SUBSPACES AND QUOTIENTS OF BANACH SPACES WITH SHRINKING UNCONDITIONAL BASES

W. B. JOHNSON AND BENTUO ZHENG

Abstract. The main result is that a separable Banach space with the weak* unconditional tree property is isomorphic to a subspace as well as a quotient of a Banach space with a shrinking unconditional basis. A consequence of this is that a Banach space is isomorphic to a subspace of a space with an unconditional basis iff it is isomorphic to a quotient of a space with an unconditional basis, which solves a problem dating to the 1970s. The proof of the main result also yields that a uniformly convex space with the unconditional tree property is isomorphic to a subspace as well as a quotient of a uniformly convex space with an unconditional finite dimensional decomposition.

1. introduction

Banach spaces with unconditional bases have much better properties than general Banach spaces, and many of these nice properties pass to subspaces. Consequently, an important and interesting question, dating from at least the 1970s, is to find the right Banach space condition under which a space can be embedded into a space with an unconditional basis.

In [JZh1], the authors gave a characterization of subspaces and quotients of reflexive spaces with unconditional bases. We showed that a reflexive separable Banach space is isomorphic to a subspace as well as a quotient of a reflexive space with an unconditional basis if and only if it has the unconditional tree property (UTP). Although we repeat the definition in the next section, here we recall that a Banach space is said to have the UTP provided that every normalized weakly null infinitely branching tree in the space admits a branch that is an unconditional basic sequence. However, the UTP is not enough to ensure that a nonreflexive space embeds into a space with an unconditional basis, as is witnessed by the James space [LT, Example 1.d.2]. Since every skipped block basis of its natural shrinking basis is equivalent to the unit vector basis of ℓ_2, it has the unconditional tree property. The James space does not embed into a space with an unconditiona l basis, since, for example, nonreflexive subspaces of spaces with unconditional bases have a subspace isomorphic to either c_0 or ℓ_1 [LT, Theorem 1.c.13].

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It turns out that the right condition to characterize subspaces and quotients of spaces with shrinking unconditional bases is the weak* unconditional tree property (w^*-UTP). A Banach space X is said to have the w^*-UTP if every normalized weak* null tree in X^* admits an unconditional branch. If X is reflexive, then the w^*-UTP is (obviously) equivalent to X^* having the UTP and (non-obviously, but by the results in [JZh1], or by Corollary 1 and results from the 1970s) also equivalent to X having the UTP. In Section 2, we prove that a separable Banach space X with the w^*-UTP is isomorphic to a subspace of a space Y with a shrinking unconditional finite dimensional decomposition (FDD) and also X is isomorphic to a quotient of a space Z with a shrinking unconditional (FDD) (see Theorem A). Moreover, Y can be chosen to be the direct sum of two quotients of X and Z can be chosen to be the direct sum of two subspaces of X. Some further reasoning yields that X is isomorphic to a subspace as well as a quotient of a space with a shrinking unconditional basis (see Corollary 1).

Theorem A of course implies that a reflexive (respectively, uniformly convex) Banach space X with the UTP is isomorphic to a subspace as well as a quotient of a reflexive (respectively, uniformly convex) Banach space Y with an unconditional FDD (actually, this also follows from the proof in [JZh1], but we neglected to say this explicitly in [JZh1]). As pointed out in [JZh1], a formal consequence of this is that a reflexive X with the UTP is isomorphic to a subspace as well as a quotient of a reflexive space with an unconditional basis, so modulo results from the 1970s, this paper recaptures, with simpler proofs, everything in [JZh1]. However, the uniformly convex case is different: a little known example discussed at the end of this paper shows that a uniformly convex space with an unconditional FDD need not embed into a uniformly convex space that has an unconditional basis. We think that this example emphasizes the necessity as well as the naturalness of working with unconditional FDDs rather than unconditional bases in embedding problems.

We should note that before obtaining the results presented here, S. Cowell and N. Kalton [CK] proved an isometric version of the equivalence (1) $\iff$ (2) in Corollary 1. We use an easy lemma from [CK], but the main part of their proof seems not to yield an isometric version of Theorem A and we see no way to derive their isometric result using the techniques we employ.

Here we want to mention that the UTP is a strong version of the unconditional subsequence property (USP), which means every normalized weakly null sequence admits an unconditional subsequence. It is known by [J] and [O] that if X is a quotient of a space with a shrinking unconditional basis, then X has the USP. Theorem A yields the improvement that X has the UTP.

We use standard Banach space terminology as can be found in [LT].

2. Main results

Theorem A. *Let X be a separable Banach space. Then the following are equivalent*

(1) *X has the w^*-UTP;*
(2) *X is isomorphic to a subspace of the direct sum of two quotients of X that have shrinking unconditional FDDs;*
(3) *X is isomorphic to a quotient of the direct sum of two subspaces of X that have shrinking unconditional FDDs.*

To prove Theorem A, we need to consider a shrinking M-basis of X, which requires that X^* is separable. The next lemma shows that if a separable Banach space X has the w^*-UTP, then X^* is separable.

Lemma 1. *Let X be a separable Banach space with the w^*-UTP. Then X^* is separable.*

Proof. By a result of Dutta and Fonf [DF], a separable Banach space X has a separable dual if and only every normalized w^*-null tree in X^* admits a boundedly complete branch. So it is enough to prove that X^* contains no copy of c_0 [LT, Theorem 1.c.10]. But if c_0 embeds into X^*, then ℓ_1 is isomorphic to a complemented subspace of X by an old result of Bessaga and Pełczyński [LT, Proposition 2.e.8]. It is easy to see that ℓ_1 fails the w^*-UTP. Therefore, X fails the w^*-UTP. $\square$

Corollary 1. *Let X be a separable Banach space. Then the following are equivalent*

(1) *X has the w^*-UTP;*
(2) *X is isomorphic to a subspace of a Banach space that has a shrinking unconditional basis;*
(3) *X is isomorphic to a quotient of a Banach space that has a shrinking unconditional basis.*

To derive Corollary 1 from Theorem 2, one needs two facts:

(I) If X has a shrinking unconditional FDD (E_n), then X embeds into a space that has a shrinking unconditional basis.
(II) If X has a shrinking unconditional FDD (E_n), then X is isomorphic to a quotient of a space that has a shrinking unconditional basis.

Lindenstrauss and Tzafriri [LT, Theorem 1.g.5] proved that if X has an unconditional FDD (E_n), then X embeds into a space Y that has an unconditional basis. One can check that if (E_n) is shrinking, then the unconditional basis for Y is also shrinking. Alternatively, one can just apply [FJT, Theorem 3.3], which implies that if X^* is separable and X embeds into a space that has an unconditional basis, then X embeds into a space that has a shrinking unconditional basis. This gives (I).

To prove (II), we need to do the dual construction of [LT, Theorem 1.g.5] and verify that X is a quotient of a space with an unconditional basis when (E_n) is unconditional. Then we use the machinery of [DFJP] and the shrinkingness of (E_n) to show X is a quotient of a space with a shrinking unconditional basis. For smoothness of reading, we postpone the proof to Section 3.

The concept of M-basis plays an important role in our proofs. The main new idea in this paper is that "killing the overlap" techniques work as well for M-bases as they do for finite dimensional decompositions. This realization allows us to simplify considerably some arguments in [JZh1]. Recall that an M-basis for a separable Banach space X is a biorthogonal system $(x_i, x_i^*) \subset X \times X^*$ such that $[x_i] = X$ and $\overline{[x_i^*]}^* = X^*$. $[x_i]$ denotes the closed linear span of the x_i's and $\overline{[x_i^*]}^*$ is the weak* closure of the linear span of the x_i^*'s. Markushevich [M1] (or see [LT, Proposition 1.f.3]) proved that every separable Banach space admits an M-basis. If X^* is separable, we can choose an M-basis (x_i, x_i^*) so that $[x_i^*] = X^*$. In such a case, we call (x_i, x_i^*) a shrinking M-basis for X. If (x_i, x_i^*) is a shrinking M-basis for X, then every normalized block sequence (y_i) of (x_i) conve rges weakly to 0. Given an M-basis (x_i, x_i^*) and $1 = n_1 < n_2 < ...$, the sequence (X_i, X_i^*) defined by $X_i = \mathrm{span}(x_j)_{j=n_i}^{n_{i+1}-1}, X_i^* = \mathrm{span}(x_j^*)_{j=n_i}^{n_{i+1}-1}$ is called a blocking of (x_i, x_i^*). Notice that if (X_i, X_i^*) is a sequence of finite dimensional spaces with $X_i \subset X, X_i^* \subset X^*, X_i \cap X_j = \{0\}$ for $i \neq j$, dim X_i=dim X_i^*, X_i^* separates the points of X_i, $[X_i] = X$ and $\overline{[X_i^*]}^* = X^*$, then one may select $(x_j)_{j=n_i}^{n_{i+1}-1} \subset X_i, (x_j^*)_{j=n_i}^{n_{i+1}-1} \subset X_i^*$ (where $n_{i+1} - n_i = \mathrm{dim} X_i = \mathrm{dim} X_i^*$) so that (x_i, x_i^*) is an M-basis for X and (X_i, X_i^*) is a blocking of (x_i, x_i^*). In contradistinction to this, an example of Szarek's [S] shows that there are finite dimensional decompositions which are not blockings of Schauder bases.

For finite dimensional decompositions, Lemma 2 goes back to [J].

Lemma 2. *Let (x_i, x_i^*) be a shrinking M-basis for X. Let (X_i, X_i^*) be a blocking of (x_i, x_i^*). Then $\forall\ \varepsilon > 0$, $\forall\ m$, $\exists\ n$ such that $\forall\ \|x^*\| = 1\ \exists\ m < t < n$, such that $\|x^*|_{X_t}\| < \varepsilon$.*

Proof. Suppose not. Then $\exists\, \varepsilon > 0$, $\exists\, m$, $\forall\, n > m$, $\exists\, \|y_n^*\| = 1$, $\forall\, m < t < n$, $\|y_n^*|_{X_t}\| \geq \varepsilon$. By Banach-Alaoglu, without loss of generality, we assume $y_n^* \xrightarrow{w^*} y^*$. So we have $\|y^*|_{X_i}\| \geq \varepsilon$, $\forall\, i > m$. Hence we get a sequence (y_i) with $y_i \in S_{X_i}$ such that $y^*(y_i) > \varepsilon/2$. Therefore (y_i) does not converge weakly to 0. But (y_i) is a normalized block sequence of (x_i) which coverges to 0 weakly since (x_i) is shrinking. $\square$

By Lemma 2 and induction, we have the following lemma.

Lemma 3. *Let (x_i, x_i^*) be a shrinking M-basis for X and let (X_i, X_i^*) be a blocking of (x_i, x_i^*). Then $\forall\, (\varepsilon_i) \downarrow 0$, $\exists$ increasing sequence $1 = m_0 < m_1 < m_2 < \cdots$ such that for all $x^* \in S_{X^*}$, there exists (t_i), $m_{i-1} < t_i < m_i$, such that $\|x^*|_{X_{t_i}}\| < \varepsilon_i$.*

A normalized tree in X is a subset (x_A) of the unit sphere of X, where A indexed over all finite subsets of the natural numbers. A normalized weakly null tree is a normalized tree in X with the additional property that $(x_{A\cup\{n\}})_{n>\max A}$ forms a weakly null sequence for every $A \in \mathbf{N}^{<\omega}$. Here $\mathbf{N}^{<\omega}$ denotes the set of all finite subsets of the natural numbers. The tree order is given by $x_A \leq x_B$ whenever A is an initial segment of B. A branch of a tree is a maximal linearly ordered subset of the tree under the tree order. A normalized weak* null tree is a tree (x_A^*) in X^* so that $(x^*_{A\cup\{n\}})_{n>\max A}$ forms a weak* null sequence for every $A \in \mathbf{N}^{<\omega}$. X is said to have the unconditional tree property (UTP) if every normalized weakly null tree in X admits an unconditional branch and X is said to have the weak* unconditional tree property (w^*-UTP) if every normalized weak* null tree in X^* admits an unconditional branch. If the branch can always be taken to have unconditional constant at most C, then X is said to have the C-UTP or C-w^*-UTP. It is shown in [OSZ] that the UTP implies the C-UTP for some $C > 0$. The same argument also proves that the w^*-UTP implies the C-w^*-UTP for some $C > 0$.

Let (X_i) be an FDD or more generally a blocking of an M-basis. A sequence (y_i) is said to be a skipped blocked basis with respect to (X_i) if there exists an increasing sequence of integers $(k_i)_{i=0}^{\infty}$ with $k_0 = 0$ and $k_i < k_{i+1} - 1$ so that $y_i \in [X_j]_{j=k_{i-1}+1}^{k_i-1}$ for $i = 0, 1, 2, \ldots$. (X_i) is said to be skipped blocked unconditional if there is a $C > 0$ so that every normalized skipped blocked basis with respect to (X_i) is C-unconditional. The smallest such C is called the skipped blocked unconditional constant for (X_i).

In [OS3], Odell and Schlumprecht proved that if X is a weak* closed subspace of a space with a boundedly complete FDD (E_i) and every weak* null tree in X has a C-unconditional branch, then there is a blocking (F_i) of (E_i) so that every small perturbation of a skipped block basis with respect to (F_i) in X is $C + \varepsilon$-unconditional.

Our next lemma shows that using the same technique as in [OS3], the result can be extended to spaces with shrinking M-bases.

Lemma 4. *Let (x_i, x_i^*) be a shrinking M-basis for X. If X^* has w^*-UTP, then there exists a blocking (X_i, X_i^*) of (x_i, x_i^*) such that (X_i^*) is skipped blocked unconditional.*

Proof. Take $C > 0$ so that every normalized weak* null tree admits a C-unconditional branch. Let us consider the following game played by two players. Player I chooses a natural number k_1. Player II chooses a vector y_1^* in $S_{[x_j^*]_{j>k_1}}$. Player I then chooses a number $k_2 > k_1$ and player II chooses a vector y_2^* in $S_{[x_i^*]_{i>k_2}}$. In this fashion, player I chooses an increasing sequence $k_1 < k_2 < k_3 < ...$ and player II chooses a sequence (y_i^*) so that $y_i^* \in [x_j^*]_{j>k_i}$. So this game can be described as player I chooses a finite codimensional tail space with respect to (x_i^*) and player II chooses a norm one vector in the space. Player I then chooses a further finite codimensional tail space and player II chooses another norm one vector in the space. We say that player I has a winning strategy $(WI(C))$ if player I can choose a sequence of finite codimensional tail spaces so that no matter how player II chooses vectors, the sequence of vectors is C-unconditional. Since the set of all C-unconditional basic sequences in X^* is closed under pointwise limits in $S_{X^*}^{\omega}$, the game is determined, by the main theorem of Martin [M2]. As a consequence, we have that the w^*-UTP implies $WI(C)$ where C is the constant associated with the w^*-UTP. Let $(z_i)_{i=1}^k$ be a finite sequence in S_{X^*}. $WI(C, (z_i)_{i=1}^k)$ means player I can choose a sequence of finite codimensional tail spaces so that no matter how player II chooses vectors (y_i) in the corresponding tail spaces, the sequence $(z_1, z_2, ..., z_k, y_1, y_2, ...)$ is C-unconditional.

By the definition of $WI(C)$, we have the following facts.

(a) $WI(C)$ is equivalent to $\exists n \in \mathbf{N}, \forall x \in S_{[x_i^*]_{i>n}}, WI(C, x)$.

(b) Let (ε_i) be a decreasing sequence of positive numbers so that $\sum \varepsilon_i < \varepsilon/2C$ and let $(u_i)_{i=1}^k$ and $(v_i)_{i=1}^k$ be two finite sequences in X^* such that $\|u_i - v_i\| < \varepsilon_i$. Then $WI(C, (u_i)_{i=1}^k)$ implies $WI(C + \varepsilon, (v_i)_{i=1}^k)$.

Let (ε_i) be a fast decreasing sequence of positive reals so that $\sum \varepsilon_i < 1$. By (a), (b) and induction, using the compactness of the unit sphere of finite dimensional spaces, we can find a sequence of increasing integers $0 = n_0 < n_1 < n_2 < ...$, so that for all $k \in \mathbf{N}, 0 \leq l < k$, and all finite normalized skipped block $(u_i)_{i=1}^l$ with respect to $(X_i^*)_{i=1}^k$, we have $WI(C + \varepsilon_k, (u_i)_{i=1}^l)$. Here $X_i^* = [x_j^*]_{j=n_{i-1}+1}^{n_i}$.

Let (y_i) be any normalized block sequence with respect to (X_i^*). For any $k \in \mathbf{N}$, we have $WI(C + \sum \varepsilon_i, (y_i)_{i=1}^k)$ which certainly implies $WI(C + 1, (y_i)_{i=1}^k)$. So there exists an extension $(y_1, ..., y_k, z_{k+1}, z_{k+2}, ...)$ of $(y_i)_{i=1}^k$ so that $(y_1, ..., y_k, z_{k+1}, z_{k+2}, ...)$ is $C+1$-unconditional. Therefore $(y_i)_{i=1}^k$ is $C+1$-unconditional. Since k is arbitrary, we conclude that (y_i) is $C+1$-unconditional. □

A Banach space X is said to have the Kadec-Klee property if whenever $x_n^* \xrightarrow{w^*} x^*$ with $(x_n^*) \subset X^*$ and $\|x_n^*\| \to \|x^*\|$, we have $\|x_n^* - x^*\| \to 0$. It is well known that any Banach space with separable dual can be renormed to have the Kadec-Klee property (see [LT, Proposition 1.b.11]).

Lemma 5. *Let (x_i, x_i^*) be a shrinking M-basis for X. If (X_i, X_i^*) is a blocking of (x_i, x_i^*) so that (X_i^*) is skipped blocked unconditional, then there is a further blocking (Y_i, Y_i^*) of (X_i, X_i^*) such that both (Y_i) and (Y_i^*) are skipped blocked unconditional.*

Proof. By renorming, without loss of generality we assume that X has the Kadec-Klee property. We claim that by taking a blocking of (X_i, X_i^*), we may assume that $([X_{t_i}]_{i=1}^\infty)^\perp = [X_j^*]_{j \neq t_i}$ for all $t_1 < t_2 < ....$ This is essentially contained in the proof of [JR, Theorem IV.4] and uses only the hypothesis that (x_i, x_i^*) is a shrinking M-basis. For the convenience of the reader, we outline the proof here. We can choose inductively finite sets $\sigma_1 \subset \sigma_2 \subset ...$ and $\eta_1 \subset \eta_2 \subset ...$ so that $\sigma = \bigcup_{n=1}^\infty \sigma_n$ and $\eta = \bigcup_{n=1}^\infty \eta_n$ are complementary infinite subsets of the positive integers and for $n = 1, 2, ...,$

(i) if $x^* \in [X_i^*]_{i \in \eta_n}$, there is an $x \in [X_i]_{i \in \eta_n \bigcup \sigma_{n+1}}$ so that $\|x\| = 1$ and $|x^*(x)| > (1 - \frac{1}{n+1})\|x^*\|$;

(ii) if $x \in [X_i]_{i \in \sigma_n}$, there is an $x^* \in [X_i^*]_{i \in \sigma_n \bigcup \eta_n}$ so that $\|x^*\| = 1$ and $|x^*(x)| > (1 - \frac{1}{n+1})\|x\|$.

Let S_n and T_n be the projections on X defined by $S_n(x) = \sum_{i \in \sigma_n} x_i^*(x) x_i$ and $T_n(x) = \sum_{i \in \eta_n} x_i^*(x) x_i$. It is easy to check that $\|T_n^*|_{[x_i]_{i \in \sigma_{n+1}}^\perp}\| \leq 1 + 1/n$. Observe that for all $x^* \in [x_i]_{i \in \sigma}^\perp, (T_n^*(x^*))$ is bounded and hence converges weak* to x^*. Thus $[x_i]_{i \in \sigma}^\perp$ is the w^* closure of $[x_i^*]_{i \in \eta}$. By [LT, Proposition 1.b.9], $\{S_n|_{[x_i]_{i \in \sigma}}\}$ defines an FDD for $[x_i]_{i \in \sigma}$. Now by the Kadec-Klee property and $\|T_n^*|_{[x_i]_{i \in \sigma_{n+1}}^\perp}\| \leq 1 + 1/n$, we have $T_n^*(x^*) \to x^*$ in norm for all $x^* \in [x_i^*]_{i \in \eta} = [x_i]_{i \in \sigma}^\perp$. This proves the claim. The above argument also works to give $[Y_j^*]_{j \notin \{t_i\}} = [Y_{t_i}]^\perp$ for any further blocking (Y_i, Y_i^*) of (X_i, X_i^*).

Let $\varepsilon > 0$ be a small number and let $(\varepsilon_i) \downarrow 0$ with $\sum \varepsilon_i < \varepsilon$. Let $(m_i)_{i=0}^\infty$ be given by Lemma 3. We define $Y_n = [X_i]_{i=m_{n-1}}^{m_n - 1}$ and $Y_n^* = [X_i^*]_{m_{n-1}}^{m_n - 1}$. It is enough to show that (Y_n) is skipped blocked unconditional. Since our proof works for any blocking of (Y_n, Y_n^*), for notational convenience we only check that for $x = \sum y_{2i}$ with $y_{2i} \in Y_{2i}$, the sum converges unconditionally. Assume that $\|x\| = 1$ and let $x^* \in S_{X^*}$ so that $\langle x, x^*\rangle = 1$. Since $\|x^*|_{X_{t_i}}\| < \varepsilon_i$, we get $d(x^*, [X_{t_i}]_{i=1}^{\infty\perp}) < 3\sum \varepsilon_i < 3\varepsilon$ by our construction.

Let $\tilde{x}^* = \sum \tilde{y}_i^* \in [X_j^*]_{j\neq t_i}$ so that $\tilde{y}_i^* \in [X_j^*]_{j=t_{2(i-1)}+1}^{t_{2i}-1}$ $(t_0 = 0)$ and $\|x^* - \tilde{x}^*\| < 3\varepsilon$. So we have $\langle x, \tilde{x}^*\rangle > 1 - 3\varepsilon$. Now let $\tilde{x} = \sum \delta_i y_{2i}$, $\delta_i \in \{-1, 1\}$. The norm of $\tilde{x}$

$$\begin{aligned}\|\tilde{x}\| &\geq |\langle \sum \delta_i y_{2i},\ \sum \delta_i \tilde{y}_i^*\rangle| \Big/ \|\sum \delta_i \tilde{y}_i^*\| \\ &= |\langle \sum y_{2i},\ \sum \tilde{y}_i^*\rangle| \Big/ \|\sum \delta_i \tilde{y}_i^*\| \\ &\geq \frac{1-3\varepsilon}{\|\sum \delta_i \tilde{y}_i\|}\end{aligned}$$

Letting λ be the skipped blocked unconditional constant for (X_i^*), we have $\|\sum \delta_i \tilde{y}_i^*\| \leq \lambda\|\sum \tilde{y}_i^*\| = \lambda\|\tilde{x}^*\| < \lambda(1 + 3\varepsilon)$. Therefore $\|\tilde{x}\| > \frac{1-3\varepsilon}{1+3\varepsilon}\lambda^{-1}$. □

In [CK], Cowell and Kalton observe that if Y is a subspace of a space with a shrinking 1-unconditional FDD, then Y has the au^*, which means for every $y^* \in Y^*$ and every $y_i^* \xrightarrow{w^*} 0$ in Y^*, $\lim \|y^* + y_i^*\| = \lim \|y^* - y_i^*\|$. It is clear that the au^* implies the w^*-UTP.

Lemma 6. *Let (x_i, x_i^*) be a shrinking M-basis for a Banach space X which is a subspace of a space with a shrinking unconditional FDD. Then there is a blocking (X_i, X_i^*) so that both (X_i) and (X_i^*) are skipped unconditional.*

Proof. By hypothesis, X is isomorphic to a subspace Y of a space with a shrinking 1-unconditional basis. By [CK], Y has the au^* and hence Y has the w^*-UTP. Since the w^*-UTP is an isomorphic condition, we get that X has the w^*-UTP. By Lemma 4, there is a blocking (X_i, X_i^*) of (x_i, x_i^*) so that (X_i^*) is skipped blocked unconditional. Using Lemma 5, we get a further blocking (Y_i, Y_i^*) so that both (Y_i) and (Y_i^*) are skipped blocked unconditional. □

A basic sequence (x_n^*) in X^* is called a w^*-basic sequence if there exists a sequence (x_n) in X for which $x_n^*(x_m) = \delta_n^m$ and such that, for every x^* in the w^* closure of $[x_n^*]$, we have

$$x^* = w^* \lim_n \sum_{i=1}^n x^*(x_i)x_i^*.$$

This is equivalent (see [LT, Proposition 1.b.9]) to saying that (x_i^*) are the biorthogonal functionals to some basis for $X/([x_i^*])^\top$. Similarly, a sequence of finite dimensional spaces (X_n^*) in X^* is a w^* FDD if and only if there is an FDD (Y_n) of $Y = X/([X_n^*])^\top$ so that $X_n^* = T^*Y_n^*$ where T is the natural quotient from X onto Y and (Y_n^*) is the biorthogonal FDD to (Y_n).

We repeat a simple lemma from [JZh1] which will be used again here.

Lemma 7. *Let X be a Banach space and X_1, X_2 be two closed subspace of X. If $X_1 \cap X_2 = \{0\}$ and $X_1 + X_2$ is closed, then X embeds into $X/X_1 \oplus X/X_2$.*

Now we prove our main results.

Proof of Theorem A. (3) implies (1) is obvious.

(1) $\Rightarrow$ (2). By Lemma 1, Lemma 4 and Lemma 5, we get a blocking (X_i, X_i^*) of a shrinking M-basis (x_i, x_i^*) of X so that both (X_i) and (X_i^*) are skipped blocked unconditional. Let $E = [X_{4i}]$, $F = [X_{4i+2}]$. Then $E + F$ is closed and $E \cap F = \{0\}$. By Lemma 7, X embeds into $X/E \oplus X/F$. We claim that X/E has a shrinking unconditional FDD. Actually, by the proof of Lemma 5, X/E has a shrinking FDD (Y_i). Moreover, the biorthogonal FDD of (Y_i) is the isometric image of $(X_{4i-3} \oplus X_{4i-2} \oplus X_{4i-1})$, which is boundedly complete and unconditional. Therefore X/E has a shrinking unconditional FDD. Similarly, X/F has a shrinking unconditional FDD.

(2) $\Rightarrow$ (3). Let (X_i, X_i^*) be given by Lemma 6. By the proof of Lemma 5, $[X_j]^{\perp}_{j\neq 4i} = [X^*_{4i}]$ and $[X_i]^{\perp}_{j\neq 4i+2} = [X^*_{4i+2}]$. Since (X_i^*) is skipped unconditional, $[X^*_{4i}] + [X^*_{4i+2}]$ is closed and $[X^*_{4i}] \cap [X^*_{4i+2}] = 0$. Hence X^* embeds into $X^*/[X^*_{4i}] \oplus X^*/[X^*_{4i+2}]$, and, by the construction, the embedding is w^* continuous. Therefore X is a quotient of $[X_j]_{j\neq 4i} \oplus [X_j]_{j\neq 4i+2}$, which has a shrinking unconditional FDD. □

If Y is a subspace or a quotient of a Banach space X, then the modulus of uniform convexity or smoothness of Y is no worse than that of X (see [LT, Section 1.e]). Thus Theorem 2 yields

Corollary 2. *If X is uniformly convex [uniformly smooth] and has the UTP, then X is isomorphic to a subspace and a quotient of a Banach space Y which has the same, up to equivalence, modulus of uniform convexity [uniform smoothness] as X.*

In Corollary 2, the space Y cannot always be taken to have an unconditional basis. Since to discuss this it is better to use isomorphic language, we recall that a space X is finitely crudely representable in a space Z provided there is a C such that every finite dimensional subspace of X is C-isomorphic to a subspace of Z. A Banach space X is superreflexive if and only if every Banach space which is finitely crudely representable in X is reflexive. The Enflo-James theorem (see [FHHMPZ, Theorem 9.18]) says that a Banach space X is superreflexive iff X is isomorphic to a uniformly convex space iff X is isomorphic to a uniformly smooth space.

In [JT, Corollary 4], an example is given of a supperreflexive space U such that if U is finitely representable in a space Z that has an unconditional basis, then also c_0 is finitely representable in Z and hence Z is not superreflexive. Let (U_n) be an increasing sequence of finite dimensional subspaces of U so that $\bigcup U_n$ is norm dense in U and let X be the ℓ_2-sum of (U_n). Then X is supperreflexive and has an unconditional FDD but X does not embed into any supperreflexive space that has an unconditional basis.

3. Proof of Fact (II)

Let (ε_n) be a fast decreasing sequence of positive reals. For any $n \in \mathbf{N}$, we choose an ε_n-net $(x_{n,j})_{j=1}^{i_n}$ of B_{E_n}. Without loss of generality, we assume the unconditional constant associated to (E_n) is 1. In this case, we can identify each E_n^* as a subspace of X^*. Define a map T from $\mathrm{span}(E_n^*)$ into c_{00} by

$$T(x^*) = (x^*(x_{1,1}), ..., x^*(x_{1,i_1}), ..., x^*(x_{n,1}), ..., x^*(x_{n,i_n}), ...).$$

Let Y_0 be the convex hull of $\cup_\theta M_\theta T(B_{\mathrm{span}(E_n^*)})$, where $\theta = (\theta_1, \theta_2, ...)$ is a sequence of signs and M_θ is defined by $M_\theta(a_1, a_2, ...) = (\theta_1 a_1, \theta_2 a_2, ...)$. We introduce a norm on span Y_0 by $\|y\| = \inf\{\lambda : y/\lambda \in Y_0\}, \forall y \in \mathrm{span}\, Y_0$. Let Y be the completion of $\mathrm{span}\, Y_0$ under this norm. It is easy to check that the natural unit vector basis (denote it by (e_n^*)) is an unconditional basis for Y. The same argument in the proof of Theorem 1.g.5 in [LT] yields the map T is an isomorphic embedding from $\mathrm{span}\,(E_n^*)$ into Y with norm less than 1. Since $\mathrm{span}\,(E_n^*)$ is dense in X^*, T extends to an isomorphism from X^* into Y. Let (e_n) be the biorthogonal functionals to (e_n^*). It is clear that (e_n) is an unconditional basic sequence. Define a map Q from $[e_n]$ into X by $Q(\sum a_i e_i) = \sum a_i x_i$, where (x_i) is the natural ordering of $(x_{n,j})_{n\in\mathbf{N}, 1\le j\le i_n}$. We show that Q is a well defined operator with norm at most 1:

$$\begin{aligned}
\|Q(\sum a_i e_i)\| = \|\sum a_i x_i\| &= \sup_{\|x^*\|\le 1}\{|x^*(\sum a_i x_i)|\} = \sup_{\|x^*\|\le 1}\{|\sum a_i x^*(x_i)|\} \\
&= \sup_{\|x^*\|\le 1}\{|\langle \sum x^*(x_i) e_i^*, \sum a_i e_i\rangle|\} \\
&\le \sup_{\|x^*\|\le 1}\{\|\sum x^*(x_i) e_i^*\| \|\sum a_i e_i\|\} \\
&\le \|\sum a_i e_i\|.
\end{aligned}$$

Next we show that Q is a quotient map by proving that $Q^* = T$.

$$\begin{aligned}
Q^*(x^*)(\sum a_i e_i) &= x^*(Q(\sum a_i e_i)) = x^*(\sum a_i x_i) \\
&= \sum a_i x^*(x_i) = \langle \sum x^*(x_i) e_i^*, \sum a_i e_i\rangle \\
&= \langle T(x^*), \sum a_i e_i\rangle.
\end{aligned}$$

Hence X is a quotient of $[e_n]$. Now it may be that (e_n) is shrinking, but we cannot prove it. To get around this, we use arguments from [DFJP] and so follow the notation of [DFJP].

Let (u_i) be an unconditional basis for a Banach space U. Let $\alpha \subset \mathbf{N}$. Then P_α denotes the projection $P_\alpha(\sum a_i u_i) = \sum_{i\in\alpha} a_i u_i$. Let V be a subset of U. V_u is defined to be $\bigcup_{\alpha\subset\mathbf{N}} P_\alpha(V)$. Our main observation is that Lemma 3 in [DFJP] works for V_u too. To be more precise, the following is true. If $V \subset U$ is convex, symmetric and $\sigma(U,\Gamma)$ compact, then so is the $\sigma(U,\Gamma)$ closure W of the convex hull of V_u. In our application, $\Gamma = [e_n]$, $u_i = e_i^*$, and $U = Y = [e_i^*]$ (Y, e_n, and e_n^* are defined in the beginning of the proof of this lemma). So Γ norms Y and $V := Q^*(B_{X^*})$ is $\sigma(Y,\Gamma)$ compact. Construct V_u, W as we defined above (see [DFJP, Lemma 2, 3] for proofs). C is obtained by the basic factorization procedure of [DFJP, Lemma 1]. Then C is $\sigma(Y,\Gamma)$ compact. Hence $\tilde{Y}$:=span C is isometric to the dual of a Banach space Z so that the $\sigma(\tilde{Y},Z)$ topology is finer than that induced by $\sigma(Y,\Gamma)$. This implies that Q^* is weak* continuous as a map from X^* into Z^*. So X is a quotient of Z. Since Y has the unconditional basis (e_n), by [DFJP, Lemma 1 (x)] (e_n) is also an unconditional basis for $\tilde{Y}$. Since $\tilde{Y}$ is a separable dual, the unconditional basis for $\tilde{Y}$ is boundedly complete. Therefore the biorthogonal functionals (E_n) form a shrinking unconditional basis for Z.

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Department of Mathematics, Texas A&M University, College Station, Texas, 77843
E-mail address: johnson@math.tamu.edu

Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin, TX 78712-0257
E-mail address: btzheng@math.utexas.edu