of this under the weaker assumption that X is reflexive and has the compact approximation property. This theorem implies that for such an X, $L_A R_B$ is weakly compact on $L(X)$ if and only if $L_{B^*} R_{A^*}$ is a weakly compact operator on $L(X^*)$. Consequently, to classify weakly compact multiplication operators on $L(L_p)$, $1 < p < \infty$, it is enough to consider the case $p > 2$. For $p \le r$ we denote the identity operator from ℓ_p into ℓ_r by $i_{p,r}$. It is immediate from [KP] that an operator T on L_p, $2 < p < \infty$, is compact if and only if $i_{2,p}$ does not factor through T.

Theorem 1 *Let $2 < p < \infty$ and let A, B be bounded linear operators on L_p. Then the multiplication operator $L_A R_B$ on $L(L_p)$ is weakly compact if and only if one of the following (mutually exclusive) conditions hold.*

(a) $i_{2,p}$ does not factor through A (i.e., A is compact)

(b) $i_{2,p}$ factors through A but $i_{p,p}$ does not factor through A (i.e., A is ℓ_p-strictly singular) and $i_{2,2}$ does not factor through B (i.e., B is ℓ_2-strictly singular)

(c) $i_{p,p}$ factors through A but $i_{2,p}$ does not factor through B (i.e., B is compact)

Proof: The proof is a straightforward application of the Kadec-Pełczyński dichotomy principle [KP]: if $\{x_n\}_{n=1}^\infty$ is a semi-normalized (i.e., bounded and bounded away from zero) weakly null sequence in L_p, $2 < p < \infty$, then there is a subsequence which is equivalent to either the unit vector basis of ℓ_p or of ℓ_2 and spans a complemented subspace of L_p. Notice that this immediately implies the "i.e.'s" in the statement of the theorem so that (a) and (c) imply that $L_A R_B$ is weakly compact. Now assume that (b) holds and let T be in $L(L_P)$. If ATB is not compact, then there is a normalized weakly null sequence $\{x_n\}_{n=1}^\infty$ in L_p so that $ATBx_n$ is bounded away from zero. By passing to a subsequence, we may assume that $\{x_n\}_{n=1}^\infty$ is equivalent to either the unit vector basis of ℓ_p or of ℓ_2. If $\{x_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_p, then since TBx_n is bounded away from zero, we can assume by passing to another subsequence that also TBx_n is equivalent to the unit vector basis of ℓ_p and similarly for $ATBx_n$, which contradicts the assumption that A is ℓ_p-strictly singular. On the other hand, if $\{x_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_2, then since B is ℓ_2-strictly singular we can assume by passing to a subsequence that Bx_n is equivalent to the unit vector basis of ℓ_p and continue as in the previous case to get a contradiction.

Now suppose that (a), (b), and (c) are all false. If $i_{p,p}$ factors through A and $i_{2,p}$ factors through B then there is sequence $\{x_n\}_{n=1}^\infty$ equivalent to the unit vector basis of ℓ_2 or of ℓ_p so that Bx_n is equivalent to the unit vector basis of ℓ_2 or of ℓ_p (of course, only three of the four cases are possible) and Bx_n spans a complemented subspace of L_p. Moreover, there is a sequence $\{y_n\}_{n=1}^\infty$ in L_p so that both y_n and Ay_n are equivalent to the unit vector basis of ℓ_p. Since Bx_n spans a complemented subspace of L_p, the mapping $Bx_n \mapsto y_n$ extends to a bounded linear operator T on L_p and ATB is not compact. Finally, suppose that $i_{2,p}$ factors through A but $i_{p,p}$ does not factor through A and $i_{2,2}$ factors through B. Then there is a sequence $\{x_n\}_{n=1}^\infty$ so that x_n and Bx_n are both equivalent to the unit vector basis of ℓ_2 and Bx_n spans a complemented subspace of L_p. There is also a sequence $\{y_n\}_{n=1}^\infty$ equivalent to the unit vector basis of ℓ_2 so that

Ay_n is equivalent to the unit vector basis of ℓ_2 or of ℓ_p. The mapping $Bx_n \mapsto y_n$ extends to a bounded linear operator T on L_p and ATB is not compact. ■

It is perhaps worthwhile to restate Theorem 1 in a way that the cases where $L_A R_B$ is weakly compact are not mutually exclusive.

Theorem 2 *Let $2 < p < \infty$ and let A, B be bounded linear operators on L_p. Then the multiplication operator $L_A R_B$ on $L(L_p)$ is weakly compact if and only if one of the following conditions hold.*

(a) A is compact

(b) A is ℓ_p-strictly singular and B is ℓ_2-strictly singular

(c) B is compact

3 ℓ_p-strictly singular operators on L_p

We recall the well known

Lemma 1 *Let W be a bounded convex symmetric subset of L_p, $1 \le p \ne 2 < \infty$. The following are equivalent:*

1. *No sequence in W equivalent to the unit vector basis for ℓ_p spans a complemented subspace of L_p.*

2. *For every C there exists n so that no length n sequence in W is C-equivalent to the unit vector basis of ℓ_p^n.*

3. *For each $\varepsilon > 0$ there is M_ε so that $W \subset \varepsilon B_{L_p} + M_\varepsilon B_{L_\infty}$.*

4. *$|W|^p$ is uniformly integrable; i.e., $\lim_{t\downarrow 0} \sup_{x\in W} \sup_{\mu(E)<t} \|\mathbf{1}_E x\|_p = 0$.*

When $p = 1$, the assumptions that W is convex and W symmetric are not needed, and the conditions in Lemma 1 are equivalent to the non weak compactness of the weak closure of W. This case is essentially proved in [KP] and proofs can also be found in books; see, e.g., [Wo, Theorem 3.C.12]). (Condition (3) does not appear in [Wo], but it is easy to check the equivalence of (3) and (4). Also, in the proof in [Wo, Theorem 3.C.12]) that not (4) implies not (1), Wojtaszczyk only constructs a basic sequence in W that is equivalent to the unit vector basis for ℓ_1; however, it is clear that the constructed basic sequence spans a complemented subspace.)

For $p > 2$, Lemma 1 and stronger versions of condition (1) can be deduced from [KP]. For $1 < p < 2$, one needs to modify slightly the proof in [Wo] for the case $p = 1$. The only essential modification comes in the proof that not (4) implies not (1), and this is where it is needed that W is convex and symmetric. Just as in [Wo], one shows that not

(4) implies that there is a sequence $\{x_n\}_{n=1}^\infty$ in W and a sequence $\{E_n\}_{n=1}^\infty$ of disjoint measurable sets so that $\inf \|1_{E_n} x_n\|_p > 0$. By passing to a subsequence, we can assume that $\{x_n\}_{n=1}^\infty$ converges weakly to, say, x. Suppose first that $x = 0$. Then by passing to a further subsequence, we may assume that $\{x_n\}_{n=1}^\infty$ is a small perturbation of a block basis of the Haar basis for L_p and hence is an unconditionally basic sequence. Since L_p has type p, this implies that there is a constant C so that for all sequences $\{a_n\}_{n=1}^\infty$ of scalars, $\|\sum a_n x_n\|_p \le C(\sum |a_n|^p)^{1/p}$. Let P be the norm one projection from L_p onto the closed linear span Y of the disjoint sequence $\{\mathbf{1}_{E_n} x_n\}_{n=1}^\infty$. Then Px_n is weakly null in a space isometric to ℓ_p and $\|Px_n\|_p$ is bounded away from zero, so there is a subsequence $\{Px_{n(k)}\}_{k=1}^\infty$ which is equivalent to the unit vector basis for ℓ_p and whose closed span is the range of a projection Q from Y. The projection QP from L_p onto the the closed span of $\{Px_{n(k)}\}_{k=1}^\infty$ maps $x_{n(k)}$ to $Px_{n(k)}$ and, because of the upper p estimate on $\{x_{n(k)}\}_{k=1}^\infty$, maps the closed span of $\{x_{n(k)}\}_{k=1}^\infty$ isomorphically onto the closed span of $\{Px_{n(k)}\}_{k=1}^\infty$. This yields that $\{x_{n(k)}\}_{k=1}^\infty$ is equivalent to the unit vector basis for ℓ_p and spans a complemented subspace. Suppose now that the weak limit x of $\{x_n\}_{n=1}^\infty$ is not zero. Choose a subsequence $\{x_{n(k)}\}_{k=1}^\infty$ so that $\inf \|1_{E_{n(2k+1)}}(x_{n(2k)} - x_{n(2k+1)})\|_p > 0$ and replace $\{x_n\}_{n=1}^\infty$ with $\{\frac{x_{n(2k)} - x_{n(2k+1)}}{2}\}_{k=1}^\infty$ in the argument above.

Notice that the argument outlined above gives that if $\{x_n\}_{n=1}^\infty$ is a sequence in L_p, $1 < p \neq 2 < \infty$, which is equivalent to the unit vector basis of ℓ_p, then there is a subsequence $\{y_n\}_{n=1}^\infty$ whose closed linear span in L_p is complemented. This is how one proves that the identity on ℓ_p factors through any operator on L_p which is not ℓ_p-strictly singular.

The Weak Tylli Conjecture for $p > 2$ is an easy consequence of the following lemma.

Lemma 2 *Let T be an operator from a $\mathcal{L}_1$ space V into L_p, $1 < p < 2$, so that $W := TB_V$ satisfies condition (1) in Lemma 1. Then for each $\varepsilon > 0$ there is an operator $S : V \to L_2$ so that $\|T - I_{2,p}S\| < \varepsilon$.*

Proof: Let $\varepsilon > 0$. By condition (3) in Lemma 1, for each norm one vector x in V there is a vector Ux in L_2 with $\|Ux\|_2 \le \|Ux\|_\infty \le M_\varepsilon$ and $\|Tx - Ux\|_p \le \varepsilon$. By the definition of $\mathcal{L}_1$ space, we can write V as a directed union $\cup_\alpha E_\alpha$ of finite dimensional spaces that are uniformly isomorphic to $\ell_1^{n_\alpha}$, $n_\alpha = \dim E_\alpha$, and let $(x_i^\alpha)_{i=1}^{n_\alpha}$ be norm one vectors in E_α which are, say, λ-equivalent to the unit vector basis for $\ell_1^{n_\alpha}$ with λ independent of α. Let U_α be the linear extension to E_α of the mapping $x_i^\alpha \mapsto Ux_i^\alpha$, considered as an operator into L_2. Then $\|T_{|E_\alpha} - I_{2,p}U_\alpha\| \le \lambda\varepsilon$ and $\|U_\alpha\| \le \lambda M_\varepsilon$. A standard Lindenstrauss compactness argument produces an operator $S : V \to L_2$ so that $\|S\| \le \lambda M_\varepsilon$ and $\|T - I_{2,p}S\| \le \lambda\varepsilon$. Indeed, extend U_α to all of V by letting $U_\alpha x = 0$ if $x \notin E_\alpha$. The net T_α has a subnet S_β so that for each x in V, $S_\beta x$ converges weakly in L_2; call the limit Sx. It is easy to check that S has the properties claimed. ■

Theorem 3 *Let T be an ℓ_p-strictly singular operator on L_p, $2 < p < \infty$, and let J be an isometric embedding of L_p into an injective Z. Then for each $\varepsilon > 0$ there is an operator $S : L_p \to Z$ so that S factors through ℓ_2 and $\|JT - S\| < \varepsilon$.*

Proof: Lemma 2 gives the conclusion when J is the adjoint of a quotient mapping from ℓ_1 or L_1 onto $L_{p'}$. The general case then follows from the injectivity of Z. ■

The next proposition, when souped-up via "abstract nonsense" and known results, gives our main result about ℓ_p-strictly singular operators on L_p. Note that it shows that an ℓ_p-strictly singular operator on L_p, $1 < p < 2$, cannot be the identity on the span of a sequence of r-stable independent random variables for any $p < r < 2$. We do not know another way of proving even this special case of our main result.

Proposition 1 *Let T be an ℓ_p-strictly singular operator on L_p, $1 < p < 2$. If X is a subspace of L_p and $T_{|X} = aI_X$ with $a \neq 0$, then X embeds into L_s for all $s < 2$.*

Proof: By making a change of density, we can by [JJ] assume that T is also a bounded linear operator on L_2, so assume, without loss of generality, that $\|T\|_p \vee \|T\|_2 = 1$, so that, in particular, $a \leq 1$. Lemma 1 gives for each $\epsilon > 0$ a constant M_ϵ so that

$$TB_{L_p} \subset \epsilon B_{L_p} + M_\epsilon B_{L_2}. \tag{1}$$

Indeed, otherwise condition (1) in Lemma 1 gives a bounded sequence $\{x_n\}_{n=1}^\infty$ in L_p so that $\{Tx_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_p. By passing to a subsequence of differences of $\{x_n\}_{n=1}^\infty$, we can assume, without loss of generality, that $\{x_n\}_{n=1}^\infty$ is a small perturbation of a block basis of the Haar basis for L_p and hence is an unconditionally basic sequence. Since L_p has type p, the sequence $\{x_n\}_{n=1}^\infty$ has an upper p estimate, which means that there is a constant C so that for all sequences $\{a_n\}_{n=1}^\infty$ of scalars, $\|\sum a_n x_n\| \leq C\|(\sum |a_n|^p)^{1/p}\|$. Since $\{Tx_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_p, $\{x_n\}_{n=1}^\infty$ also has a lower p estimate and hence $\{x_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_p. This contradicts the ℓ_p strict singularity of T.

Iterating this we get for every n and $0 < \epsilon < 1/2$

$$a^n B_X \subset T^n B_{L_p} \subset \epsilon^n B_{L_p} + 2M_\epsilon B_{L_2}$$

or, setting $A := 1/a$,

$$B_X \subset A^n \epsilon^n B_{L_p} + 2A^n M_\epsilon B_{L_2}.$$

For f a unit vector in X write $f = f_n + g_n$ with $\|f_n\|_2 \leq 2A^n M_\epsilon$ and $\|g_n\|_p \leq (A\epsilon)^n$. Then $f_{n+1} - f_n = g_n - g_{n+1}$, and since evidently f_n can be chosen to be of the form $(f \vee -k_n) \wedge k_n$ (with appropriate interpretation when the set $[f_n = \pm k_n]$ has positive measure), the choice of f_n, g_n can be made so that

$$\|f_{n+1} - f_n\|_2 \leq \|f_{n+1}\|_2 \leq 2M_\epsilon A^{n+1}$$

$$\|g_n - g_{n+1}\|_p \leq \|g_n\|_p \leq (A\epsilon)^n.$$

(Alternatively, to avoid thinking, just take any $f = f_n + g_n$ so that $\|f_n\|_2 \leq 2A^n M_\epsilon$ and $\|g_n\|_p \leq (A\epsilon)^n$. Each left side of the two displayed inequalities is less than twice the corresponding right side as long as $A\varepsilon \leq 1$.)

For $p < s < 2$ write $\frac{1}{s} = \frac{\theta}{2} + \frac{1-\theta}{p}$. Then

$$\|f_{n+1} - f_n\|_s \leq \|f_{n+1} - f_n\|_2^\theta \|g_n - g_{n+1}\|_p^{1-\theta} \leq (2M_\epsilon A)^\theta (A\epsilon^{1-\theta})^n$$

which is summable if $\epsilon^{1-\theta} < 1/A$. But $\|f - f_n\|_p \to 0$ so $f = f_1 + \sum_{n=1}^\infty f_{n+1} - f_n$ in L_p and hence also in L_s if $\epsilon^{1-\theta} < 1/A$. So for some constant C_s we get for all $f \in X$ that $\|f\|_p \leq \|f\|_s \leq C_s \|f\|_p$. ■

We can now prove our main theorem. For background on ultrapowers of Banach spaces, see [DJT, Chapter 8].

Theorem 4 *Let T be an ℓ_p-strictly singular operator on L_p, $1 < p < 2$. If X is a subspace of L_p and $T_{|X}$ is an isomorphism, then X embeds into L_r for all $r < 2$.*

Proof: In view of Rosenthal's theorem [Ro3], it is enough to prove that X has type s for all $s < 2$. By virtue of of the Krivine-Maurey-Pisier theorem, [Kr] and [MP] (or, alternatively, Aldous' theorem, [Al] or [KM]), we only need to check that for $p < s < 2$, X does not contain almost isometric copies of ℓ_s^n for all n. (To apply the Krivine-Maurey-Pisier theorem we use that the second condition in Lemma 1, applied to the unit ball of X, yields that X has type s for some $p < s \leq 2$). So suppose that for some $p < s < 2$, X contains almost isometric copies of ℓ_s^n for all n. By applying Krivine's theorem [Kr] we get for each n a sequence $(f_i^n)_{i=1}^n$ of unit vectors in X which is $1+\epsilon$-equivalent to the unit vector basis for ℓ_s^n and, for some constant C (which we can take independently of n), the sequence $(CTf_i^n)_{i=1}^n$ is also $1+\epsilon$-equivalent to the unit vector basis for ℓ_s^n. By replacing T by CT, we might as well assume that $C = 1$. Now consider an ultrapower $T_\mathcal{U}$, where $\mathcal{U}$ is a free ultrafilter on the natural numbers. The domain and codomain of $T_\mathcal{U}$ is the (abstract) L_p space $(L_p)_\mathcal{U}$, and $T_\mathcal{U}$ is defined by $T_\mathcal{U}(f_1, f_2, \dots) = (Tf_1, Tf_2, \dots)$ for any (equivalence class of a) bounded sequence $(f_1, f_2, \dots)$. It is evident that $T_\mathcal{U}$ is an isometry on the ultraproduct of span $(f_i^n)_{i=1}^n$; $n = 1, 2, \dots$, and hence $T_\mathcal{U}$ is an isometry on a subspace of $(L_p)_\mathcal{U}$ which is isometric to ℓ_s. Since condition 2 in Lemma 1 is obviously preserved when taking an ultrapower of a set, we see that $T_\mathcal{U}$ is ℓ_p-strictly singular. Finally, by restricting $T_\mathcal{U}$ to a suitable subspace, we get an ℓ_p-strictly singular operator S on L_p and a subspace Y of L_p so that Y is isometric to ℓ_s and $S_{|Y}$ is an isometry. By restricting the domain of S, we can assume that Y has full support and the functions in Y generate the Borel sets. It then follows from the Plotkin-Rudin theorem [Pl], [Ru] (see [KK, Theorem 1]) that $S_{|Y}$ extends to an isometry W from L_p into L_p. Since any isometric copy of L_p in L_p is norm one complemented (see [La, §17]), there is a norm one operator $V : L_p \to L_p$ so that $VW = I_{L_p}$. Then $VS_{|Y} = I_Y$ and VS is ℓ_p-strictly singular, which contradicts Proposition 1. ■

Remark 1 *The ℓ_1-strictly singular operators on L_1 also form an interesting class. They are the weakly compact operators on L_1. In terms of factorization, they are just the closure in the operator norm of the integral operators on L_1 (see, e.g., the proof of Lemma 2).*

4 The example

Rosenthal [Ro1] proved that if $\{x_n\}_{n=1}^\infty$ is a sequence of three valued, symmetric, independent random variables, then for all $1 < p < \infty$, the closed span in L_p of $\{x_n\}_{n=1}^\infty$ is complemented by means of the orthogonal projection P, and $\|P\|_p$ depends only on p, not on the specific sequence $\{x_n\}_{n=1}^\infty$. Moreover, he showed that if $p > 2$, then for any sequence $\{x_n\}_{n=1}^\infty$ of symmetric, independent random variables in L_p, $\|\sum x_n\|_p$ is equivalent (with constant depending only on p) to $(\sum \|x_n\|_p^p)^{1/p} \vee (\sum \|x_n\|_2^2)^{1/2}$. Thus if $\{x_n\}_{n=1}^\infty$ is normalized in L_p, $p > 2$, and $w_n := \|x_n\|_2$, then $\|\sum a_n x_n\|_p$ is equivalent to $\|\{a_n\}_{n=1}^\infty\|_{p,w} := (\sum |a_n|^p)^{1/p} \vee (\sum |a_n|^2 w_n^2)^{1/2}$. The completion of the finitely non zero sequences of scalars under the norm $\|\cdot\|_{p,w}$ is called $X_{p,w}$. It follows that if $w = \{w_n\}_{n=1}^\infty$ is any sequence of numbers in $[0, 1]$, then $X_{p,w}$ is isomorphic to a complemented subspace of L_p. Suppose now that $w = \{w_n\}_{n=1}^\infty$ and $v = \{v_n\}_{n=1}^\infty$ are two such sequences of weights and $v_n \ge w_n$, then the diagonal operator D from $X_{p,w}$ to $X_{p,v}$ that sends the nth unit vector basis vector e_n to $\frac{w_n}{v_n} e_n$ is contractive and it is more or less obvious that D is ℓ_p-strictly singular if $\frac{w_n}{v_n} \to 0$ as $n \to \infty$. Since $X_{p,w}$ and $X_{p,v}$ are isomorphic to complemented subspaces of L_p, the adjoint operator D^* is $\ell_{p'}$ strictly singular and (identifying $X_{p,w}^*$ and $X_{p,v}^*$ with subspaces of $L_{p'}$) extends to a $\ell_{p'}$ strictly singular operator on $L_{p'}$. Our goal in this section is produce weights w and v so that D^* is an isomorphism on a subspace of $X_{p,v}^*$ which is not isomorphic to a Hilbert space.

For all $0 < r < 2$ there is a positive constant c_r such that

$$|t|^r = c_r \int_0^\infty \frac{1 - \cos tx}{x^{r+1}} dx$$

for all $t \in \mathbb{R}$. It follows that for any closed interval $[a, b] \subset (0, \infty)$ and for all $\varepsilon > 0$ there are $0 < x_1 < x_2 < \cdots < x_{n+1}$ such that $\max_{1 \le j \le n} \left| \frac{x_{j+1} - x_j}{x_j^{r+1}} \right| \le \varepsilon$ and

$$\left| c_r \sum_{j=1}^n \frac{x_{j+1} - x_j}{x_j^{r+1}} (1 - \cos t x_j) - |t|^r \right| < \varepsilon \tag{2}$$

for all t with $|t| \in [a, b]$.

Let $0 < q < r < 2$ and define v_j and a_j, $j = 1, \dots, n$, by

$$v_j^{\frac{2q}{2-q}} = c_r \frac{x_{j+1} - x_j}{x_j^{r+1}} \ , \qquad \frac{a_j}{v_j^{\frac{2}{2-q}}} = x_j.$$

Let Y_j, $j = 1, \dots, n$, be independent, symmetric, three valued random variables such that $|Y_j| = v_j^{\frac{-2}{2-q}} \mathbf{1}_{B_j}$ with $\text{Prob}(B_j) = v_j^{\frac{2q}{2-q}}$, so that in particular $\|Y_j\|_q = 1$ and $v_j = \|Y_j\|_q / \|Y_j\|_2$. Then the characteristic function of Y_j is

$$\varphi_{Y_j}(t) = 1 - v_j^{\frac{2q}{2-q}} + v_j^{\frac{2q}{2-q}} \cos(t v_j^{\frac{-2}{2-q}}) = 1 - v_j^{\frac{2q}{2-q}} (1 - \cos(t v_j^{\frac{-2}{2-q}}))$$

and

$$\begin{aligned} \varphi_{\sum a_j Y_j}(t) = & \prod_{j=1}^n (1 - v_j^{\frac{2q}{2-q}}(1-\cos(ta_j v_j^{\frac{-2}{2-q}}))) \\ = & \prod_{j=1}^n (1 - c_r \tfrac{x_{j+1}-x_j}{x_j^{r+1}}(1-\cos(tx_j))) \end{aligned} \tag{3}$$

To evaluate this product we use the estimates on $\frac{x_{j+1}-x_j}{x_j^{r+1}}$ to deduce that, for each j

$$\begin{aligned} |\log(1 - c_r \tfrac{x_{j+1}-x_j}{x_j^{r+1}}(1-\cos(tx_j))) + & \ c_r \tfrac{x_{j+1}-x_j}{x_j^{r+1}}(1-\cos(tx_j))| \\ & \leq C\varepsilon c_r^2 \tfrac{x_{j+1}-x_j}{x_j^{r+1}}(1-\cos(tx_j)) \end{aligned}$$

for some absolute $C < \infty$. Then, by (2),

$$\begin{aligned} |\textstyle\sum_{j=1}^n \log(1 - c_r \tfrac{x_{j+1}-x_j}{x_j^{r+1}}(1-\cos(tx_j))) + & \ c_r \textstyle\sum_{j=1}^n \tfrac{x_{j+1}-x_j}{x_j^{r+1}}(1-\cos(tx_j))| \\ & \leq C\varepsilon c_r(\varepsilon + b^r). \end{aligned}$$

Using (2) again we get

$$|\sum_{j=1}^n \log(1 - c_r \frac{x_{j+1}-x_j}{x_j^{r+1}}(1-\cos(tx_j))) + |t|^r| \leq (C+1)\varepsilon(\varepsilon + b^r)$$

(assuming as we may that $b \geq 1$), and from (3) we get

$$\varphi_{\sum a_j Y_j}(t) = (1 + O(\varepsilon))\exp(-|t|^r)$$

for all $|t| \in [a, b]$, where the function hiding under the O notation depends on r and b but on nothing else. It follows that, given any $\eta > 0$, one can find a, b and ε, such that for the corresponding $\{a_j, Y_j\}$ there is a symmetric r-stable Y (with characteristic function $e^{-|t|^r}$) satisfying

$$\|Y - \sum_{j=1}^n a_j Y_j\|_q \leq \eta.$$

This follows from classical translation of various convergence notions; see e.g. [Ro2, p. 154].

Let now $0 < \delta < 1$. Put $w_j = \delta v_j$, $j = 1, \ldots, n$, and let Z_j, $j = 1, \ldots, n$, be independent, symmetric, three valued random variables such that $|Z_j| = w_j^{\frac{-2}{2-q}} \mathbf{1}_{C_j}$ with $\mathrm{Prob}(C_j) = w_j^{\frac{2q}{2-q}}$, so that in particular $\|Z_j\|_q = 1$ and $w_j = \|Z_j\|_q / \|Z_j\|_2$. In a similar manner to the argument above we get that,

$$\begin{aligned} \varphi_{\sum \delta a_j Z_j}(t) = & \prod_{j=1}^n (1 - w_j^{\frac{2q}{2-q}}(1-\cos(t\delta a_j w_j^{\frac{-2}{2-q}}))) \\ = & \prod_{j=1}^n (1 - \delta^{\frac{2q}{2-q}} v_j^{\frac{2q}{2-q}}(1-\cos(t\delta^{\frac{-q}{2-q}} a_j v_j^{\frac{-2}{2-q}}))) \\ = & (1 + O(\varepsilon))\exp(-\delta^{\frac{q(2-r)}{2-q}}|t|^r) \end{aligned}$$

for all $|t| \in [\delta^{\frac{q}{2-q}}a, \delta^{\frac{q}{2-q}}b]$, where the O now depends also on δ.

Assuming $\delta^{\frac{q(2-r)}{2-q}} > 1/2$ and for a choice of a, b and ε, depending on δ, r, q and η we get that there is a symmetric r-stable random variable Z (with characteristic function $e^{-\delta^{\frac{q(2-r)}{2-q}}|t|^r}$) such that

$$\|Z - \sum_{j=1}^{n} \delta a_j Z_j\|_q \leq \eta.$$

Note that the ratio between the L_q norms of Y and Z are bounded away from zero and infinity by universal constants and each of these norms is also universally bounded away from zero. Consequently, if ε is small enough the ratio between the L_q norms of $\sum_{j=1}^{n} a_j Y_j$ and $\sum_{j=1}^{n} \delta a_j Z_j$ are bounded away from zero and infinity by universal constants.

Let now δ_i be any sequence decreasing to zero and r_i any sequence such that $q < r_i \uparrow 2$ and satisfying $\delta_i^{\frac{q(2-r_i)}{2-q}} > 1/2$. Then, for any sequence $\varepsilon_i \downarrow 0$ we can find two sequences of symmetric, independent, three valued random variables $\{Y_i\}$ and $\{W_i\}$, all normalized in L_q, with the following additional properties:

- put $v_j = \|Y_j\|_q / \|Y_j\|_2$ and $w_j = \|Z_j\|_q / \|Z_j\|_2$. Then there are disjoint finite subsets of the integers σ_i, $i = 1, 2, \ldots$, such that $w_j = \delta_i v_j$ for $j \in \sigma_i$.
- There are independent random variables $\{\bar{Y}_i\}$ and $\{\bar{Z}_i\}$ with $\bar{Y}_i$ and $\bar{Z}_i$ r_i stable with bounded, from zero and infinity, ratio of L_q norms and there are coefficients $\{a_j\}$ such that

$$\|\bar{Y}_i - \sum_{j \in \sigma_i} a_j Y_j\|_q < \varepsilon_i \quad \text{and} \quad \|\bar{Z}_i - \sum_{j \in \sigma_i} \delta_i a_j Z_j\|_q < \varepsilon_i.$$

From [Ro1] we know that the spans of $\{Y_j\}$ and $\{Z_j\}$ are complemented in L_q, $1 < q < 2$, and the dual spaces are naturally isomorphic to the spaces $X_{p,\{v_j\}}$ and $X_{p,\{w_j\}}$ respectively; both the isomorphism constants and the complementation constants depend only on q. Here $p = q/(q-1)$ and

$$\|\{\alpha_j\}\|_{X_{p,\{u_j\}}} = \max\{(\sum |\alpha_j|^p)^{1/p}, (\sum u_j^2 \alpha_j^2)^{1/2}\}.$$

Under this duality the adjoint D^* to the operator D that sends Y_j to $\delta_i Z_j$ for $j \in \sigma_i$ is formally the same diagonal operator between $X_{p,\{w_i\}}$ and $X_{p,\{v_i\}}$. The relation $w_j = \delta_i v_j$ for $j \in \sigma_i$ easily implies that this is a bounded operator. $\delta_i \to 0$ implies that this operator is ℓ_q strictly singular. If $\varepsilon_i \to 0$ fast enough, D^* preserves a copy of $\text{span}\{\bar{Y}_i\}$. Finally, if r_i tend to 2 not too fast this span is not isomorphic to a Hilbert space. Indeed, let $1 \leq s_j \uparrow 2$ be arbitrary and let $\{n_j\}_{j=1}^{\infty}$ be a sequence of positive integers with $n_j^{\frac{1}{s_j}-\frac{1}{2}} \geq j$, $j = 1, 2, \ldots$, say. For $1 \leq k \leq n_j$, put $r_{n_1+n_2+\cdots+n_{j-1}+k} = s_j$. Then the span of $\{Y_i\}_{i=n_1+\cdots+n_{j-1}+1}^{n_1+\cdots+n_j}$ is isomorphic, with constant independent of j, to $\ell_{s_j}^{n_j}$ and this last space is of distance at least j from a Euclidean space.

It follows that if $J : L_q \to \ell_\infty$ is an isometric embedding, then JD^* cannot be arbitrarily approximated by an operator which factors through a Hilbert space, and hence the Weak Tylli Conjecture is false in the range $1 < q < 2$.

5 Convolution by a biased coin

In this section we regard L_p as $L_p(\Delta)$, where $\Delta = \{-1,1\}^{\mathbb{N}}$ is the Cantor group and the measure is the Haar measure μ on Δ; i.e., $\mu = \prod_{n=1}^\infty \mu_n$, where $\mu_n(-1) = \mu_n(1) = 1/2$. For $0 < \varepsilon < 1$, let ν_ε be the ε- biased coin tossing measure; i.e., $\nu_\varepsilon = \prod_{n=1}^\infty \nu_{\varepsilon,n}$, where $\nu_{\varepsilon,n}(1) = \frac{1+\varepsilon}{2}$ and $\nu_{\varepsilon,n}(-1) = \frac{1-\varepsilon}{2}$. Let T_ε be convolution by ν_ε, so that for a μ-integrable function f on Δ, $(T_\varepsilon f)(x) = (f * \nu_\varepsilon)(x) = \int_\Delta f(xy)\, d\nu_\varepsilon(y)$. The operator T_ε is a contraction on L_p for all $1 \le p \le \infty$. Let us recall how T_ε acts on the characters on Δ. For $t = \{t_n\}_{n=1}^\infty \in \Delta$, let $r_n(t) = t_n$. The characters on Δ are finite products of these *Rademacher functions* r_n (where the void product is the constant one function). For A a finite subset of $\mathbb{N}$, set $w_A = \prod_{n\in A} r_n$ and let W_n be the linear span of $\{w_A : |A| = n\}$. Then $T_\varepsilon w_A = \varepsilon^{|A|} w_A$.

We are interested in studying T_ε on L_p, $1 \le p < 2$. The background we mention below is all contained in Bonami's paper [Bo] (or see [Ro4]). On L_p, $1 < p < 2$, T_ε is ℓ_p-strictly singular; in fact, T_ε even maps L_p into L_r for some $r = r(p, \varepsilon) > p$. Indeed, by interpolation it is sufficient to check that T_ε maps L_s into L_2 for some $s = s(\varepsilon) < 2$. But there is a constant C_s which tends to 1 as $s \uparrow 2$ so that for all $f \in W_n$, $\|f\|_2 \le C_s^n \|f\|_s$ and the orthogonal projection P_n onto (the closure of) W_n satisfies $\|P_n\|_p \le C_s^n$. From this it is easy to check that if $\varepsilon C_s^2 < 1$, then T_ε maps L_s into L_2. We remark in passing that Bonami [Bo] found for each p (including $p \ge 2$) and ε the largest value of $r = r(p, \varepsilon)$ such that T_ε maps L_p into L_r.

Thus Theorem 4 yields that if X is a subspace of L_p, $1 < p < 2$, and T_ε (considered as an operator from L_p to L_p) is an isomorphism on X, then X embeds into L_s for all $s < 2$. Since, as we mentioned above, T_ε maps L_s into L_2 for some $s < 2$, it then follows from an argument in [Ro4] that X must be isomorphic to a Hilbert space. (Actually, as we show after the proof, Lemma 3 is strong enough that we can prove Theorem 5 without using Theorem 4.) Since [Ro4] is not generally available, we repeat Rosenthal's argument in Lemma 3 below.

Now T_ε is not ℓ_1 strictly singular on L_1. Nevertheless, we still get that if X is a reflexive subspace of L_1 and T_ε (considered as an operator from L_1 to L_1) is an isomorphism on X, then X is isomorphic to a Hilbert space. Indeed, Rosenthal showed (see Lemma 3) that then there is another subspace X_0 of L_1 which is isomorphic to X so that X_0 is contained in L_p for some $1 < p < 2$, the L_p and L_1 norms are equivalent on X_0, and T_ε is an isomorphism on X_0. This implies that as an operator on L_p, T_ε is an isomorphism on X_0 and hence X_0 is isomorphic to a Hilbert space. (To apply Lemma 3, use the fact [Ro3] that if X is a relexive subspace of L_1, then X embeds into L_p for some $1 < p < 2$.)

We summarize this discussion in the first sentence of Theorem 5. The case $p = 1$ solves Problem B from Rosenthal's 1976 paper [Ro4].

Theorem 5 *Let $1 \leq p < 2$, let $0 < \varepsilon < 1$, and let T_ε be considered as an operator on L_p. If X is a reflexive subspace of L_p and the restriction of T_ε to X is an isomorphism, then X is isomorphic to a Hilbert space. Moreover, if $p > 1$, then X is complemented in L_p.*

We now prove Rosenthal's lemma [Ro4, proof of Theorem 5] and defer the proof of the "moreover" statement in Theorem 5 until after the proof of the lemma. .

Lemma 3 *Suppose that T is an operator on L_p, $1 \leq p < r < s < 2$, X is a subspace of L_p which is isomorphic to a subspace of L_s, and $T_{|X}$ is an isomorphism. Then there is another subspace X_0 of L_p which is isomorphic to X so that X_0 is contained in L_r, the L_r and L_p norms are equivalent on X_0, and T is an isomorphism on X_0.*

Proof: We want to find a measurable set E so that

(1) $X_0 := \{\mathbf{1}_E x : x \in X\}$ is isomorphic to X,

(2) $X_0 \subset L_r$,

(3) $T_{|X_0}$ is an isomorphism.

(We did not say that $\|\cdot\|_p$ and $\|\cdot\|_r$, are equivalent on X_0 since that follows formally from the closed graph theorem. The isomorphism $X \to X_0$ guaranteed by (a) is of course the mapping $x \mapsto \mathbf{1}_E x$.)

Assume, without loss of generality, that $\|T\| = 1$. Take $a > 0$ so that $\|Tx\|_p \geq a\|x\|_p$ for all x in X. Since ℓ_p does not embed into L_s we get from (4) in Lemma 1 that there is $\eta > 0$ so that if E has measure larger than $1 - \eta$, then $\|\mathbf{1}_{\sim E} x\|_p \leq \frac{a}{2}\|x\|_p$ for all x in x. Obviously (1) and (3) are satisfied for any such E. It is proved in [Ro3] that there is strictly positive g with $\|g\|_1 = 1$ so that $\frac{x}{g}$ is in L_r for all x in X. Now simply choose $t < \infty$ so that $E := [g < t]$ has measure at least $1 - \eta$; then E satisfies (1), (2), and (3). ∎

Next we remark how to avoid using Theorem 4 in proving Theorem 5. Suppose that T_ε is an isomorphism on a reflexive subspace X of L_p, $1 \leq p < 2$. Let s be the supremum of those $r \leq 2$ such that X is isomorphic to a subspace of L_r, so $1 < s \leq 2$. It is sufficient to show that $s = 2$. But if $s < 2$, we get from the interpolation formula that if $r < s$ is sufficiently close to s, then T_ε maps L_r into L_t for some $t > s$ and hence, by Lemma 3, X embeds into L_t.

Finally we prove the "moreover" statement in Theorem 5. We now know that X is isomorphic to a Hilbert space. In the proof of Lemma 3, instead of using Rosenthal's result from [Ro3], use Grothendieck's theorem [DJT, Theorem 3.5], which implies that there is strictly positive g with $\|g\|_1 = 1$ so that $\frac{x}{g}$ is in L_2 for all x in X. Choosing E the same way as in the proof of Lemma 3 with $T := T_\varepsilon$, we get that (1), (2), and (3) are true with $r = 2$. Now the L_2 and L_p norms are equivalent on both X_0 and on $T_\varepsilon X_0$. But it is clear that the only way that T_ε can be an isomorphism on a subspace X_0 of L_2 is for the orthogonal projection P_n onto the closed span of W_k, $0 \leq k \leq n$, to be an isomorphism on X_0 for some finite n. But then also in the L_p norm the restriction of P_n to X_0 is an isomorphism because the L_p norm and the L_2 norm are equivalent on the span of W_k, $0 \le k \le n$ and P_n is bounded on L_p (since $p > 1$). It follows that the operator $S := P_n \circ \mathbf{1}_E$ on L_p maps X_0 isomorphically onto a complemented subspace of L_p, which implies that X_0 is also complemented in L_p.

Here is the problem that started us thinking about ℓ_p-strictly singular operators:

Problem 1 *Let $1 < p < 2$ and $0 < \varepsilon < 1$. On $L_p(\Delta)$, does T_ε satisfy the conclusion of the Tylli Conjecture?*

After we submitted this paper, G. Pisier [Pi] answered problem 1 in the affirmative.

Although the example in section 4 shows that the Tylli Conjecture is false, something close to it may be true:

Problem 2 *Let $1 < p < r < 2$. Is every ℓ_p-strictly singular operator on L_p in the closure of the operators on L_p that factor through L_r?*

6 Appendix

In this appendix we prove a theorem that is essentially due to Saksman and Tylli. The only novelty is that we assume the compact approximation property rather than the approximation property.

Theorem 6 *Let X be a reflexive Banach space and let A, B be in $L(X)$. Then*

(a) If ATB is a compact operator on X for every T in $L(X)$, then $L_A R_B$ is a weakly compact operator on $L(X)$.

(b) If X has the compact approximation property and $L_A R_B$ is a weakly compact operator on $L(X)$, then ATB is a compact operator on X for every T in $L(X)$.

Proof: To prove (a), recall [Kal] that for a reflexive space X, on bounded subsets of $K(X)$ the weak topology is the same as the weak operator topology (the operator $T \mapsto f_T \in C((B_X, \text{ weak}) \times (B_{X^*}, \text{weak}))$, where $f_T(x, x^*) := \langle x^*, Tx\rangle$, is an isometric isomorphism from $K(X)$ into a space of continuous functions on a compact Hausdorff space). Now if (T_α) is a bounded net in $L(X)$, then since X is reflexive there is a subnet (which we still denote by (T_α)) which converges in the weak operator topology to, say $T \in L(X)$. Then $AT_\alpha B$ converges in the the weak operator topology to ATB. But since all these operators are in $K(X)$, $AT_\alpha B$ converges weakly to ATB by Kalton's theorem. This shows that $L_A R_B$ is a weakly compact operator on $L(X)$.

To prove (b), suppose that we have a $T \in L(X)$ with ATB not compact. Then there is a weakly null normalized sequence $\{x_n\}_{n=1}^\infty$ in X and $\delta > 0$ so that for all n, $\|ATBx_n\| > \delta$. Since a reflexive space with the compact approximation property also has the compact metric approximation property [CJ], there are $C_n \in K(X)$ with $\|C_n\| < 1 + 1/n$, $C_nBx_i = Bx_i$ for $i \le n$. Since the C_n are compact, for each n, $\|C_nBx_m\| \to 0$ as $m \to \infty$. Thus $A(TC_n)Bx_i = ATBx_i$ for $i \le n$ and $\|A(TC_n)Bx_m\| \to 0$ as $m \to \infty$. This implies that no convex combination of $\{A(TC_n)B\}_{n=1}^{\infty}$ can converge in the norm of $L(X)$ and hence $\{A(TC_n)B\}_{n=1}^{\infty}$ has no weakly convergent subsequence. This contradicts the weak compactness of L_AR_B and completes the proof. ∎

References

[Al] Aldous, David J. Subspaces of L^1, via random measures. Trans. Amer. Math. Soc. 267 (1981), no. 2, 445–463.

[Bo] Bonami, Aline. Etude des coefficients de Fourier des fonctions de $L^p(G)$. (French) Ann. Inst. Fourier (Grenoble) 20 1970 fasc. 2 335–402 (1971).

[CJ] Cho, Chong-Man; Johnson, William B. A characterization of subspaces X of l_p for which $K(X)$ is an M-ideal in $L(X)$. Proc. Amer. Math. Soc. 93 (1985), no. 3, 466–470.

[DJT] Diestel, Joe; Jarchow, Hans; Tonge, Andrew. Absolutely summing operators. Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995.

[JJ] Johnson, W. B.; Jones, L. Every L_p operator is an L_2 operator. Proc. Amer. Math. Soc. 72 (1978), no. 2, 309–312.

[Jo] Johnson, William B. Operators into L_p which factor through ℓ_p. J. London Math. Soc. (2) 14 (1976), no. 2, 333–339.

[KP] Kadec, M. I.; Pełczyński, A. Bases, lacunary sequences and complemented subspaces in the spaces L_p. Studia Math. 21 1961/1962 161–176.

[Kal] Kalton, N. J. Spaces of compact operators. Math. Ann. 208 (1974), 267–278.

[KK] Koldobsky, Alexander; König, Hermann. Aspects of the isometric theory of Banach spaces. Handbook of the geometry of Banach spaces, Vol. I, 899–939, North-Holland, Amsterdam, 2001.

[Kr] Krivine, J. L. Sous-espaces de dimension finie des espaces de Banach reticules. Ann. of Math. (2) 104 (1976), no. 1, 1–29.

[KM] Krivine, J.-L.; Maurey, B. Espaces de Banach stables. (French) [Stable Banach spaces] Israel J. Math. 39 (1981), no. 4, 273–295.

[La] Lacey, H. Elton. The isometric theory of classical Banach spaces. Die Grundlehren der mathematischen Wissenschaften, Band 208. Springer-Verlag, New York-Heidelberg, 1974.

[LT] Lindenstrauss, Joram; Tzafriri, Lior. Classical Banach spaces I&II. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92 & 97. Springer-Verlag, Berlin-New York, 1977 & 1979

[MP] Maurey, Bernard; Pisier, Gilles. Series de variables aleatoires vectorielles independantes et proprietes geometriques des espaces de Banach. (French) Studia Math. 58 (1976), no. 1, 45–90.

[PR] Pełczyński, A.; Rosenthal, H. P. Localization techniques in L^p spaces. Studia Math. 52 (1974/75), 263–289.

[Pi] Pisier, Gilles. Remarks on hypercontractive semigroups and operator ideals. arXiv:0708.3423.

[Pl] Plotkin, A. I. An algebra that is generated by translation operators, and L^p-norms. (Russian) Functional analysis, No. 6: Theory of operators in linear spaces (Russian), pp. 112–121. Ulyanovsk. Gos. Ped. Inst., Ulyanovsk, 1976.

[Ro1] Rosenthal, Haskell P. On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables. Israel J. Math. 8 (1970), 273–303.

[Ro2] Rosenthal, Haskell P. On the span in L^p of sequences of independent random variables. II. Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory, pp. 149–167. Univ. California Press, Berkeley, Calif., 1972.

[Ro3] Rosenthal, Haskell P. On subspaces of L^p. Ann. of Math. (2) 97 (1973), 344–373.

[Ro4] Rosenthal, Haskell P. Convolution by a biased coin. The Altgeld Book 1975/76.

[Ru] Rudin, Walter. L^p-isometries and equimeasurability. Indiana Univ. Math. J. 25 (1976), no. 3, 215–228.

[ST1] Saksman, Eero; Tylli, Hans-Olav. Weak compactness of multiplication operators on spaces of bounded linear operators. Math. Scand. 70 (1992), no. 1, 91–111.

[ST2] Saksman, Eero; Tylli, Hans-Olav. Multiplications and elementary operators in the Banach space setting. In: Methods in Banach space theory. (Proc. V Conference on Banach spaces, Caceres, September 13-18, 2004; J.F.M. Castillo and W.B. Johnson, eds.) London Mathematical Society Lecture Note Series 337 (Cambridge University Press, 2006), pp. 253-292.

[We] Weis, L. On perturbations of Fredholm operators in $L_p(\mu)$-spaces. Proc. Amer. Math. Soc. 67 (1977), no. 2, 287–292.

[Wo] Wojtaszczyk, P. Banach spaces for analysts. Cambridge Studies in Advanced Mathematics, 25. Cambridge University Press, Cambridge, 1991.

William B. Johnson
Department Mathematics
Texas A&M University
College Station, TX, USA
E-mail: johnson@math.tamu.edu

Gideon Schechtman
Department of Mathematics
Weizmann Institute of Science
Rehovot, Israel
E-mail: gideon@weizmann.ac.il