Lipschitz quotients from metric trees and from Banach spaces containing ℓ_1

William B. Johnson*, Joram Lindenstrauss†, David Preiss
and Gideon Schechtman†

1 Introduction

A Lipschitz map f between the metric spaces X and Y is called a Lipschitz quotient map if there is a $C > 0$ (the smallest such C, the co-Lipschitz constant, is denoted $coLip(f)$, while $Lip(f)$ denotes the Lipschitz constant of f) so that for every $x \in X$ and $r > 0$, $fB_X(x, r) \supset B_Y(f(x), r/C)$. Thus Lipschitz quotient maps are surjective maps which by definition have the property ensured by the open mapping theorem for surjective linear operators between Banach spaces.

If there is a Lipschitz quotient mapping f from X onto Y with $Lip(f) \cdot coLip(f) \leq C$, we say that Y is a C-Lipschitz quotient of X. If Y is a C-Lipschitz quotient of X for some C, we say that Y is a Lipschitz quotient of X.

Lipschitz quotient maps were introduced and studied in [4]. Among other results the following proposition was proved in [4] (see also [5], Theorem 11.18).

Proposition 1. Let X be a superreflexive Banach space and let the Banach space Y be a Lipschitz quotient of X. Then Y^* is crudely finitely representable in X^*.

The conclusion of the Proposition means that there is a constant λ so that for every finite dimensional (linear) subspace F of Y^* there is a finite dimensional subspace E of X^* so that $d(E, F) \leq \lambda$ where d denotes here the Banach Mazur distance. An equivalent way to express the conclusion of Proposition 1 is that Y^* is linearly isomorphic to a subspace of an ultraproduct of X^*. An immediate consequence of Proposition 1 is that any Banach space Y which is a Lipschitz quotient of a Hilbert space must be linearly isomorphic to a Hilbert space.

Proposition 1 was proved in [4] by using a special result on approximating Lipschitz functions from a superreflexive Banach space into finite dimensional Banach spaces (the so-called UAAP,

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see also [5], Proposition 11.16). It was proved in [4] that this approximation property actually characterizes superreflexive spaces and thus this proof of Proposition 1 does not work beyond the class of superreflexive spaces.

In [12] it was shown that if X is asymptotically uniformly smooth (AUS) then every Lipschitz map from X to a finite dimensional space has for every $\varepsilon > 0$ points of ε-Fréchet differentiability. We do not intend to recall here the definitions of AUS or of ε-Fréchet differentiability. We just mention that the class of spaces having an equivalent AUS norm is much richer than that of superreflexive spaces and contains, for example, the non-reflexive space c_0. On the other hand any space X with an AUS norm is an Asplund space (any separable subspace of X has a separable dual). The existence of points of ε-Fréchet differentiability for every $\varepsilon > 0$ provides an approximation of Lipschitz functions by affine functions which also suffices for proving the conclusion of Proposition 1. Thus Proposition 1 is valid also if X has an AUS norm. In [12] the result on existence of points of ε-Fréchet differentiability and thus Proposition 1 is proved also for some spaces X which do not admit an equivalent AUS norm (for example, every X which is a $C(K)$ space with K compact countable; unless such X is isomorphic to c_0 it does not have an equivalent AUS norm).

It is possible that the ε-Fréchet differentiability result (and thus Proposition 1) holds for every Asplund space X. The ε-Fréchet differentiability result is known to be false for every non-Asplund space (see e.g. [5], Proposition 4.12, or Chapter III of [7]). In this connection we mention also that in [4] it is proved that a Banach space which is a Lipschitz quotient of an Asplund space is itself an Asplund space.

All the results mentioned so far leave open the question whether the conclusion of Proposition 1 is valid for an arbitrary Banach space X. Recall that in the formally dual setting of Lipschitz embeddings the dual version of Proposition 1 is valid for an arbitrary Banach space X; i.e., if X Lipschitz embeds in Y then X is crudely finitely representable in Y. This fact is due to Ribe [17] (see also [5], Corollary 7.10).

We prove in this paper that Proposition 1 may fail to hold for many common Banach spaces. We do this by proving in Theorem 1 that if X is any separable Banach space containing ℓ_1 then there is a Lipschitz quotient map from X onto any separable Banach space Y. In particular, there is a Lipschitz quotient map from $C[0,1]$ onto ℓ_1; in fact, known results in the linear theory reduce the general theorem to this special case. This provides also the first known examples of pairs of separable Banach spaces X and Y so that there is a Lipschitz quotient map from X to Y but no such linear quotient map.

Theorem 1 can be considered in a way as a dual result to a theorem of Aharoni [1] (see also [3] or [5], Theorem 7.11) that every separable metric space is Lipschitz equivalent to a subset of c_0. There is however no obvious connection between the proof of Theorem 1 and the known proofs of Aharoni's theorem. The proof of Theorem 1 depends strongly on the special structure of ℓ_1.

Theorem 1 actually holds for a general complete separable metrically convex metric space Y.

Recall that a metric space Y is *metrically convex* provided that for each pair $\{x, y\}$ of points in Y and $0 < \lambda < 1$ there exists a point z in Y so that $d(x, z) = \lambda d(x, y)$ and $d(y, z) = (1 - \lambda)d(x, y)$. In other words, for each x and y in Y there is a geodesic arc from x to y (i.e., an arc which is the range of an isometry from an interval). It is not simply for the sake of generalization that we mention metrically convex spaces. The Lipschitz quotient maps f constructed in the proof of Theorem 1 factor through a special metrically convex metric space which we call an ℓ_1 *tree.* This factorization forces the mapping f to have the property that whenever the Gâteaux derivative $f'(x)$ of f exists at a point $x \in X$ then the rank of $f'(x)$ is ≤ 1 (see Proposition 5 below). Note that if X has an AUS norm and $1 < \dim Y < \infty$ such a situation is impossible by the results in [12] on ε-Fréchet differentiability mentioned above.

An ℓ_1 tree is a special kind of *metric tree.* Metric trees are used in geometry, metric topology, and group theory, but, as far as we know, this is the first time they have been used in the geometry of Banach spaces.

There is a weakening of the notion of Lipschitz quotient map which has been studied by David and Semmes [6]. Say that a Lipschitz map f from the metric space X into the metric space Y is *ball non collapsing* provided there is a $c > 0$ so that for every $r > 0$, the image under f of any ball of radius r contains a ball of radius cr. If f is a Lipschitz map from a metric space X into $\mathbb{R}^n$, say that f is *measure non collapsing* provided there is a $c > 0$ so that for every $r > 0$, the image under f of any ball of radius r has Lebesgue measure at least cr^n. David and Semmes [6] showed that that if X is finite dimensional and $f : X \to \mathbb{R}^n$ is measure non collapsing then f is ball non collapsing. In [4] this was extended to the case where X is any superreflexive Banach space. In Theorem 4 we show that if X is any Banach space which contains a subspace isomorphic to ℓ_1, then there is a measure non collapsing map from X onto the plane whose range has empty interior (and hence the mapping is not ball non collapsing).

In Section 3 we prove non separable versions of Theorem 1. Theorem 5 implies in particular that if Y is a Banach space (or more generally a complete metrically convex space) whose density character is at most that of the continuum then there is a Lipschitz quotient map from ℓ_∞ onto Y.

2 Proofs of the separable results

We begin by recalling the definition of a metric tree. A metric space X is a *metric tree* provided it is complete, metrically convex, and there is a unique arc (which then by metric convexity must be a geodesic arc) joining each pair of points in X. For an introduction to metric trees see [10]. However, except for the proof of Proposition 5 we do not use the theory of metric trees. Instead we work with a different notion of 'separable metric tree' which we denote by SMT and a different notion of 'metric tree' which we denote by MT. In Section 4 we further discuss metric trees and observe that a metric space is an SMT if and only if it is a separable metric tree and that a metric space is a MT if and only if it is a metric tree.

The concept of SMT is defined by describing how to construct one. In order to explain the construction, we need (a special case of) the concept of the ℓ_1-union of two metric spaces. Suppose that (X, d_X) and (Y, d_Y) are two metric spaces whose intersection is a single point, p. We define the ℓ_1 union $X \cup_1 Y$ of X and Y to be $(X \cup Y, d)$, where the metric d agrees with d_X on X, d agrees with d_Y on Y, and if $x \in X$, $y \in Y$, then $d(x, y)$ is defined to be $d_X(x, p) + d_Y(p, y)$. We can now describe the construction of an SMT. Let I_1 be a closed interval or a closed ray and define $T_1 := I_1$. The metric space T_1 is the first approximation to our SMT. Having defined T_n, let I_{n+1} be a closed interval or a closed ray whose intersection with T_n is an end point, p_n, of I_n, and define $T_{n+1} := T_n \cup_1 I_{n+1}$. The completion, T, of $\bigcup_{n=1}^{\infty} T_n$ is an SMT. If each I_n is a ray with end point p_{n-1} for $n > 1$ and the set $\{p_n\}_{n=1}^{\infty}$ of nodal points is dense in T, then we call T an 'ℓ_1 tree' and say that $\{I_n\}_{n=1}^{\infty}$, $\{T_n\}_{n=1}^{\infty}$ describe an allowed construction of T. (Of course, typically there are many different allowed constructions of a given SMT.)

The relevance of SMTs to Lipschitz quotients is made clear by Proposition 2.

Proposition 2. Let T be an ℓ_1 tree. Then every separable, complete, metrically convex metric space is a 1-Lipschitz quotient of T.

Proof. Let $\{I_n\}_{n=1}^{\infty}$, $\{T_n\}_{n=1}^{\infty}$ describe an allowed construction of T and let the nodes be $\{p_n\}_{n=1}^{\infty}$. Let Y be a separable, complete, metrically convex metric space. We build the desired Lipschitz quotient map by defining it on T_n by induction. That we can do this is clear from the following. Suppose you have a 1-Lipschitz map $f : T_n \to Y$, and y is taken from some countable dense subset Y_0 of Y. Extend f to T_{n+1} by mapping I_{n+1} to a geodesic arc $[f(p_n), y]$ which joins $f(p_n)$ to y; f is an isometry on $\{z \in I_{n+1} : d(p_n, z) \leq d(f(p_n), y)\}$ and f maps points on I_{n+1} whose distance to p_n is larger than $d(f(p_n), y)$ to y. This makes f act like a Lipschitz quotient at p_n relative to $[f(p_n), y]$. Since the nodal points are dense in T, it is evident that a judicious selection of the points from Y_0 will produce a 1-Lipschitz quotient mapping. For example, order Y_0 as a sequence and for each node p pick a sequence of nodes $\{p_{k_n}\}$ tending to p so that the sequences corresponding to different p-s are disjoint. Then, at the $k_n + 1$ step, extend f to T_{k_n+1} by mapping I_{k_n+1} to a geodesic arc which joins $f(p_{k_n})$ to the n-th element of Y_0. (We use the fact that to build a 1-Lipschitz quotient mapping from a complete metric space X onto a complete metric space Y it is enough to construct a function f from a dense subset X_0 of X into Y so that for each x in X_0 and for a dense set of r in $\mathbb{R}^+$, the closure in Y of $f[B_X(x, r)] \cap X_0$ is $B_Y(f(x), r)$. See, for example, the argument for Proposition 3.15 in [4].) $\square$

For the proof of Theorem 1 we use the fact that an SMT is a 1-absolute Lipschitz retract. A metric space X is called a λ-absolute Lipschitz retract (where $1 \leq \lambda < \infty$) provided that whenever X is isometrically contained in a metric space Y, there is a retraction, f, from Y onto X with $Lip(f) \leq \lambda$. A metric space X is a 1-absolute Lipschitz retract if and only if X is metrically convex and every collection of mutually intersecting closed balls in X have a common point (see e.g. [5], Proposition 1.4).

An isomorphic version of Lemma 1 was proved in a more general context in [13].

Lemma 1. Assume that X and Y are 1-absolute Lipschitz retracts which intersect in a single point, p. Then $X \cup_1 Y$ is also a 1-absolute Lipschitz retract.

Proof. It is evident that $X \cup_1 Y$ is metrically convex. We check that any collection of mutually intersecting closed balls in $X \cup_1 Y$ have a point in common. Let

$$\{B(x_\alpha, r_\alpha), x_\alpha \in X, \alpha \in \Gamma_1\} \cup \{B(y_\beta, \rho_\beta), y_\beta \in Y, \beta \in \Gamma_2\}$$

be such a collection in $X \cup Y$. We may clearly assume that both Γ_1 and Γ_2 are non-empty. By assumption we have for every $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$

$$r_\alpha + \rho_\beta \geq d(x_\alpha, p) + d(y_\beta, p).$$

If $r_\alpha \geq d(x_\alpha, p)$ for every $\alpha \in \Gamma_1$ and $\rho_\beta \geq d(y_\beta, p)$ for every $\beta \in \Gamma_2$ then p is a point common to all the balls in the collection. Otherwise there is, say, an $\alpha_0 \in \Gamma_1$ so that $r_{\alpha_0} < d(x_{\alpha_0}, p)$. Then $R = \inf_{\beta \in \Gamma_2} (\rho_\beta - d(y_\beta, p)) > 0$. The collection of balls $\{B(x_\alpha, r_\alpha),\ \alpha \in \Gamma_1\} \cup \{B(p, R)\}$ in X will be mutually intersecting and thus have a point x in common. It is trivial to check that this point x is a common point to the given collection of balls in $X \cup_1 Y$. $\square$

That every metric tree is a 1-absolute Lipschitz retract is basic to the theory of metric trees. We now show, independently of this fact, that the same holds for SMTs.

Corollary 1. Let T be an SMT. Then T is a 1-absolute Lipschitz retract.

Proof. Let $\{I_n\}_{n=1}^\infty$, $\{T_n\}_{n=1}^\infty$ describe an allowed construction of T and assume that T is isometrically contained in a metric space S. By Lemma 1 and induction, for each n there is a nonexpansive retraction P_n from S onto T_n. It is natural to try to get a cluster point of the mappings P_n in the space of functions from S to T under the topology of pointwise convergence. Since, except in some degenerate situations, T is not locally compact, this presents a problem. However, the space T can be isometrically embedded into ℓ_1 in an obvious way as a weak* closed subset. (Map I_1 isometrically to an interval or ray in the direction e_1. Having defined the mapping, f, on T_n, extend f to T_{n+1} by mapping I_{n+1} isometrically into $f(p_n) + \mathbb{R}^+ e_{n+1}$.) Any cluster point of $\{P_n\}_{n=1}^\infty$ in the space of functions from S to T under the topology of weak*-pointwise convergence is a nonexpansive retraction from S onto T. $\square$

Let Δ be the Cantor set $\{-1, 1\}^{\mathbb{N}}$ and for each n let r_n be the nth coordinate projection on Δ. In the space $C(\Delta)$, the sequence $\{r_n\}_{n=1}^\infty$ is isometrically equivalent to the unit vector basis of ℓ_1. For $n = 1, 2, \ldots,$ let E_n be the functions in $C(\Delta)$ which depend only on the first n coordinates.

Notice that if x is in E_n and $m > n$ then for all real t, $\|x + tr_m\| = \|x\| + |t|$. In other words, if I is a ray in the direction of r_m emanating from a point p in E_n, then, in $C(\Delta)$, the set $E_n \cup I$ is an ℓ_1 union of E_n and I. That $\{r_n\}_{n=1}^{\infty}$ acts like the ℓ_1 basis *over* $C(\Delta)$ is the key to proving Proposition 3.

Proposition 3. Every SMT is a 1-Lipschitz quotient of $C(\Delta)$.

Proof. Let $\{I_n\}_{n=1}^{\infty}$, $\{T_n\}_{n=1}^{\infty}$ describe an allowed construction of the SMT T. The desired Lipschitz quotient map is defined by induction by defining at step n the map on E_n. Let $\{x_n\}_{n=1}^{\infty}$ be a dense sequence in $\bigcup_{n=1}^{\infty} E_n$ so that for each n, x_n is in E_n, and let Y_0 be a subset of $\bigcup_{n=1}^{\infty} T_n$ such that $Y_0 \cap T_n$ is dense in T_n for each n. Suppose that we have a 1-Lipschitz mapping, f, from E_n into T_n and y is in $Y_0 \cap T_{n+1}$. We can extend f to a 1-Lipschitz map, $\tilde{f}$, from $E_n \cup (x_n + \mathbb{R}^+ r_{n+1})$ by mapping the set $\{x_n + te_{n+1} : 0 \le t \le d(f(x_n), y)\}$ isometrically onto the geodesic arc from $f(x_n)$ to y and setting $\tilde{f}(x_n + te_{n+1}) = y$ for $t > d(f(x_n), y)\}$. Then, since T_{n+1} is a 1-absolute Lipschitz retract by Corollary 1, we can extend $\tilde{f}$ to a 1-Lipschitz map from E_{n+1} into T_{n+1}. Since the sequence $\{x_n\}_{n=1}^{\infty}$ is dense in $C(\Delta)$, it is evident that a judicious selection of the points from Y_0 will produce a 1-Lipschitz quotient mapping from $C(\Delta)$ onto T. $\square$

Corollary 2 is an immediate consequence of Propositions 3 and 2.

Corollary 2. If Y is a separable, complete, metrically convex metric space, then Y is a 1-Lipschitz quotient of $C(\Delta)$.

Remark. Corollary 2 can be proved without using Proposition 2 and in fact without mentioning the concept of SMT. One defines the Lipschitz quotient on E_n by induction as in the proof of Proposition 3. To extend the 1-Lipschitz mapping $\tilde{f} : E_n \cup (x_n + \mathbb{R}^+ r_{n+1}) \to Y$ to a 1-Lipschitz mapping from E_{n+1} into Y, one uses the fact that $E_n \cup (x_n + \mathbb{R}^+ r_{n+1})$ is a 1-absolute Lipschitz retract, which follows from Lemma 1 because the space E_n is isometrically isomorphic to ℓ_∞^n and thus is a 1-absolute Lipschitz retract.

Theorem 1. Let X be a separable Banach space which contains a subspace isomorphic to ℓ_1 and let $\varepsilon > 0$. Then every separable, complete, metrically convex metric space is a $(1+\varepsilon)$-Lipschitz quotient of X.

Proof. By a theorem of James [11] X contains a subspace $(1+\varepsilon)$-isomorphic to ℓ_1. Pełczyński [15], [16] proved that this implies that $C(\Delta)$ is $(1+\varepsilon)$-isomorphic to a linear quotient of X. Indeed, let T be a $(1+\varepsilon)$ linear quotient from the subspace of X $(1+\varepsilon)$-isomorphic to ℓ_1 onto $C(\Delta)$. Extend T to a map S, with norm $\|T\|$, from X to $L_\infty(\Delta)$ and let Y denote the (separable) closure of the range of S. The main result of [16] implies that $C(\Delta)$ contains a space isometric to $C(\Delta)$ which is norm one complemented in Y. Denote the projection by P. Then PS is a $(1+\varepsilon)$-linear quotient map.

Theorem 1 follows from this and Corollary 2. $\square$

An alternative way to prove Theorem 1 without referring to [15] is sketched in the proof of Theorem 5 below.

Recall that a metric space Y is C-*quasiconvex* provided that any two points x and y in Y can be joined by an arc whose length is at most $Cd(x,y)$. Non essential modifications of the proofs given above yield Theorem 2.

Theorem 2. Let X be a separable Banach space which contains a subspace isomorphic to ℓ_1, let $\varepsilon > 0$, and let $1 \le C < \infty$. Then every separable, complete, C-quasiconvex metric space is a $(1+\varepsilon)C$-Lipschitz quotient of X.

We do not know if a Banach space X with the property that every separable, complete, C-quasiconvex metric space is a Lipschitz quotient of X must contain an isomorphic image of ℓ_1. However the next proposition says that such an X must contain a metric space eqivalent to an ℓ_1 tree.

Proposition 4. If X is a Banach space that has an ℓ_1 tree as a Lipschitz quotient space then X contains a subset which is Lipschitz equivalent to an ℓ_1 tree.

Proof. By [4], since X admits a non-Asplund Banach space as a Lipschitz quotient, X is non-Asplund. Let Y be any separable subspace of X whose dual is non separable. Then by a theorem of Stegall ([18] or [8] p. 194), there is a bounded linear operator from Y to $C(\Delta)$ such that the image of the unit ball of Y contains a Haar system in $C(\Delta)$. Recall that in this context a Haar system is a collection of indicator functions of sets which form a dyadic tree under the order of inclusion. Passing to a subsequence of the sets we get a collection of sets $\{A_{\bar{n}}\}$ indexed by all the finite sequences of integers $\bar{n} = (n_1, \ldots, n_k)$ and such that $A_{\bar{n}} \supset A_{\bar{m}}$ whenever $\bar{n}$ is an initial segment of $\bar{m}$ and $A_{(n_1,\ldots,n_k,r)} \cap A_{(n_1,\ldots,n_k,s)} = \emptyset$ whenever $r \neq s$. Let $\{r_i\}$ be a dense sequence in $[0,\infty)$. For $\bar{t} = (r_{i_1}, \ldots, r_{i_k}, s)$ with $s \in [0,\infty)$ let $u_{\bar{t}} = r_{i_1}\mathbf{1}_{A_\emptyset} + r_{i_2}\mathbf{1}_{A_{i_1}} + \cdots + r_{i_k}\mathbf{1}_{A_{(i_1,\ldots,i_{k-1})}} + s\mathbf{1}_{A_{(i_1,\ldots,i_k)}}$, where $\mathbf{1}_A$ denotes the indicator function of the set A. It is not difficult to check that the closure in $C(\Delta)$ of the collection of all such $u_{\bar{t}}$ is 2-Lipschitz equivalent to an ℓ_1 tree (with nodes of the form $r_{i_1}\mathbf{1}_{A_\emptyset} + r_{i_2}\mathbf{1}_{A_{i_1}} + \cdots + r_{i_k}\mathbf{1}_{A_{(i_1,\ldots,i_{k-1})}}$). For each $\bar{n}$ let $x_{\bar{n}}$ be an element in the unit ball of Y which is mapped by Stegall's map to $\mathbf{1}_{A_{\bar{n}}}$ and extend this discrete map (of $\{x_{\bar{n}}\}$ to $\{\mathbf{1}_{A_{\bar{n}}}\}$) linearly, denoting the inverse image of $u_{\bar{t}}$ by $x_{\bar{t}}$. The set $\{x_{\bar{t}}\}$ is easily seen to be Lipschitz equivalent to an ℓ_1 tree. $\square$

Remark. We do not know if every Banach space X containing (Lipschitz equivalent image of) an ℓ_1 tree has the property that every separable, complete, metrically convex metric space is a Lipschitz quotient of X.

It is known, and more or less clear from the unique arc property, that every connected subset of a metric tree is contractible. Using this fact it is very simple to prove Proposition 5.

Proposition 5. Let f be a Lipschitz mapping from the Banach space X into the Banach space Y. Assume that f factors continuously through a metric tree; i.e., there is a metric tree T and continuous functions $g : X \to T$ and $h : T \to Y$ so that $hg = f$. If f is Gâteaux differentiable at some point x in X, then the Gâteaux derivative $f'(x)$ of f at x has rank at most one.

Proof. Assume, without loss of generality, that $x = 0$ and $f(0) = 0$. Suppose, for contradiction, that $f'(0)$ has rank at least two. By composing with a linear projection onto a two dimensional subspace of Y we can assume that Y is two dimensional and that $f'(0)$ has rank two. By restricting to a suitable two dimensional subspace of X, we may further assume that X is two dimensional. Finally, by composing with suitable isomorphisms from $\mathbf{R}^2$ to X and from Y to $\mathbf{R}^2$, we may assume that $X = Y = \mathbf{R}^2$ and $f'(0)$ is the identity mapping on $\mathbf{R}^2$.

Now that we are in the finite dimensional setting we know that $f'(0)$ is a Fréchet derivative because f is a Lipschitz function (this is the only place where the Lipschitz assumption on f is used). So there exists $r > 0$ so that if x is in $\mathbf{R}^2$ and $|x| = r$, then $|x - f(x)| < r/2$ (recall that $f(0) = 0$ and $f'(0)$ is the identity). In particular, zero is not in $f[rS^1]$. However, $g[rS^1]$, hence also $f[rS^1] = hg[rS^1]$, is contractible in itself. This implies the absurdity that rS^1 is also contractible in $\mathbf{R}^2 \sim \{0\}$. $\square$

Using the fact, to be explained in Section 4, that every SMT is a metric tree, we get from Propositions 2, 3, and 5 that:

Theorem 3. Let X be a separable Banach space which contains a subspace isomorphic to ℓ_1 and let Y be any separable Banach space. There there is a Lipschitz quotient map from X onto Y so that whenever the Gâteaux derivative of f exists, it has rank at most one.

It is well known (see e.g. [5], p. 155) that if Y has the Radon Nikodym property then any Lipschitz function from a separable space into Y is Gâteaux differentiable outside a Gauss null set.

One can prove Theorem 3 without referring to the theory of metric trees. One needs only to prove that any connected subset of an SMT is contractible and use the proof of Proposition 5 to prove an analogous result for SMTs. The direct proof of the contractability of connected sets in an SMT is not hard but quite lengthy so we prefer not to include it here.

We end this section with an application of Theorem 1 to concepts studied by David and Semmes [6].

Theorem 4. Let X be a separable Banach space which has a subspace isomorphic to ℓ_1. Then there is a measure non collapsing mapping f from X into $\mathbf{R}^2$ whose range is a closed set with empty interior.

Proof. First note that if $\{B_n\}_{n=1}^{\infty}$ is a sequence of non overlapping open balls in $\mathbf{R}^2$ then $\mathbf{R}^2 \sim \bigcup_{n=1}^{\infty} B_n$ is π-quasiconvex. To prove this it is sufficient to check that for each $N = 1, 2, \dots,$ $\mathbf{R}^2 \sim$

$\bigcup_{n=1}^{N} B_n$ is π-quasiconvex. To prove this latter statement, draw the line segment between two points in $\mathbf{R}^2 \sim \bigcup_{n=1}^{N} B_n$. If the segment intersects B_n with $n \le N$, then it intersects $\overline{B_n}$ in a chord. Replace the chord by the arc on the boundary of B_n cut by the chord, which has length at most π times the length of the chord.

Now let $\{x_n\}_{n=1}^{\infty}$ be dense in the plane. We choose by recursion open balls B_n centered at x_n so that for each n, B_n does not intersect any B_i with $i < n$ and with the convention that B_n is the "empty ball" exactly when x_n is in the closure of $\bigcup_{i=1}^{n} B_i$. The balls are chosen so that for each N and each x in $\mathbf{R}^2 \sim \bigcup_{n=1}^{N} B_n$, the measure of $B(x,r) \sim \bigcup_{n=1}^{N} B_n$ is larger than $\frac{\pi}{2}r^2$. The first ball can be any ball centered at x_1. Assume that B_1, B_2, ... , B_n have been chosen to satisfy the stated conditions. In the non trivial case that $D := d(x_{n+1}, \bigcup_{n=1}^{N} B_n) > 0$, we need to choose $\varepsilon > 0$ sufficiently small so that if we set $B_{n+1} := B(x_{n+1}, \varepsilon)$, then $\bigcup_{n=1}^{N+1} B_n$ satisfies the desired conditions. Now balls with radius at most $D/4$ cannot intersect both $B(x_{n+1}, D/2)$ and $\bigcup_{n=1}^{N} B_n$. So as long as we choose $\varepsilon < D/2$, we only have to worry about balls with radius r at least $D/4$. For balls in that range it is evident that for every x in $\mathbf{R}^2 \sim \bigcup_{n=1}^{N} B_n$, the measure of $B(x,r) \sim \bigcup_{n=1}^{N} B_n$ is larger than $\frac{\pi}{2}r^2$ by an amount $\delta > 0$ independent of x and $r \ge D/2$. So we can take care of such balls by choosing $\varepsilon > 0$ so that $\pi\varepsilon^2$ is less than δ. This completes the inductive construction. It is clear that if x is in $\mathbf{R}^2 \sim \bigcup_{n=1}^{\infty} B_n$, then for every $r > 0$ the measure of $B(x,r) \sim \bigcup_{n=1}^{\infty} B_n$ is at least $(\pi/2)r^2$.

Now the set $\mathbf{R}^2 \sim \bigcup_{n=1}^{\infty} B_n$ is π-quasiconvex and complete, so by Theorem 2 there is a Lipschitz quotient mapping f from X onto $\mathbf{R}^2 \sim \bigcup_{n=1}^{\infty} B_n$. But then f, when considered as map into $\mathbf{R}^2$, is measure non collapsing, and of course the range of f is closed and has empty interior. $\square$

3 The nonseparable case

We prove in this section non separable versions of Theorem 1. First we discuss the notion of MT. The concept of MT is defined by describing how an MT is constructed. The construction is a transfinite version of the construction of an SMT. Let $\aleph$ be a cardinal number and identify $\aleph$ with the set of all ordinals whose cardinality is less than $\aleph$. Let I_1 be a closed interval or a closed ray and define $T_1 := I_1$. Having defined T_α for all $\alpha < \beta$, where $\beta < \aleph$, let I_β be a closed interval or a closed ray whose intersection with the completion $\tilde{T}_\beta$ of $\bigcup_{\alpha<\beta} T_\alpha$ is an end point, p_β, of I_β. Define T_β to be the ℓ_1 union of $\tilde{T}_\beta$ and I_β. The completion, T, of $\bigcup_{\alpha<\aleph} T_\alpha$ is an MT. If each I_α is a ray with end point p_α for $\alpha > 1$ and for every non empty open subset U of T the set $\{\alpha < \aleph : p_\alpha \in U\}$ has cardinality $\aleph$, then we call T an $\ell_1(\aleph)$ tree and say that $\{I_\alpha\}_{\alpha<\aleph}$, $\{T_\alpha\}_{\alpha<\aleph}$ describe an allowed construction of T. It is more or less clear that there are $\ell_1(\aleph)$ trees for each infinite cardinal $\aleph$.

In Section 4 we explain why a metric space is an MT if and only if it is a metric tree. From the general theory of metric trees we can then conclude that every MT is a 1-absolute Lipschitz retract.

Alternatively, this can be proved directly: Let $\{I_\alpha\}_{\alpha<\aleph}$, $\{T_\alpha\}_{\alpha<\aleph}$ describe an allowed construction of the tree T and show by transfinite induction, using an argument similar to that in the proof of Corollary 1, that each T_α is a 1-absolute Lipschitz retract.

By replacing the induction argument in Proposition 2 by a transfinite induction argument, we get a non separable version:

Proposition 6. Let T be an $\ell_1(\aleph)$ tree. Then every complete, metrically convex metric space whose density character is at most $\aleph$ is a 1-Lipschitz quotient of T.

There is also a non separable version of Proposition 3. However, since Pełczyński's theorem [15] does not have a non separable analogue, there are limited applications of the non separable extension of Proposition 3. As a replacement in the non separable setting, we have Proposition 7. Let $\Delta_\aleph$ be the group $\{-1,1\}^\aleph$ (where $\aleph$ is an uncountable cardinal) and let μ be the normalized Haar measure on the Borel subsets of $\Delta_\aleph$; i. e., μ is the product measure $\nu^\aleph$ where ν is normalized counting measure on $\{-1,1\}$. For each $\alpha<\aleph$, let r_α be the projection of $\Delta_\aleph$ onto its α-th coordinate. Then $\{r_\alpha\}_{\alpha<\aleph}$ is '$\ell_1(\aleph)$ over $L_\infty(\Delta_\aleph,\mu)$' in an obvious sense. Specifically, if X is a subspace of $L_\infty(\Delta_\aleph,\mu)$ with density character strictly less than $\aleph$, then (since every function in $L_\infty(\Delta_\aleph,\mu)$ depends on only countably many coordinates) there is a subset Δ_0 of $\Delta_\aleph$ of cardinality strictly less than $\aleph$ so that each function in x depends only on the coordinates in Δ_0. It follows that if α is in the (non empty) set $\Delta_\aleph \sim \Delta_0$ and x is in X, then $\|x+tr_\alpha\| = \|x\|+|t|$ for all real t. Thus transfinite induction and the same considerations used in the proof of Proposition 3 yield:

Proposition 7. Let X be a subspace of $L_\infty(\Delta_\aleph,\mu)$ (where $\aleph$ is an uncountable cardinal) with density character $\aleph$ and assume that $\{r_\alpha\}_{\alpha<\aleph}\subset X$. Then every MT of density character at most $\aleph$ is a 1-Lipschitz quotient of X.

If $\aleph^{\aleph_0}=\aleph$, then the density character of $L_\infty(\Delta_\aleph,\mu)$ itself is $\aleph$ and hence $L_\infty(\Delta_\aleph,\mu)$ is isometrically isomorphic to a quotient space of $\ell_1(\aleph)$. Now if X is a Banach space which contains a subspace isomorphic to $\ell_1(\aleph)$ then for each $\varepsilon>0$, X contains a subspace $1+\varepsilon$-isomorphic to $\ell_1(\aleph)$ by the non separable version of a theorem of James [11] (this requires only that $\aleph$ be infinite). Since $L_\infty(\Delta_\aleph,\mu)$ is injective, it follows that $L_\infty(\Delta_\aleph,\mu)$ is $1+\varepsilon$-isomorphic to a linear quotient of X for every $\varepsilon>0$. Therefore, from Propositions 7 and 6 we get our first non separable version of Theorem 1.

Theorem 5. Let X be a Banach space which contains a subspace isomorphic to $\ell_1(\aleph)$ and let $\varepsilon>0$. Assume that $\aleph=\aleph^{\aleph_0}$. Then every complete, metrically convex metric space of density character at most $\aleph$ is a $(1+\varepsilon)$-Lipschitz quotient of X.

In case $\aleph<\aleph^{\aleph_0}$ we can prove a weaker version of Theorem 5:

Theorem 6. Let X be a Banach space of density character $\aleph$ which contains a subspace isomorphic to $\ell_1(\aleph)$ and let $\varepsilon>0$. Then every complete, metrically convex metric space of density character at most $\aleph$ is a $(1+\varepsilon)$-Lipschitz quotient of X.

Proof. Let $\{x_\alpha\}_{\alpha<\aleph}$ be unit vectors in X which are $1+\varepsilon$-equivalent to the unit vector basis of $\ell_1(\aleph)$. Since $L_\infty(\Delta_\aleph,\mu)$ is injective, the mapping $x_\alpha \to r_\alpha$ extends to a linear operator L of norm at most $1+\varepsilon$ from X into $L_\infty(\Delta_\aleph,\mu)$. The range of this linear operator has density character at most $\aleph$ and L is an isomorphism from the closed space of $\{x_\alpha\}_{\alpha<\aleph}$ onto the closed span of $\{r_\alpha\}_{\alpha<\aleph}$, so the same considerations as in Proposition 7 yield that every $\ell_1(\aleph)$ tree is a $1+\varepsilon$-Lipschitz quotient of X. Now apply Proposition 6. $\square$

Remark. 1. It is well known that ℓ_∞ has a subspace isometric to $\ell_1(2^{\aleph_0})$. Indeed, $\ell_\infty \supset C(0,1)^*$ (like the dual of any separable space) and the atomic measures in $C(0,1)^*$ are isometric to $\ell_1(2^{\aleph_0})$. Hence by Theorem 5 any complete metrically convex metric space of density character $\leq 2^{\aleph_0}$ is an image of ℓ_∞ by a map whose Lipschitz and co-Lipschitz constants are 1. In particular there is a Lipschitz quotient map from ℓ_∞ onto c_0 (the question of the validity of this result was raised in several places, see e.g. [5], p. 279).

2. It is a formal consequence of the preceding remark that there is retraction from ℓ_∞ onto c_0 that is a Lipschitz quotient map. To see this, let P be a Lipschitz retraction from ℓ_∞ onto c_0; for a proof of the existence of such a mapping see [5], p. 14. Let f be a Lipschitz quotient mapping from ℓ_∞ onto c_0 with $f(0)=0$. Define a function g on $\ell_\infty \oplus \ell_\infty$ by $g(x,y) := (P(x)+f(y),0)$. The map g is easily seen to be a retraction and Lipschitz quotient map from $\ell_\infty \oplus \ell_\infty (\equiv \ell_\infty)$ onto its subspace $c_0 \oplus \{0\}$. By the subspace homogeneity of ℓ_∞ (see [14]), there is also a retraction from ℓ_∞ onto c_0 that is a Lipschitz quotient map.

3. In the literature there are several non-obvious criteria for deciding for which uncountable cardinal $\aleph$ does $\ell_1(\aleph)$ embed isomorphically into a given Banach space X. See [2] and its references. These criteria depend often on the axioms one uses in set theory. From these criteria and Theorem 5 one may deduce some results on the existence of non-linear quotient maps between suitable spaces. We do not enter into this here.

4 Metric trees

Recall that a complete metric space X is a metric tree (also called an $\mathbf{R}$-tree) provided that for any two points x and y in X, there is a unique arc joining x to y, and this arc is a geodesic arc. See [10] for an overview of the study of metric trees and [9] for statements and sometimes proofs of the most basic properties of metric trees. References to original sources are contained in these two papers.

One particularly elegant characterization of metric trees and subsets thereof is given by the four point condition. A metric space X is said to satisfy the *four point condition* provided that for each set x, y, w, z of four points in X, two of the three numbers $d(x,y)+d(w,z)$, $d(x,w)+d(y,z)$, $d(x,z)+d(y,w)$ are the same, and that number is at least as large as the third. The characterizations are that

1. A metric space embeds isometrically into some metric tree if and only if it satisfies the four point condition.

2. A metric space is a metric tree if and only if it is complete, connected, and satisfies the four point condition.

This characterization of metric trees makes Proposition 8 very easy to prove.

Proposition 8. 1. A metric space T is an SMT if and only if T is a separable metric tree.

2. A metric space T is an MT if and only if T is a metric tree.

Proof. Assume that T is an SMT and suppose that $\{I_n\}_{n=1}^{\infty}$, $\{T_n\}_{n=1}^{\infty}$ describe an allowed construction of T. It is easy to check by induction that each T_n satisfies the four point condition and hence so does $\bigcup_{n=1}^{\infty} T_n$, whence so does T because the four point condition obviously passes to completions. Furthermore, $\bigcup_{n=1}^{\infty} T_n$ is arcwise connected, hence its completion T is connected. Thus T is a metric tree by the characterization (2) above. Of course, an SMT is separable, so this completes the proof of the "only if" part of (1). To go the other way, assume that T is a separable metric tree and let $\{x_n\}_{n=1}^{\infty}$ be a countable dense subset of T. We need to build subsets $\{I_n\}_{n=1}^{\infty}$, $\{T_n\}_{n=1}^{\infty}$ of T which describe an allowed construction of T. Let $I_1 = T_1 = \{x_1\}$. Having defined T_n, let $\tilde{I}_{n+1}$ be the geodesic arc from x_1 to x_n, let $\tilde{x}_n$ be the last point on this arc which is in T_n (set $\tilde{x}_n = x_n$ if x_n is in T_n), let I_{n+1} be the geodesic arc from $\tilde{x}_n$ to x_n, and set $T_{n+1} = T_n \cup I_{n+1}$. It is easy to check that this works.

The second statement is proved similarly. ☐

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W.B. Johnson
Department of Mathematics
Texas A&M University
College Station, TX 77843 U.S.A.
johnson@math.tamu.edu

J. Lindenstrauss
Institute of Mathematics
Hebrew University of Jerusalem
Jerusalem, Israel
joram@math.huji.ac.il

D. Preiss
Department of Mathematics
University College London
London, Great Britain
dp@math.ucl.ac.uk

G. Schechtman
Department of Mathematics
Weizmann Institute of Science
Rehovot, Israel
gideon@wisdom.weizmann.ac.il