Very tight embeddings of subspaces of L_p, $1 \le p < 2$, into ℓ_p^n

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Abstract

We prove that for $1 \le p < r < 2$, every n-dimensional subspace E of L_r, in particular ℓ_r^n, well-embeds into ℓ_p^m for some $m \le (1+\epsilon)n$, where "well" depends on p, r, and the arbitrary $\epsilon > 0$, but not on n.

1 Introduction

Given two normed spaces X and Y we say that X K-embeds into Y provided there exists a subspace $\bar{X}$ of Y and an isomorphism T from X onto $\bar{X}$ with $\|T\|\|T^{-1}\| \le K$. Starting with Dvoretzky's Theorem [D], a lot of attention was given to the question of what is the smallest dimension of a space, out of a certain family, into which ℓ_2^d K-embeds. For example, it is shown in [FLM] that, for every $\epsilon > 0$, ℓ_2^d $(1+\epsilon)$-embeds into ℓ_p^n whenever $d \le c(\epsilon)n$, for $1 \le p < 2$, and whenever $d \le c(p,\epsilon)n^{2/p}$, for $2 < p < \infty$. The proofs in [FLM] are based on concentration inequalities, following Milman's [Mi] method for proving Dvoretzky-like theorems. As such the proofs do not give better results (i.e., smaller n) for $\epsilon > 1$ than what they give for $\epsilon = 1$, say. There is however a different method of Kashin [K], based on volume considerations, which only works in some restricted class of spaces but gives better results for large K: For every $\mu > 1$ and $1 \le p < 2$ there is a $K = K(p,\mu)$ such that for every d ℓ_2^d, K-embeds into $\ell_p^{\mu d}$.

Concerning embeddings of other classical spaces, the first result not involving Euclidean spaces is contained in [JS1], where it was proved that for all $\epsilon > 0$ and $1 < p < 2$, ℓ_p^d $(1+\epsilon)$-embeds into ℓ_1^n whenever $d \le c(p,\epsilon)n$. This was extended by replacing ℓ_p^d with more general subspaces of L_1, in [BLM] and [T], using the random sampling method introduced in [S]. There is an extensive literature on this subject with more extensions and refinements. We refer the reader to [JS3] for a survey of this topic.

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All these results were proved by using some kind of concentration results and as such did not give better estimates, in terms of the dimension of the containing space, when the constant of the embedding was allowed to be large. The exception is a result of Naor and Zvavitch [NZ]. They proved that for every $\mu > 1$ and $1 < p < 2$, ℓ_p^d $K(d,p,\mu)$-embeds into $\ell_1^{\mu d}$ where $K(d,p,\mu)$ is a power of $\log d$ depending only on p and μ.

Our purpose here is to strengthen this to the status of Kashin's theorem and show that for every $\mu > 1$ and $1 < p < 2$ there is a $K = K(p,\mu)$ such that for every d, ℓ_p^d K-embeds into $\ell_1^{\mu d}$. The critical reader may notice that at first glance this may look contradictory: Since any two d-dimensional subspaces of a μd dimensional space have a $(2-\mu)d$-dimensional common subspace, it follows that any two $\ell_{p_i}^d$ spaces, $1 < p_1, p_2 < 2$ have $(2-\mu)d$-dimensional subspaces which are $2K$-isomorphic (and $(2-\mu) < 1$ is arbitrarily close to 1). Of course there is no contradiction here, the common subspace can be an Euclidean space, by Kashin's theorem (or any $\ell_r^{(2-\mu)d}$, $\max\{p_1,p_2\} < r < 2$ by ours).

In Theorem 2 below we actually show more.

Main Result *For every $1 \le p < q \le 2$ and every $\mu > 1$ there is a constant $K = K(p,q,\mu)$ such that if X is a d-dimensional subspace of L_q then X K-embeds into ℓ_p^n for $n = [\mu d]$.*

Theorem 2 also contains information about general subspaces of L_p; i.e., subspaces which do not necessarily embed into L_q for some $q > p$.

2 The main result

We begin by stating a theorem from [BKT] which will be our main tool. The case $p = 1$ of this theorem was known earlier; as was mentioned in [BKT], it appears implicitly in [CP]. We denote by $\{e_i\}_{i=1}^n$ the canonical basis of ℓ_p^n and by $[e_i]_{i\in\sigma}$ the subspace of ℓ_p^n spanned by $\{e_i\}_{i\in\sigma}$. Given an operator Q from ℓ_p^n, $Q_{|[e_i]_{i\in\sigma}}$ denotes the operator Q restricted to $[e_i]_{i\in\sigma}$.

Theorem 1 *([BKT]) There is a constant $1 \ge c > 0$ such that if $1 \le p < 2$, $Q : \ell_p^n \to Y$ is a quotient map and $a\,\mathrm{dim}Y \ge n$ for some $a > 1$, then there is a subset σ of $\{1, 2, \ldots, n\}$ of cardinality at least $c^a n$ so that $Q_{|[e_i]_{i\in\sigma}}$ is a c^{-a} isomorphism.*

The proof is contained in the proof of Theorem 2.1 in [BKT] using the estimates in Elton's theorem [E] (see [MV] for the best estimates, but they are not needed here).

Corollary 1 *Let X be a d-dimensional subspace of ℓ_p^n, $1 \le p < 2$ and put $a = n/(n-d)$. Then X is $1 + c^{-a}$ isomorphic to a subspace of $\ell_p^{[(1-c^a)n]}$.*

Proof: Put $Y = \ell_p^n/X$ and denote by Q the quotient map from ℓ_p^n onto Y with kernel X. Let σ be the set from the conclusion of Theorem 1 and let $P : \ell_p^n \to [e_i]_{i\in\tilde{\sigma}}$ be the natural restriction map, where $\tilde{\sigma}$ is the complementary set to σ. For $x \in X$,

$$\|Px\| = \|x-(I-P)x\| \geq \|Q(I-P)x\| \geq c^a\|(I-P)x\|$$

and thus

$$\|x\| \leq \|Px\| + \|(I-P)x\| \leq (1+c^{-a})\|Px\|.$$

This means that X is $1+c^{-a}$ isomorphic to a subspace of $[e_i]_{i\in\tilde{\sigma}}$ which in turn is isometric to ℓ_p^k with $k \leq n - c^a n$. ■

Proposition 1 *For every $\lambda > 0$ there is a $K < \infty$ such that if X is a subspace of ℓ_p^n, $1 \leq p < 2$, then X is K isomorphic to a subspace of ℓ_p^m with $m \leq \dim X + \lambda n$.*

Proof: Put $d = \dim X$. Note that if $n \geq k > d + \lambda n$, then $\frac{k}{k-d} \leq \frac{1}{\lambda}$ and thus $c^{\frac{k}{k-d}} \geq c^{\frac{1}{\lambda}}$. (Recall that c is the constant from Theorem 1 and thus satisfies $0 < c \leq 1$.) Consequently, if $n \geq k \geq d + \lambda n$ and X is C isomorphic to a subspace of ℓ_p^k, then, by Corollary 1, X is $C(1+c^{-\frac{1}{\lambda}})$ isomorphic to a subspace of $\ell_p^{(1-c^{\frac{1}{\lambda}})k}$.

By iterating we see that if s is the first integer for which $m \equiv (1-c^{\frac{1}{\lambda}})^s n < d + \lambda n$, then X is $(1+c^{-\frac{1}{\lambda}})^s$ isomorphic to a subspace of ℓ_p^m.

Since s can be estimated as a function of λ, the proposition is proved. ■

Recall that the type q constant of a normed space X is the smallest T for which for all n and all choices of $\{x_i\}_{i=1}^n \subset X$

$$(\mathrm{Ave}\|\sum_{i=1}^n \epsilon_i x_i\|^2)^{1/2} \leq T(\sum_{i=1}^n \|x_i\|^q)^{1/q}$$

where Ave denotes the average over all choices of signs $\{\epsilon_i\} \in \{-1,1\}^n$. Recall that for $1 \leq q \leq 2$ the type q constant of L_q over any measure space is finite and bounded by a constant depending only on q. Consequently, the result stated in the abstract follows from the following theorem, which is the main result of this paper.

Theorem 2 *(a) For every $1 \leq p < q \leq 2$, $T < \infty$, and every $\mu > 1$ there is a constant $K = K(p,q,T,\mu)$ such that if X is a d-dimensional subspace of L_p whose type q constant is at most T, then X K-embeds into ℓ_p^n for $n = [\mu d]$.*
(b) For every $1 \leq p < 2$, and every $\mu > 1$ there is a constant $K = K(p,\mu)$ such that if X is a d-dimensional subspace of L_p, then X $K(\log(d+2))^{1/p}$-embeds into ℓ_p^n for $n = [\mu d]$.

Proof: (a) follows from Proposition 1 and the fact, first proved in [BLM], that the conclusion of (a) holds for *some* $\mu = \mu(p,q,T)$ and $K = 2$ (say).
(b) Using Proposition 1, it is enough to prove the claim for some μ, depending only on p. We first 2-embed X into $\ell_{1/2}^{\mu d}$ for some μ (or into any other $\ell_r^{\mu d}$ for some $r < p$). The possibility of doing that is essentially contained in [BLM], although, for $p = 1$, it is first

stated and proved in [JS2]. [JS3] contains a proof for the whole range which is probably the shortest available one. We are assuming, as we may, that μd is an integer. We denote the image of X under this embedding by Y. Now there are two ways to proceed, both via some change of density argument. We shall describe one way based on a change of density due to Pisier and remark about the other one, based on a result of Maurey, later.

It follows from Theorem 1.1 in [P] that there is a probability measure ν on $\{1,2,\dots,\mu d\}$ such that

$$\|x\|_{p,\infty} \le C\|x\|_{1/2}$$

for all x in the the image Z of Y under the natural isometry between $\ell_{1/2}^{\mu d}$ and the space $L_{1/2}(\{1,2,\dots,\mu d\},\nu)$. Here C is an absolute constant, $\|x\|_{1/2}$ denotes the norm of x in $L_{1/2}(\{1,2,\dots,\mu d\},\nu)$ and $\|x\|_{p,\infty} = \max_{t>0} t(\nu(\{i;\, \|x(i)\| > t\}))^{1/p}$. Since ν is a probability measure we also have that $\|x\|_{p,\infty} \ge c\|x\|_{1/2}$ for all x and some absolute $c > 0$. In particular Z C/c-embeds into $L_{p,\infty}(\nu)$.

Let A be a subset of $\{1,2,\dots,\mu d\}$ of positive measure. Applying the inequality $\|x\|_{p,\infty} \ge c\|x\|_{1/2}$ in the probability space $(A, \frac{\nu_{|A}}{\nu(A)})$ we get that

$$\|x_{|A}\|_{1/2} \le c^{-1}\nu(A)^{2-1/p}\|x_{|A}\|_{p,\infty} \le (C/c)\nu(A)^{2-1/p}\|x\|_{1/2}$$

for all $x \in Z$. It follows that if $(C/c)\nu(A)^{2-1/p} < 1/2$ then the restriction to $\tilde{A}$, the complement of A, is a 2-isomorphism on Z. Note also that the equivalence of the $L_{1/2}$ norm with the $L_{p,\infty}$ norm on this restriction remains valid, although we have to change the constants to some other absolute constants. Since the total measure of $A = \{i;\, \nu(i) < 1/(M\mu d)\}$ is at most $1/M$ we may assume, by restricting to $\tilde{A}$, that, for an appropriate absolute M, $\nu(i) \ge 1/(M\mu d)$.

The reason for eliminating the small atoms is that now one can easily deduce that the (natural) distance between $L_{p,\infty}(\nu)$ and $L_p(\nu)$ is an absolute constant multiple of $(\log(\mu d))^{1/p}$. This concludes the proof.

As we remarked before, instead of using Pisier's result one can use a much earlier result of Maurey [Ma] which is similar in nature and ensures, for every $\epsilon > 0$, an embedding of Y into $L_{p-\epsilon}$ with constant depending only on ϵ. Computing the exact dependence on ϵ in this result and optimizing the product of this constant with the distance between $L_{p-\epsilon}^{\mu d}$ and $L_p^{\mu d}$, one gets the same result. ∎

Remarks 1. Let p, q, T and X be as in the statement of Theorem 2a. Applying the theorem with $\mu = 2$ we get a good embedding of X into ℓ_p^{2d}. Call the image of this embedding Y. Perturbing Y a bit we may assume that the restriction P_1 (resp. P_2) to the first (resp. last) d coordinates is an isomorphism on Y (i.e., these restrictions are of rank d when considered as operators on Y. We do not assume anything about the constant of isomorphism). Note that for $x \in Y$, $\|x\|$ is well equivalent to $\max\{\|P_1 x\|_p, \|P_2 x\|_p\}$. Geometrically this means that the unit ball of Y (and thus of X) is well approximated by the intersection of two linear images of the ball of ℓ_p^d. More precisely, there are two invertible linear operators T_1 and T_2 such that

$$T_1(B_{\ell_1^d}) \cap T_2(B_{\ell_1^d}) \subset B_X \subset K(T_1(B_{\ell_1^d}) \cap T_2(B_{\ell_1^d})) \qquad (1)$$

where $K = K(p, q, T)$.

Recall that for $q = 2$, i.e., when $X = \ell_2^d$, a somewhat stronger assertion (where the linear maps involved are multiples of orthogonal transformations) is a well known theorem of Kashin [K], the proof of which is similar to the proof of the the result of Kashin stated in the introduction.

Without the assumption on type, (1) holds with K replaced with $K(p)(\log(2d+2))^{1/p}$.

2. We do not know if the statement Theorem 2 also holds for $0 < p < 1$. The missing part is a variant of Theorem 1 to this range. The problem of extending the proof of this theorem for the range $0 < p < 1$ is that the natural extension of Elton's theorem for p-normed spaces is false; this follows easily from the fact observed in [Po] that $L_{p,\infty}$ is a p-normed space. We would like to thank Nigel Kalton for bringing this to our attention.

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