ON COMPLEMENTED VERSIONS OF JAMES'S DISTORTION THEOREMS

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Abstract. Examples are given to show that two natural questions asked in [5] about complemented versions of James's distortion theorems have negative answers.

1. Introduction

The James's distortion theorem for ℓ_1 (respectively, for c_0) states that whenever a Banach space contains a subspace isomorphic to ℓ_1 (respectively, c_0) then the Banach space contains subspaces that are almost isometric to ℓ_1 (respectively, c_0). In [5], complemented versions of James's distortion theorems were considered in the following senses:

Theorem 1. *Let X be a Banach space whose dual unit ball is weak*-sequentially compact and $\varepsilon > 0$. If X contains a subspace isomorphic to c_0, then there exists a subspace Z of X and a projection P from X onto Z such that Z is $(1+\varepsilon)$-isometric to c_0 and $\|P\| \leq 1+\varepsilon$. Moreover, if X contains a subspace isometric to c_0, then there exists a subspace Z of X and a projection P from X onto Z such that Z is isometric to c_0 and $\|P\| = 1$.*

Theorem 2. *Let X be a Banach space which contains a complemented subspace isomorphic to ℓ_1 and $\varepsilon > 0$. Then there exists a subspace Y of X and a projection P from X onto Y such that Y is $(1+\varepsilon)$-isometric to ℓ_1 and $\|P\| \leq 1+\varepsilon$.*

While Theorem 2 can be viewed as the exact analogue of the James's distortion theorem for complemented copies of ℓ_1, Theorem 1 may be interpreted as combination of the James's distortion theorem for c_0 and the classical Sobczyk Theorem. These led to the following natural questions (see [5, Question 1, Question 2]):

Question 1. *If a Banach space X contains a complemented copy of c_0 and if $\varepsilon > 0$, does there exist a subspace Z of X and a projection P from X onto Z such that Z is $(1+\varepsilon)$-isometric to c_0 and $\|P\| \leq 1+\varepsilon$?*

1991 *Mathematics Subject Classification.* [2000]Primary: 46B20, 46B25. Secondary: 46B03, 46B45.

Key words and phrases. ℓ_1, c_0, renorming, James's distortion theorems.

†Research partially supported by NSF DMS–0503688.

‡Research partially supported by NSF DMS–0456781.

January 31, 2007.

Question 2. *If a Banach space X contains a complemented subspace isometric to ℓ_1, does there exists a subspace Z of X and a projection P from X onto Z such that Z is isometric to ℓ^1 and $\|P\| = 1$?*

The aim of this note is to provide examples showing that, as expected, the answers to both questions are negative.

2. The examples

2.1. **The c_0-case.** In this subsection, we exhibit a Banach space X with the property that any complemented subspace of X that is almost isometric to c_0 has large projection constant, thus answering Question 1 negatively. The space X is a renorming of $\ell_\infty \oplus_\infty c_0$.

We denote by $\|\cdot\|$ the usual norm on $\ell_\infty \oplus_\infty c_0$. In order to define the new norm on X, let $J : \ell_1 \to \ell_\infty$ be an isometric embedding of ℓ_1 into ℓ_∞ and $Q : \ell_1 \to c_0$ be a quotient map. For $\delta > 0$, a norm $|||\cdot|||$ on X is defined by fixing its unit ball:

$$B_{(X,|||\cdot|||)} := \big\{(Jf, Qf); f \in \ell_1, \|f\|_1 \leq 1\big\} + \delta B_{(X,\|\cdot\|)}.$$

It is clear that $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms on X and X contains a complemented subspace isomorphic to c_0.

Proposition 3. *Let $\varepsilon > 0$ and Z be a subspace of $(X, |||\cdot|||)$ that is $(1+\varepsilon)$-isometric to c_0 and is complemented in X. If P is a projection from X onto Z then*

$$|||P||| \geq \frac{1+\delta}{4\delta(1+\varepsilon)^3}.$$

Proof. Throughout, we also denote by $|||\cdot|||$ the corresponding dual norm on X^*. Let $(V_n)_{n\geq 1}$ be a basic sequence $(1+\varepsilon)$-equivalent to the unit vector basis of c_0 and whose closed linear span is Z. Let P be a projection from X onto Z. Then P is of the form

$$P = \sum_{n=1}^{\infty} V_n^* \otimes V_n$$

where $(V_n^*)_{n\geq 1}$ is a weak*-null sequence in X^* and the sum can be taken with respect to the strong operator topology. Observe that $X^* = (\ell_\infty)^* \oplus_1 \ell_1$ isomorphically and thus for every $n \geq 1$, $V_n^* = (x_n^*, a_n^*)$ where $(x_n^*)_{n\geq 1}$ (respectively, $(a_n^*)_{n\geq 1}$) is a weak*-null sequence in $(\ell_\infty)^*$ (respectively, ℓ_1). Since weak*-null sequences are weakly-null in $(\ell_\infty)^*$ (see for instance [1, Theorem 15, p.103]), we have

$$\text{weak} - \lim_{n\to\infty} x_n^* = 0.$$

There exists a convex block $(y_n^*)_{n\geq 1}$ of $(x_n^*)_{n\geq 1}$ with

$$\lim_{n\to\infty} \|y_n^*\| = 0. \tag{2.1.1}$$

There exists a strictly increasing sequence of integers $(k_n)_{n\geq 0}$ and positive scalars $\alpha_j^{(n)}$, where $k_{n-1}+1 \leq j \leq k_n$, $\sum_{j=k_{n-1}+1}^{k_n} \alpha_j^{(n)} = 1$ for $n \geq 1$, and

$$y_n^* = \sum_{j=k_{n-1}+1}^{k_n} \alpha_j^{(n)} x_j^*.$$

For $n \geq 1$, consider the corresponding block sequences:

$$\begin{aligned} W_n &= \sum_{j=k_{n-1}+1}^{k_n} V_j \\ W_n^* &= \sum_{j=k_{n-1}+1}^{k_n} \alpha_j^{(n)} V_j^* \end{aligned} \tag{2.1.2}$$

Then $(W_n)_{n\geq 1}$ is equivalent to the unit vector basis of c_0. Moreover, for every $n, k \geq 1$, $\langle W_n^*, W_k \rangle = \delta_n^k$, $W_n^* = (y_n^*, b_n^*)$ where $b_n^* = \sum_{j=k_{n-1}+1}^{k_n} \alpha_j^{(n)} a_j^*$, and $|||W_n||| \leq 1+\varepsilon$. The latter implies that for $n \geq 1$, W_n can be decomposed as

$$W_n = \big(Jf_n, Qf_n\big) + \big(x_n, a_n\big) \tag{2.1.3}$$

with

(i) $f_n \in \ell_1$ satisfying $\|f_n\|_1 \leq 1+\varepsilon$;
(ii) $\|(x_n, a_n)\| \leq \delta(1+\varepsilon)$.

Since $(f_n)_{n\geq 1}$ is a bounded sequence in ℓ_1, we may assume (by passing to a subsequence if necessary) that for every $n \geq 1$,

$$f_n = f_0 + g_n + h_n$$

where

(a) $\lim_{n\to\infty} \|f_n\|_1$ exists;
(b) $\text{weak}^* - \lim_{n\to\infty} f_n = f_0$;
(c) $(g_n)_{n\geq 1}$ is a disjointly supported sequence in ℓ_1;
(d) $\lim_{n\to\infty} \|h_n\|_1 = 0$.

We claim that

$$\lim_{n\to\infty} \|f_n\|_1 \leq 3\delta(1+\varepsilon). \tag{2.1.4}$$

To see this claim, let $N \geq 1$, then

$$\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N}\|f_n\|_1 &\leq \big\|f_0\big\|_1 + \frac{1}{N}\sum_{n=1}^{N}\big\|g_n\big\|_1 + \frac{1}{N}\sum_{n=1}^{N}\big\|h_n\big\|_1\\
&= \big\|f_0\big\|_1 + \frac{1}{N}\Big\|\sum_{n=1}^{N}g_n\Big\|_1 + \frac{1}{N}\sum_{n=1}^{N}\|h_n\|_1\\
&\leq 2\|f_0\|_1 + \frac{1}{N}\Big\|\sum_{n=1}^{N}f_n\Big\|_1 + \frac{2}{N}\sum_{n=1}^{N}\big\|h_n\big\|_1.
\end{aligned}$$

Note that $\|f_0\|_1 \leq \underline{\lim}_{N\to\infty}N^{-1}\big\|\sum_{n=1}^{N}f_n\big\|_1$ and $\lim_{N\to\infty}N^{-1}\sum_{n=1}^{N}\|h_n\|_1 = 0$. We deduce that

$$\lim_{n\to\infty}\big\|f_n\big\|_1 \leq 3\,\overline{\lim}_{N\to\infty}\frac{1}{N}\Big\|\sum_{n=1}^{N}f_n\Big\|_1.$$

We observe that

$$\begin{aligned}
\frac{1}{N}\Big\|\sum_{n=1}^{N}f_n\Big\|_1 &= \frac{1}{N}\Big\|\sum_{n=1}^{N}Jf_n\Big\|_\infty\\
&\leq \frac{1}{N}\Big\|\sum_{n=1}^{N}\big(Jf_n, Qf_n\big)\Big\|\\
&\leq \frac{1}{N}\Big\|\sum_{n=1}^{N}W_n\Big\| + \frac{1}{N}\Big\|\sum_{n=1}^{N}\big(x_n, a_n\big)\Big\|\\
&\leq \frac{1}{N}\Big\|\sum_{n=1}^{N}W_n\Big\| + \delta(1+\varepsilon).
\end{aligned}$$

Since (W_n) is equivalent to the unit vector basis of c_0, we have $\lim\limits_{N\to\infty}N^{-1}\Big\|\sum\limits_{n=1}^{N}W_n\Big\| = 0$. Combining all the above estimates, we get inequality (2.1.4).

We now show that if Π is the projection from X onto the closed linear span of $(W_n)_{n\geq 1}$ defined by $\Pi = \sum_{n=1}^{\infty}W_n^*\otimes W_n$, then

$$|||\Pi||| \geq \frac{1+\delta}{4\delta(1+\varepsilon)^2}. \tag{2.1.5}$$

To see this, we first observe that for every $y^* \in \ell_1$,

$$|||(0,y^*)||| = \|y^*\|_1(1+\delta). \tag{2.1.6}$$

For every $n \geq 1$, we have

$$\begin{aligned} 1 &= \langle W_n^*, W_n \rangle \\ &= \langle (y_n^*, b_n^*), (Jf_n + x_n, Qf_n + a_n) \rangle \\ &= \langle y_n^*, Jf_n + x_n \rangle + \langle b_n^*, Qf_n + a_n \rangle \\ &\leq (\|y_n^*\| + \|b_n^*\|)(\|f_n\|_1 + \delta(1+\varepsilon)). \end{aligned}$$

Taking limits as $n \to \infty$, we deduce that

$$\underline{\lim}_{n\to\infty} \|b_n^*\|_1 \geq \frac{1}{4\delta(1+\varepsilon)}. \tag{2.1.7}$$

We can estimate $|||\Pi|||$ as follows:

$$\begin{aligned} |||\Pi||| &\geq \sup_{n\geq 1} \frac{|||W_n^*|||}{1+\varepsilon} \\ &= \sup_{n\geq 1} \frac{|||(y_n^*, b_n^*)|||}{1+\varepsilon} \\ &\geq \underline{\lim}_{n\to\infty} \frac{|||(y_n^*, b_n^*)|||}{1+\varepsilon} \\ &= \underline{\lim}_{n\to\infty} \frac{|||(0, b_n^*)|||}{1+\varepsilon}. \end{aligned}$$

Thus (2.1.5) follows by combining (2.1.7) and (2.1.6). We conclude the proof by observing that $|||\Pi||| \leq (1+\varepsilon)|||P|||$. □

2.2. **The ℓ_1-case.** Now we provide an example showing that Theorem 2 does not extend to the isometric case. In particular, the answer to Question 2 is negative.

First, recall that a norm $\|\cdot\|$ on a Banach space E is said to be *strictly convex* if $\mathrm{Ext}(B_E) = S_E$. This is equivalent to the following property (see for instance [6, p. 246]):

$$\text{If } x, y \in S_E \text{ satisfy } \|x+y\| = 2, \text{ then } x = y. \tag{2.2.1}$$

It is clear from (2.2.1) that if $\|\cdot\|$ is strictly convex then $(E, \|\cdot\|)$ does not contain any ℓ_∞^2 (the two dimensional ℓ_∞) isometrically. Indeed, if $e_1 = (1,0)$ and $e_2 = (0,1)$ are the unit vector basis of ℓ_∞^2 then $x = e_1$ and $y = e_1 + e_2$ fail to satisfy (2.2.1).

Define a norm $|\cdot|$ on ℓ_1 that is equivalent to the usual norm and such that its dual norm $|\cdot|^*$ is strictly convex. Such a dual norm on ℓ_∞ can be taken by setting:

$$|(a_i)|^* := \Big(\|(a_i)\|_\infty^2 + \sum_{i=1}^{\infty} 2^{-i}|a_i|^2\Big)^{1/2}.$$

Details on the existence of the norm $|\cdot|$ can be found in [6, pp. 241–254]. We define a Banach space Y by setting:

$$Y := (C[0,1], \|\cdot\|_\infty) \oplus_\infty (\ell_1, |\cdot|).$$

Proposition 4. *The Banach space Y contains a complemented subspace that is isometric to ℓ_1 but ℓ_1 is not isometric to a quotient of Y.*

Let (e_n) be the unit vector basis of ℓ_1 and for $n \geq 1$, set $v_n := e_n/|e_n|$. Fix a sequence $(f_n)_{n\geq 1}$ in $C[0,1]$ that is isometrically equivalent to the unit vector basis of ℓ_1 and for $n \geq 1$, define $U_n := (f_n, v_n) \in Y$. We claim that $(U_n)_{n\geq 1}$ is isometrically equivalent to the unit vector basis of ℓ_1 and its closed linear span is complemented. In fact, for any finite sequence $(a_n)_{n\geq 1}$ of scalars,

$$\begin{aligned}\sum_{n\geq 1} |a_i| &= \Big\| \sum_{n\geq 1} a_n f_n \Big\|_\infty \\ &\leq \Big\| \sum_{n\geq 1} a_n U_n \Big\|_Y \\ &\leq \sum_{n\geq 1} |a_i|,\end{aligned}$$

therefore, $\sum_{n\geq 1} |a_i| = \big\| \sum_{n\geq 1} a_n U_n \big\|_Y$. Moreover, if we denote by Z the closed linear span of $(U_n)_{n\geq 1}$ then Z is a complemented subspace of Y. Indeed, let $T : (\ell_1, |\cdot|) \to Y$ be defined by setting $T(v_n) = U_n$ for all $n \geq 1$ and Π be the second projection from Y onto $(\ell_1, |\cdot|)$ then $T \circ \Pi$ is a projection from Y onto Z.

The fact that ℓ_1 is not isometric to a quotient of Y follows from the next lemma, which we assume is well known.

Lemma 5. *Let E and F be Banach spaces and $T : c_0 \to E \oplus_1 F$ be an isometry. Then there exists $c_j \geq 0$, $j = 1, 2$ with:*

(a) $c_1 + c_2 = 1$;
(b) *if* $T = (T_1, T_2)$ *then* $\|T_j(e)\| = c_j \|e\|$ *for* $j = 1, 2$ *and all* $e \in c_0$.

In particular, if $E \oplus_1 F$ contains an isometric copy of c_0 then either E or F contains an isometric copy of c_0.

Proof. Denote by c_{00} the space of finitely supported sequences of scalars and let $(e_n)_{n\geq 1}$ be the unit vector basis of c_0. Write $T = (T_1, T_2)$ with $T_1 : c_0 \to E$ and $T_2 : c_0 \to F$. We shall verify that for every $x \in c_{00}$ with $\|x\| = 1$,

$$\|T_j(x)\| = \|T_j(e_1)\| \text{ for } j = 1, 2.$$

To see this, we will show first that if x and y are disjointly supported unit vectors then

$$\|T_j(x)\| = \|T_j(y)\| \text{ for } j = 1, 2. \tag{2.2.2}$$

Write $2x = (x - y) + (x + y)$. Then

$$\begin{aligned} 2 = \|T(2x)\| &= \|T_1(2x)\| + \|T_2(2x)\| \\ &\leq (\|T_1(x-y)\| + \|T_1(x+y)\|) + (\|T_2(x-y)\| + \|T_2(x+y)\|) \\ &= \|T(x-y)\| + \|T(x+y)\| = 2. \end{aligned}$$

For $j = 1, 2$, set $a_j = \|T_j(x-y)\| + \|T_j(x+y)\| - 2\|T_j(x)\|$. Then (a_1, a_2) is a positive element of ℓ_1^2 whose norm is equal to zero so

$$2\|T_j(x)\| = \|T_j(x-y)\| + \|T_j(x+y)\|, \quad j = 1, 2.$$

By reversing the role of x and y, we get (2.2.2).

Now, let $x \in c_{00}$ with $\|x\| = 1$. Choose, $n > 1$ so that e_n and x are disjointly supported. From (2.2.2),

$$\|T_j(x)\| = \|T_j(e_n)\| = \|T_j(e_1)\|, \quad j = 1, 2.$$

Setting $c_j = \|T_j(e_1)\|$ for $j = 1, 2$ proves the lemma. □

End of the proof of Proposition 4. If ℓ_1 is isometric to a quotient of Y then the dual space $Y^* = C[0,1]^* \oplus_1 (\ell_\infty, |\cdot|^*)$ contains an isometric copy of $\ell_\infty = \ell_1^*$ and hence of c_0. But since $|\cdot|^*$ is strictly convex and $C[0,1]^*$ is a L_1-space, this is in contradiction with Lemma 5 and thus completes the proof. □

3. Concluding remarks

The notion of asymptotically isometric copies of ℓ_1 (respectively, c_0) is closely related to James's distortion theorems. We recall that a Banach space E is said to contain an asymptotically isometric copy of ℓ_1 (respectively, c_0) if there exist a null sequence $(\varepsilon_n)_{n\geq 1}$ in $(0,1)$ and a sequence $(x_n)_{n\geq 1}$ in E such that for all finite sequence $(t_n)_n$ of scalars:

$$\sum_n (1-\varepsilon_n)|t_n| \leq \Big\| \sum_n t_n x_n \Big\| \leq \sum_n |t_n|,$$

respectively,

$$\sup_n (1-\varepsilon_n)|t_n| \leq \Big\| \sum_n t_n x_n \Big\| \leq \sup_n |t_n|,$$

The norm introduced in the definition of the Banach space Y can be used to provide examples confirming the optimality of the James's distortion theorems. We refer to [4] for earlier examples.

Proposition 6. *(a) $(\ell_1, |\cdot|)$ does not contain any subspace asymptotically isometric to ℓ_1.*

(b) $(\ell_\infty, |\cdot|^*)$ does not contain any subspace asymptotically isometric to c_0.

Proof. These statements follow from the norm $|\cdot|^*$ being strictly convex and some known results. First, containing an asymptotically isometric copy of c_0 and containing an isometric copy of c_0 is equivalent in a dual space ([3]). Second, according to [2], a Banach space contains an asymptotically isometric copy of ℓ_1 if and only if its dual contains an isometric copy of $L_1[0,1]$. But since $(\ell_\infty, |\cdot|^*)$ is strictly convex, it does not contain any isometric copy of ℓ_1 and therefore it can not contain any isometric copy of $L_1[0,1]$. Part (a) was already observed in [2] where an explicit formula for $|\cdot|$ was given (see [2, Corollary 12]). □

The proof of Proposition 4 yields that the last part of the conclusion of Proposition 4 can be strengthened to "c_0 is not isometric to a subspace of Y^*". However, in [3] Dowling proved that ℓ_1 is a quotient of X if and only if c_0 embeds isometrically into X^*, so the more natural statement involving ℓ_1 is only formally weaker.

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