A CHARACTERIZATION OF SUBSPACES AND QUOTIENTS OF REFLEXIVE BANACH SPACES WITH UNCONDITIONAL BASES

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Abstract. We prove that the dual or any quotient of a separable reflexive Banach space with the unconditional tree property has the unconditional tree property. This is used to prove that a separable reflexive Banach space with the unconditional tree property embeds into a reflexive Banach space with an unconditional basis. This solves several long standing open problems. In particular, it yields that a quotient of a reflexive Banach space with an unconditional finite dimensional decomposition embeds into a reflexive Banach space with an unconditional basis.

1. Introduction

It has long been known that Banach spaces with unconditional bases as well as their subspaces are much better behaved than general Banach spaces, and that many of the reflexive spaces (including $L_p(0,1), 1 < p < \infty$) that arise naturally in analysis have unconditional bases. It is however difficult to determine whether a given Banach space has an unconditional basis or embeds into a space which has an unconditional basis. Two problems, considered important since at least the 1970's, stand out.

(a) Give an intrinsic condition on a Banach space X which is equivalent to the embeddability of X into a space with an unconditional basis.
(b) Does every complemented subspace of a space with an unconditional basis have an unconditional basis?

Problem (b) remains open, but in this paper we provide a solution to problem (a) for reflexive Banach spaces. This characterization also yields that a quotient of a reflexive space with an unconditional basis embeds into a reflexive space with unconditional basis, which solves another problem from the 1970's. Here some condition on the space with an unconditional basis is needed because every separable Banach space is a quotient of ℓ_1.

There is, of course, quite a lot known around problems (a) and (b). For example, Pełczyński and Wojtaszczyk [15] proved that if X has an unconditional expansion of identity (i.e., a sequence (T_n) of finite rank operators such that $\sum T_n$ converges unconditionally in the strong operator topology to the identity on X), then X is isomorphic to a complemented subspace of a space that has an unconditional finite

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dimensional decomposition (UFDD). Later, Lindenstrauss and Tzafriri [11] showed that every space with an UFDD embeds (not necessarily complementably) into a space with an unconditional basis. As regards reflexive spaces, it was shown in [4] using a result from [1] (and answering a question from that paper) that if a reflexive Banach space embeds into a space with an unconditional basis, then it embeds into a reflexive space with an unconditional basis. As regards the quotient problem we mentioned above, Feder [2] gave a partial solution by proving that if X is a quotient of a reflexive space which has an UFDD and X has the approximation property, then X embeds into a space with an unconditional basis.

It is well known and easy to see that if a Banach space X embeds into a space with an unconditional basis, then X has the unconditional subsequence property; that is, there exists a $K > 0$ so that every normalized weakly null sequence in X has a subsequence which is K-unconditional. In fact, failure of the unconditional subsequence property is the only known criterion for proving that a given reflexive space does not embed into a space with an unconditional basis. However, in the last section we construct a Banach space which has the unconditional subsequence property but does not embed into a Banach space that has an unconditional basis. This is not surprising, given previous examples of E. Odell and Th. Schlumprecht [12]. Moreover, Odell and Schlumprecht have taught us that by replacing a subsequence property by the corresponding "branch of a tree" property, you get a stronger property that sometimes can be used to give a characterization of spaces that embed into a space with some kind of structure. The property relevant for us is the unconditional tree property and Odell and Schlumprecht's beautiful results are essential tools for us.

We use standard Banach space theory terminology, as can be found in [11].

2. Main results

Definition 2.1. Let $[\mathbf{N}]^{<\omega}$ denote all finite subsets of the positive integers. By a normalized weakly null tree, we mean a family $(x_A)_{A\in[\mathbf{N}]^{<\omega}} \subset S_X$ with the property that every sequence $(x_{A\bigcup\{n\}})_{n\in\mathbf{N}}$ is weakly null. Let $A = \{n_1, ..., n_m\}$ with $n_1 < ... < n_m$ and $B = \{j_1, ..., j_r\}$ with $j_1 < ... < j_r$. Then we say A is an initial segment of B if $m \leq r$ and $n_i = j_i$ when $1 \leq i \leq m$. The tree order on $(x_A)_{A\in[\mathbf{N}]^{<\omega}}$ is given by $x_A \leq x_B$ if A is an initial segment of B. A branch of a tree is a maximal linearly ordered subset of the tree under the tree order. We say X has the C-UTP if every normalized weakly null tree in X has a C-unconditional branch for some $C > 0$. X has the UTP if X has the C-UTP for some $C > 0$.

Remark 2.2. E. Odell, Th. Schlumprecht and A. Zsak proved in [14] that if every normalized weakly null tree in X admits a branch which is unconditional, then X has the C-UTP for some $C > 0$. A simpler proof will appear in the forthcoming paper of R. Haydon, E. Odell and Th. Schlumprecht [5]. So there is no ambiguity when using the term "UTP".

Given an FDD (E_n), (x_n) is said to be a block sequence with respect to (E_n) if there exists a sequence of integers $0 = m_1 < m_2 < m_3 < ...$ such that $x_n \in \bigoplus_{j=m_n}^{m_{n+1}-1} E_j, \forall n \in \mathbf{N}$. (x_n) is said to be a skipped block sequence with respect to (E_n) if there exists a sequence of increasing integers $0 = m_1 < m_2 < m_3 < ...$ such that $m_n + 1 < m_{n+1}$ and $x_n \in \bigoplus_{j=m_n+1}^{m_{n+1}-1} E_j, \forall n \in \mathbf{N}$. Let $\delta = (\delta_i)$ be a sequence of positive numbers decreasing to 0. We say (y_n) is a δ-skipped block sequence with respect to (E_n) if there is a skipped block sequence (x_n) so that $\|y_n - x_n\| < \delta_n \|y_n\|$ for all $n \in \mathbf{N}$. We say (F_n) is a blocking of (E_n) if there is a sequence of increasing integers $0 = k_0 < k_1 < ...$ so that $F_n = \oplus_{j=k_{n-1}+1}^{k_n} E_j$.

Definition 2.3. Let X be a Banach space with an FDD (E_n). If there exists a $C > 0$ so that every skipped block sequence with respect to (E_n) is C-unconditional, then we say (E_n) is an unconditional skipped blocked FDD (USB FDD).

The following is a blocking lemma of W. B. Johnson and M. Zippin (see [10] or Proposition 1.g.4(a) in [11]) which will be used later.

Lemma 2.4. *Let $T : X \to Y$ be a bounded linear operator. Let (B_n) be a shrinking FDD of X and let (C_n) be an FDD of Y. Let (δ_n) be a sequence of positive numbers tending to 0. Then there are blockings $(B_n^{'})$ of (B_n) and $(C_n^{'})$ of (C_n) so that, for every $x \in B_n^{'}$, there is a $y \in C_{n-1}^{'} \bigoplus C_n^{'}$ such that $\|Tx - y\| \leq \delta_n \|x\|$.*

The lemma above actually works for any further blockings of $(B_n^{'})$ and $(C_n^{'})$. To be more precise, we have the following stronger result which is actually a formal consequence of Lemma 2.4 as stated.

Lemma 2.5. *Let $T : X \to Y$ be a bounded linear operator. Let (B_n) be a shrinking FDD of X and let (C_n) be an FDD of Y. Let (δ_n) be a sequence of positive numbers tending to 0. Then there are blockings $(B_n^{'})$ of (B_n) and $(C_n^{'})$ of (C_n) so that, for any further blockings $(\tilde{B_n})$ of $(B_n^{'})$ with $\tilde{B_n} = \oplus_{i=k_n}^{k_{n+1}-1} B_i^{'}$ and $(\tilde{C_n})$ of $(C_n^{'})$ with $\tilde{C_n} = \oplus_{i=k_n}^{k_{n+1}-1} C_i^{'}$ and for any $x \in \tilde{B_n}$, there is a $y \in \tilde{C_{n-1}} \oplus \tilde{C_n}$ such that $\|Tx - y\| \leq \delta_n \|x\|$.*

Proof. Let (δ_i) be a sequence of positive numbers decreasing to 0. Let $(\tilde{\delta_i})$ be another sequence of positive numbers which go to 0 so fast that $\sum_{j=i}^{\infty} \tilde{\delta_j} < 2\lambda\delta_i$, where λ is the basis constant for (B_n). By Lemma 2.4, we get blockings $(B_n^{'})$ of (B_n) and $(C_n^{'})$ of (C_n) so that for every $x \in B_n^{'}$, there is a $y \in C_{n-1}^{'} \oplus C_n^{'}$ such that $\|Tx - y\| \leq \tilde{\delta_n} \|x\|$. Let $\tilde{B_n} = \oplus_{i=k_n}^{k_{n+1}-1} B_i^{'}$ and $\tilde{C_n} = \oplus_{i=k_n}^{k_{n+1}-1} C_i^{'}$ be blockings of $(B_n^{'})$ and $(C_n^{'})$. Let $x \in \tilde{B_n}$. Then we can write $x = \sum_{i=k_n}^{k_{n+1}-1} x_i, x_i \in B_i^{'}$. So by our construction of $(B_n^{'})$ and $(C_n^{'})$, there are $y_i \in C_{i-1}^{'} \oplus C_i^{'}, k_n \leq i \leq k_{n+1} - 1$ so that $\|Tx_i - y_i\| \leq \tilde{\delta_i} \|x_i\|, k_n \leq i \leq k_{n+1} - 1$. Let $y = \sum_{i=k_n}^{k_{n+1}-1} y_i \in \tilde{C_{n-1}^{'}} \oplus \tilde{C_n^{'}}$. Then we have

$$\|Tx - y\| \leq \sum_{i=k_n}^{k_{n+1}-1} \tilde{\delta_i} \|x_i\| \leq \sum_{i=k_n}^{k_{n+1}-1} 2\lambda \tilde{\delta_i} \delta_i \|x\| \leq \delta_n \|x\|.$$

□

The following convenient reformulation of Lemma 2.4 will also be used (see [9] and [10] or [13]).

Lemma 2.6. *Let $T : X \mapsto Y$ be a bounded linear operator. Let (B_n) be a shrinking FDD for X and (C_n) be an FDD for Y. Let (δ_i) be a sequence of positive numbers decreasing to 0. Then there is a blocking (B_n') of (B_n) and a blocking (C_n') of (C_n) so that for any $x \in B_n'$ and any $m \neq n, n-1$,*

$$\|Q_m(Tx)\| < \delta_{\max\{m,n\}} \|x\|,$$

where Q_j is the canonical projection from Y onto C_j'.

Remark 2.7. The qualitative content of Lemma 2.6 is that there are blockings (B'_n) of (B_n) and (C'_n) of (C_n) so that TB'_n is essentially contained in $C'_{n-1} + C'_n$.

Our first theorem says that the unconditional tree property for reflexive Banach spaces passes to quotients. It plays a key role in this paper and involves the lemmas above as well as results and ideas of Odell and Schlumprecht.

Let us explain the sketch of the proof of the special case when Y is a reflexive space with the UTP and Y has an FDD (E_n), while X is a quotient of Y which has an FDD (V_n). Since Y has the UTP, by Odell and Schlumprecht's fundamental result [12], there is a blocking (F_n) of the (E_n) which is an USB FDD. Then we use the "killing the overlap" technique of [6] to get a further blocking (G_n) so that any norm one vector y is a small perturbation of the sum of a skipped block sequence (y_i) with respect to (F_n) and $y_i \in G_{i-1} \oplus G_i$. Let $Q : Y \mapsto X$ be the quotient map. Using Lemma 2.5 and passing to a further blocking, without loss of generality, we assume that QG_i is essentially contained in $H_{i-1} + H_i$, where (H_i) is the corresponding blocking of (V_n). Let (x_A) be a normalized weakly null tree in X. We then choose a branch (x_{A_i}) so lacunary that (x_{A_i}) is a small perturbation of a block sequence of (H_n) and for each i, there is at least one H_{k_i} between the essential support of x_{A_i} and $x_{A_{i+1}}$. Let $x = \sum a_i x_{A_i}$ with $\|x\| = 1$. Considering a preimage y of x under the quotient Q from Y onto X (with $\|y\| = 1$), by our construction, we can essentially write y as the sum of (y_i) where (y_i) is a skipped block sequence with respect to (F_n). Since (F_n) is USB, (y_i) is unconditional. By passing to a suitable blocking (z_i) of (y_i) and using Lemma 2.5, it is not hard to show that Qz_i is essentially equal to $a_i x_{A_i}$. Noticing that (z_i) is unconditional, we conclude that (x_{A_i}) is also unconditional.

For the general case when X and Y do not have an FDD, we have to embed them into some superspaces with FDD. The difficulty is that when we decompose a vector in Y as the sum of disjointly supported vectors in the superspace, we do not know that the summands are in Y. The same problem occurs for vectors in X. This makes the proof rather technical and a lot of computations appear.

Theorem 2.8. *Let X be a quotient of a separable reflexive Banach space Y with the UTP. Then X has the UTP.*

Proof. By Zippin's result [16], Y embeds isometrically into a reflexive space Z with an FDD. A key point in the proof is that Odell and Schlumprecht proved (Proposition 2.4 in [13]) that there is a further blocking (G_n) of the FDD for Z, $\delta = (\delta_i)$ and a $C > 0$ so that every δ-skipped block sequence $(y_i) \subset Y$ with respect to (G_i) is C-unconditional. Let λ be the basis constant for (G_n).

Since X is separable, we can regard X as a subspace of L_∞. Let $\epsilon > 0$. We may assume that

(a) $\sum_{j>i} \delta_j < \delta_i$,
(b) $i\delta_i < \delta_{i-1}$,
(c) $\sum \delta_i < \epsilon$.

Let Q be a quotient map from Y onto X, which can be extended to a norm one map from Z into L_∞ and we still denote it by Q. QZ, as any separable subspace of L_∞, is contained in some super space isometric to $C(\Delta)$ with monotone basis (v_i). Here Δ is the Cantor set.

Let (x_A) be a normalized weakly null tree in X. Then we let (E_n) and (F_n) be blockings of (G_i) and (v_i) respectively which satisfy the conclusions of Lemma 2.5

and Lemma 2.6. Using the "killing the overlap" technique (see Proposition 2.6 in [13]), we can find a further blocking $(\tilde{E_n} = \oplus_{i=l(n)+1}^{l(n+1)} E_i)$ so that for every $y \in S_Y$, there exists $(y_i) \subset Y$ and integers (t_i) with $l(i-1) < t_i \leq l(i)$ for all i such that

(I) $y = \sum y_i$,
(II) For $i \in \mathbf{N}$, either $\|y_i\| < \delta_i$ or $\| \sum_{j=t_{i-1}+1}^{t_i-1} P_j y_i - y_i\| < \delta_i \|y_i\|$,
(III) $\| \sum_{j=t_{i-1}+1}^{t_i-1} P_j y - y_i\| < \delta_i$,
(IV) $\|P_{t_i} y\| < \delta_i$ for $i \in \mathbf{N}$,

where P_j is the canonical projection from Y onto E_j. Let $\tilde{F_n} = \oplus_{i=l(n)+1}^{l(n+1)} F_i$ and let $\tilde{P_j}$ be the canonical projection from X onto $\tilde{F_j}$. Since (x_A) is a weakly null tree, we can pick inductively a branch (x_{A_i}) and an increasing sequence of integers $1 = k_0 < k_1 < ...$ such that for any $i \in \mathbf{N}$

(i) $\| \sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P_j} x_{A_i} - x_{A_i}\| < \delta_i$,
(ii) $\| \sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P_j} x_{A_t}\| < \delta_{\max\{i,t\}}$, for any $t \neq i$.

We will prove that (x_{A_i}) is unconditional. Let $x = \sum a_i x_{A_i}$, $\|x\| = 1$. Let $y \in S_Y$ so that $Q(y) = x$. Then y can be written as $\sum y_j$ where (y_j) satisfy (I), (II), (III) and (IV). Define $k_{-1} = -1$ and let $z_i = \sum_{j=k_{2i-3}+2}^{k_{2i-1}+1} y_j$. We will prove that $\|Qz_i - a_i x_{A_i}\|$ is small.

$$\begin{aligned} \|Qz_i - a_i x_{A_i}\| \leq{} & \|Q(\sum_{j=t_{k_{2i-3}+1}}^{t_{k_{2i-1}+1}} P_j y) - (\sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P_j})x\| \\ & + \|z_i - \sum_{j=t_{k_{2i-3}+1}}^{t_{k_{2i-1}+1}} P_j y\| \\ & + \|a_i x_{A_i} - (\sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P_j})x\|. \end{aligned} \tag{2.1}$$

Hence we need to estimate the three terms in the right hand side of the above inequality. By the construction, for $i > 1$, we have

(2.2)
$$\begin{aligned} \|z_i - \sum_{j=t_{k_{2i-3}+1}}^{t_{k_{2i-1}+1}} P_j y\| &< \sum_{j=k_{2i-3}+2}^{k_{2i-1}+1} (\| \sum_{l=t_{j-1}+1}^{t_j-1} P_l y - y_j\| + \|P_{t_{j-1}} y\|) + \|P_{t_{k_{2i-1}+1}} y\| \\ &< \sum_{j=k_{2i-3}+2}^{k_{2i-1}+1} \delta_j + \sum_{j=k_{2i-3}+2}^{k_{2i-1}+2} \delta_{j-1} \\ &< \delta_{k_{2i-3}+1} + \delta_{k_{2i-3}} \\ &< \delta_i. \end{aligned}$$

By direct calculation, for $i = 1$, we have

$$\|z_1 - \sum_{j=1}^{t_{k_1+1}} P_j y\| < 2\delta_1. \tag{2.3}$$

This gives an estimate of the second term. For the third term, we have

(2.4)

$$
\begin{aligned}
\|a_i x_{A_i} - (\sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P}_j)x\| &< \|(\sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P}_j)(a_i x_{A_i} - x)\| + \|a_i(x_{A_i} - (\sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P}_j)x_{A_i}\| \\
&< 2(k_{2i-2}\delta_{k_{2i-2}} + \sum_{j\geq k_{2i-1}} \delta_j) + 2\delta_i \\
&< 2(\delta_{k_{2i-2}-1} + \delta_{k_{2i-1}-1}) + 2\delta_i \\
&< 4\delta_i.
\end{aligned}
$$

For the first term, let Q_j be the canonical projection from X onto F_j and let $J_1 = [t_{k_{2i-3}+1}, t_{k_{2i-1}+1}]$, $J_2 = [l_{k_{2i-2}} + 1, l_{k_{2i-1}}]$ and $J_1' = (t_{k_{2i-3}+1}, t_{k_{2i-1}+1})$. Then we have

(2.5)

$$
\begin{aligned}
\|Q(\sum_{j\in J_1} P_j y) - (\sum_{j\in J_2} Q_j)Qy\| &\leq \|Q(\sum_{j\in J_1} P_j y) - (\sum_{j\in J_1} Q_j)Qy\| + \|(\sum_{j\in J_1} Q_j)Qy - (\sum_{j\in J_2} Q_j)Qy\| \\
&= \|Q(\sum_{j\in J_1} P_j y) - (\sum_{j\in J_1} Q_j)Qy\| + \|\sum_{j\in J_1 - J_2} Q_j(\sum a_i x_{A_i})\| \\
&< \|Q(\sum_{j\in J_1} P_j y) - (\sum_{j\in J_1} Q_j)Qy\| + 4\delta_i \\
&\leq \|(\sum_{j\notin J_1} Q_j)Q(\sum_{j\in J_1} P_j y)\| + \|(\sum_{j\in J_1} Q_j)Q(\sum_{j\notin J_1} P_j y)\| + 4\delta_i \\
&< \|(\sum_{j\notin J_1} Q_j)Q(\sum_{j\in J_1'} P_j y)\| + \|(\sum_{j\in J_1'} Q_j)Q(\sum_{j\notin J_1} P_j y)\| + 6\delta_i \\
&< 2\lambda\delta_i + 2\lambda\delta_i + 6\delta_i \\
&= (4\lambda + 6)\delta_i.
\end{aligned}
$$

¿From Inequality 2.2, 2.3, 2.4 and 2.5, we conclude that

$$\|Qz_i - a_i x_{A_i}\| < (4\lambda + 12)\delta_i.$$

Let $(\epsilon_i) \subset \{-1, 1\}^{\mathbf{N}}$. Let $I \subset \mathbf{N}$ be the set of indices $i \in \mathbf{N}$ for which $\|y_i\| < \delta_i$ and let $I_i = [k_{2i-3} + 2, k_{2i-1} + 1]$. So $z_i = \sum_{j\in I_i} y_j$. Let $z_i' = \sum_{j\in I_i - I} y_j$. It is easy to verify that $\|z_i - z_i'\| < \delta_i$. Hence $\|Qz_i' - a_i x_{A_i}\| < (4\lambda + 13)\delta_i$. Now by (II), we know that (z_i') is a δ-skipped block sequence. Hence, (z_i') is unconditional. So we have

$$
\begin{aligned}
\|\sum \epsilon_i a_i x_{A_i}\| &\leq \|Q(\sum \epsilon_i z_i')\| + (4\lambda + 13)\epsilon \\
&\leq C\|\sum z_i'\| + (4\lambda + 13)\epsilon \\
&< C(\|\sum z_i\| + \sum \delta_i) + (4\lambda + 13)\epsilon \\
&\leq C + (C + 4\lambda + 13)\epsilon,
\end{aligned}
$$

This shows (x_{A_i}) is an unconditional sequence. □

Remark 2.9. If the original space Y has the $1+\epsilon$-UTP for any $\epsilon > 0$, then any quotient of Y has the $1+\epsilon$-UTP for any $\epsilon > 0$.

The following is an elementary lemma which will be used later. We omit the standard proof.

Lemma 2.10. *Let X be a Banach space and X_1, X_2 be two closed subspace of X. If $X_1 \cap X_2 = \{0\}$ and $X_1 + X_2$ is closed, then X embeds into $X/X_1 \oplus X/X_2$.*

In [7], W. B. Johnson and H. P. Rosenthal proved that any separable Banach space X admits a subspace Y so that both Y and X/Y have a FDD. The proof uses Markuschevich bases. A Markuschevich basis for a separable Banach space X is a biorthogonal system $\{x_n, x_n^*\}_{n\in\mathbf{N}}$ for which the span of the x_n's is dense in X and the x_n^*'s separate the points of X. By Theorem 1.f.4 in [11], every separable Banach space X has a Markuschevich basis $\{x_n, x_n^*\}_{n\in\mathbf{N}}$ so that $[x_n^*]$ contains any designated separable subspace of X^*. The following lemma is a stronger form of the result of Johnson and Rosenthal which follows from the original proof. For the convenience of the readers, we give a sketch of the proof. We use $[x_i]_{i\in I}$ to denote the closed linear span of $(x_i)_{i\in I}$.

Lemma 2.11. *Let X be a separable Banach space. Then there exists a subspace Y with FDD (E_n) so that for any blocking (F_n) of (E_n) and for any sequence $(n_k) \subset \mathbf{N}$, $X/\overline{span\{(F_{n_k})_{k=1}^\infty\}}$ admits an FDD (G_n). Moreover, if X^* is separable, (E_n) and (G_n) can be chosen to be shrinking.*

Proof. Let $\{x_i, x_i^*\}$ be a Markuschevich basis for X so that $[x_i^*]$ is a norm determining subspace of X^* and even $[x_i^*] = X^*$ if X^* is separable. Then we can choose inductively finite sets $\sigma_1 \subset \sigma_2 \subset ...$ and $\eta_1 \subset \eta_2 \subset ...$ so that $\sigma = \bigcup_{n=1}^\infty \sigma_n$ and $\eta = \bigcup_{n=1}^\infty \eta_n$ are complementary infinite subsets of the positive integers and for $n = 1, 2, ...,$

(i) if $x^* \in [x_i^*]_{i\in\eta_n}$, there is a $x \in [x_i]_{i\in\eta_n \bigcup \sigma_{n+1}}$ so that $\|x\| = 1$ and $|x^*(x)| > (1 - \frac{1}{n+1})\|x^*\|$;

(ii) if $x \in [x_i]_{i\in\sigma_n}$, there is a $x^* \in [x_i^*]_{i\in\sigma_n \bigcup \eta_n}$ so that $\|x^*\| = 1$ and $|x^*(x)| > (1 - \frac{1}{n+1})\|x\|$.

Once we have this, by the proof of Theorem IV.4 in [7], we have $[x_i]_{i\in\sigma}^\perp$ is the w^* closure of $[x_i^*]_{i\in\eta}$. Put $Y = [x_i^*]_{i\in\eta}^\perp = [x_i]_{i\in\sigma}$. By the analogue of Proposition II.1(a) in [7], we deduce that X/Y has an FDD and that $([x_i]_{i\in\sigma_n})_{n=1}^\infty$ forms an FDD for Y. So to prove Lemma 2.11, it is enough to prove that for any blocking (Σ_n) of (σ_n) or any subsequence (σ_{n_k}) of (σ_n) (this of course needs the redefining of (η_n)), (i) and (ii) still hold. But this is more or less obvious because if $\Sigma_n = \bigcup_{i=k_{n-1}+1}^{k_n} \sigma_i$, then we define $\Delta_n = \bigcup_{i=k_{n-1}+1}^{k_n} \eta_i$ and it is easy to check $\{\Sigma_n, \Delta_n\}$ satisfy (i) and (ii). For a subsequence (σ_{n_k}), if we let $\Sigma_k = \sigma_{n_k}$ and define $\Delta_k = \bigcup_{i=n_k}^{n_{k+1}-1} \eta_i$, then $\{\Sigma_n, \Delta_n\}$ satisfy (i) and (ii). The rest is exactly the same as in the proof of Theorem IV.4 in [7].

□

The next lemma shows that for a reflexive space with an USB FDD, its dual also has an USB FDD.

Lemma 2.12. *Let X be a reflexive Banach space with an USB FDD (E_n). Then there is a blocking (F_n) of (E_n) so that (F_n^*) is an USB FDD for X^*.*

Proof. Without loss of generality, we assume (E_n) is monotone. Let (δ_i) be a sequence of positive numbers deceasing fast to 0. By the "killing the overlap" technique, we get a blocking (F_n) of (E_n) with $F_n = \sum_{i=k_{n-1}+1}^{k_n} E_i$ so that given any $x = \sum x_i$ with $x_i \in E_i, \|x\| = 1$, there is an increasing sequence (t_n) with $k_{n-1} < t_n < k_n$ such that $\|x_{t_i}\| < \delta_i$, where $0 = k_0 < k_1 < ....$ Let (F_n^*) be the dual FDD of (F_n) and let (x_i^*) be a normalized skipped block sequence with respect to (F_n^*) so that $x_i^* \in \oplus_{j=m_{i-1}+1}^{m_i-1} F_j^*$ where $0 = m_0 < m_1 < ....$ Let $x^* = \sum a_i x_i^*$ with $\|x^*\| = 1$. Let $x = \sum x_i$ be a norming functional of x^* with $x_i \in E_i$. By the definition of (F_n), we get an increasing sequence (t_i) with $k_{i-1} < t_i < k_i$ so that $\|x_{t_i}\| < \delta_i$. We define $y_1 = \sum_{j=1}^{t_{m_1}-1} x_j$ and $y_i = \sum_{j=t_{m_{i-1}}+1}^{t_{m_i}-1} x_j$ for $i > 1$. Let $y = \sum y_i$. So by the triangle inequality,

$$\|x - y\| \le \|\sum x_{t_{m_i}}\| \le \sum \|x_{t_{m_i}}\| < \sum \delta_{t_{m_i}}.$$

Let $(\epsilon_i) \subset \{-1, 1\}^{\mathbf{N}}$ and let $\tilde{x^*} = \sum \epsilon_i a_i x_i^*$. We will estimate $\tilde{x^*}(\sum \epsilon_i y_i)$.

$$\begin{aligned} |\tilde{x^*}(\sum \epsilon_i y_i)| &= |\sum \epsilon_i a_i x_i^*(\sum \epsilon_i y_i)| \\ &= |\sum a_i x_i^*(\sum y_i)| \\ &= |x^*(y)| \\ &\ge 1 - \sum \delta_{t_{m_i}}. \end{aligned}$$

Since (y_i) is a skipped block sequence with respect to (E_i), (y_i) is unconditional. Hence

$$\|\sum \epsilon_i y_i\| \le C\|\sum y_i\| < C(1 + \sum \delta_{m_i}),$$

where C is the unconditional constant associated with the USB FDD (E_n). If we let $\sum \delta_i < \epsilon/2$, then we conclude that

$$\|\tilde{x^*}\| > (1 - \epsilon)/C(1 + \epsilon).$$

Therefore, (x_i^*) is unconditional with unconditional constant less than $(1 + 3\epsilon)C$ if ϵ is sufficiently small. Hence (F_n^*) is an USB FDD.

□

Theorem 2.13. *Let X be a separable reflexive Banach space. Then the following are equivalent.*

(a) *X has the UTP.*
(b) *X embeds into a reflexive Banach space with an USB FDD.*
(c) *X^* has the UTP.*

Proof. It is obvious that (b) implies (a). If we can prove (a) implies (b), and X satisfies (b), then by Lemma 2.12, X^* is a quotient of a reflexive space with an USB FDD. So by Theorem 2.8, X^* has the UTP. Hence we only need to show that (a) implies (b). Let X_1 be a subspace of X with an FDD (E_n) given by Lemma 2.11. By Proposition 2.4 in [13], we get a blocking (F_n) of (E_n) so that (F_n) is an USB FDD. Let $Y_1 = [F_{4n}]$ and $Y_2 = [F_{4n+2}]$. Then (F_{4n}) and (F_{4n+2}) form unconditional FDDs for Y_1 and Y_2. By Lemma 2.11, X/Y_i has an FDD. Since X has the UTP, by Theorem 2.8, X/Y_i has the UTP. Now using Proposition 2.4 in [13] again, we know that X/Y_i has an USB FDD. Noticing that $Y_1 \cap Y_2 = \{0\}$ and $Y_1 + Y_2$ is closed, by Lemma 2.10, we have that X embeds into $X/Y_1 \oplus X/Y_2$. Hence X embeds into a reflexive space with an USB FDD. □

Corollary 2.14. *Let X be a separable reflexive Banach space with the UTP. Then X embeds into a reflexive Banach space with an unconditional basis.*

Proof. By Theorem 2.13, X embeds into a reflexive space Y with an USB FDD (E_n). We prove that Y embeds into a reflexive space with an unconditional FDD. Then as was mentioned in the introduction, Y embeds into a reflexive space with an unconditional basis and so X does.

By Lemma 2.12, there is a blocking (F_n) of (E_n) so that (F_n^*) is an USB FDD for Y^*. Now let $Y_1 = \oplus F_{4n}$ and let $Y_2 = \oplus F_{4n+2}$. Then we have $Y_1 \cap Y_2 = \{0\}$ and $Y_1 + Y_2$ is closed because (F_{2n}), being a skipped blocking of (E_n), is unconditional. By Lemma 2.10, Y embeds into $Y/Y_1 \oplus Y/Y_2$. Since $(Y/Y_i)^*$ is isomorphic to $Y_i^{\perp}$, it is enough to prove $Y_i^{\perp}$ has an unconditional FDD. Let $G_n^* = F_{4n-3}^* \oplus F_{4n-2}^* \oplus F_{4n-1}^*$. It is easy to see that (G_n^*) forms an FDD for $Y_1^{\perp}$. Noticing that (G_n) is a skipped blocking of (F_n^*), we conclude that (G_n) is unconditional. Similarly, we can show that $Y_2^{\perp}$ admits an unconditional FDD. This finishes the proof. $\square$

Corollary 2.15. *Let X be a quotient of a reflexive Banach space with an unconditional FDD. Then X embeds into a reflexive Banach space with an unconditional basis.*

Proof. Combine Theorem 2.8 and Corollary 2.14. $\square$

We mention again that in 1974 Davis, Figiel, Johnson and Pełczyński proved [1] that a reflexive Banach space X which embeds into a Banach space with a shrinking unconditional basis embeds into a reflexive space X with an unconditional basis. The next year, Figiel, Johnson and Tzafriri [4] got a stronger result by removing the shrinkingness of the unconditional basis in the hypothesis. Our next corollary gives a parallel result for quotients.

Corollary 2.16. *Let X be a separable reflexive Banach space. If X is a quotient of a Banach space with a shrinking unconditional basis, then X is isomorphic to a quotient of a reflexive Banach space with an unconditional basis.*

Proof. Since X is a quotient of a Banach space with a shrinking unconditional basis, X^* is a subspace of a Banach space with an unconditional basis. Hence, by [4], X^* is isomorphic to a subspace of a reflexive Banach space with an unconditional basis. Therefore, X is isomorphic to a quotient of a reflexive Banach space with an unconditional basis. $\square$

Remark 2.17. Corollary 2.16 is different from the result of Figiel, Johnson and Tzafriri in that the shrinkingness in our result cannot be removed. The reason is more or less obvious since every separable Banach space is a quotient of ℓ_1 which has an unconditional basis.

Gluing Theorem 2.13, Corollary 2.14, Corollary 2.15 and Corollary 2.16 together, we have the following long list of equivalences.

Theorem 2.18. *Let X be a separable reflexive Banach space. Then the following are equivalent.*

(a) *X has the UTP.*
(b) *X is isomorphic to a subspace of a Banach space with an unconditional basis.*

(c) *X is isomorphic to a subspace of a reflexive space with an unconditional basis.*
(d) *X is isomorphic to a quotient of a Banach space with a shrinking unconditional basis.*
(e) *X is isomorphic to a quotient of a reflexive space with an unconditional basis.*
(f) *X is isomorphic to a subspace of a quotient of a reflexive space with an unconditional basis.*
(g) *X is isomorphic to a subspace of a reflexive quotient of a Banach space with a shrinking unconditional basis.*
(h) *X is isomorphic to a quotient of a subspace of a reflexive space with an unconditional basis.*
(i) *X is isomorphic to a quotient of a reflexive subspace of a Banach space with a shrinking unconditional basis.*

3. Example

In this section we give an example of a reflexive Banach space for which there exists a $C > 0$ so that every normalized weakly null sequence admits an C-unconditional subsequence while for any $C > 0$ there is a normalized weakly null tree such that every branch is not C-unconditional. The construction is an analogue of Odell and Schlumprecht's example (see Example 4.2 in [12]).

We will first construct an infinite sequence of reflexive Banach spaces X_n. Each X_n is infinite dimensional and has the property that for $\epsilon > 0$, every normalized weakly null sequence has a $1+\epsilon$-unconditional basic subsequence, while there is a normalized weakly null tree for which every branch is at least C_n-unconditional and C_n goes to infinity when n goes to infinity. Then the ℓ_2 sum of X_n's is a reflexive Banach space with the desired property.

Let $[\mathbf{N}]^{\leq n}$ be the set of all subsets of the positive integers with cardinality less than or equal to n. Let $c_{00}([\mathbf{N}]^{\leq n})$ be the space of sequences with finite support indexed by $[\mathbf{N}]^{\leq n}$ and denote its canonical basis by $(e_A)_{A\in[\mathbf{N}]^{\leq n}}$. Let (h_i) be any normalized conditional basic sequence which satisfies a block lower ℓ_2 estimate, for example, the boundedly complete basis of James' space (see problem 6.41 in [3]). Let $\sum a_A e_A$ be an element of $c_{00}([\mathbf{N}]^{\leq n})$. Let $(\beta_k)_{k=1}^m$ be disjoint segments. By a segment in $[\mathbf{N}]^{\leq n}$, we mean a sequence $(A_i)_{i=1}^k \in [\mathbf{N}]^{\leq n}$ with $A_1 = \{n_1, n_2, ..., n_l\}, A_2 = \{n_1, n_2, ..., n_l, n_{l+1}\}, ..., A_k = \{n_1, n_2, ..., n_l, ..., n_{l+k-1}\}$, for some $n_1 < n_2 < ... < n_{l+k-1}$. Let $\beta_k = \{A_{1,k}, A_{2,k}, ..., A_{j_k,k}\}$ with $A_{i,k} < A_{i+1,k}$ under the tree order in $[\mathbf{N}]^{\leq n}$. Now we define X_n to be the completion of $c_{00}([\mathbf{N}]^{\leq n})$ under the norm

$$\|\sum a_A e_A\|_{X_n} = \sup\{(\sum_{k=1}^m (\|\sum_{A_{i,k}\in\beta_k} a_{A_{i,k}} h_i\|)^2)^{1/2} : (\beta_k)_{k=1}^m \quad \text{are disjoint segments}\}.$$

Let $X = (\sum X_n)_2$. We first show that for any $C > 0$ there is a normalized weakly null tree in X so that every branch is at least C-unconditional. Let M be so big that the unconditional constant of $(h_i)_{i=1}^M$ is greater than C. Actually the normalized weakly null tree $(e_A)_{A\in[\mathbf{N}]^{\leq M}}$ in X_M has the property above because any branch of it is 1-equivalent to $(h_i)_{i=1}^M$ since (h_i) has a block lower ℓ_2 estimate.

Next we need to verify that for every $\epsilon > 0$, every normalized weakly null sequence in X has an $1+\epsilon$-unconditional basic subsequence. Actually, we will prove that there is a subsequence which is $1+\epsilon$-equivalent to the unit vector basis of ℓ_2. By a gliding-hump argument, it is not hard to verify the following fact.

Fact. Let (Y_k) be a sequence of reflexive Banach spaces. And let $Y = (\sum Y_k)_{\ell_2}$. If for every $\epsilon > 0, k \in \mathbf{N}$, every normalized weakly null sequence in Y_k has a subsequence which is $1+\epsilon$-equivalent to the unit vector basis of ℓ_2, then for every $\epsilon > 0$, every normalized weakly null sequence in Y has a subsequence which is $1+\epsilon$-equivalent to the unit vector basis of ℓ_2.

Considering the fact, it is enough to show that for every $\epsilon > 0, k \in \mathbf{N}$, every normalized weakly null sequence in X_k has a subsequence which is $1+\epsilon$ equivalent to the unit vector basis of ℓ_2. We prove this by induction.

For $k = 1$, X_1 is isometric to ℓ_2, so the conclusion is obvious.

Assume the conclusion is true for X_k. By the definition of X_{k+1}, X_{k+1} is isometric to $(\sum(\mathbf{R} \oplus X_k))_{\ell_2}$ (where $\mathbf{R} \oplus X_k$ has some norm so that $\{0\} \oplus X_k$ is isometric to X_k). Hence by hypothesis and the fact we mentioned above, it is easy to see the conclusion is true in X_{k+1}. This finishes the proof.

Remark 3.1. The proof of the corresponding induction step in Example 4.2 in [12] is more complicated than the very simple induction argument in the previous paragraph. Schlumprecht realized after [12] was written that the induction could be done so simply and his argument works in our context.

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