ℓ_p $(p > 2)$ Does Not Coarsely Embed Into a Hilbert Space

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A (not necessarily continuous) map f between two metric spaces (X, d) and (Y, δ) is called a *coarse embedding* (see [G, 7.G]) if there exist two non-decreasing functions $\varphi_1 : [0, \infty) \to [0, \infty)$ and $\varphi_2 : [0, \infty) \to [0, \infty)$ such that

1. $\varphi_1(d(x, y)) \leq \delta(f(x), f(y)) \leq \varphi_2(d(x, y))$
2. $\varphi_1(t) \to \infty$ as $t \to \infty$.

Nowak [N], improving a theorem due to A. N. Dranishnikov, G. Gong, V. Lafforgue, and G. Yu [DGLY], gave a characterization of coarse embeddability of general metric spaces into a Hilbert space using a result of Schoenberg on negative definite kernels. He used this characterization to show that the spaces $L_p(\mu)$ coarsely embed into a Hilbert space for $p < 2$. In this article, we show that ℓ_p does not coarsely embed into a Hilbert space when $p > 2$. It was already proved in [DGLY] that the Lipschitz universal space c_0 (see [A]) does not coarsely embed into a Hilbert space.

In its full generality, the statement of our result is as follows:

Theorem 1 *Suppose that a Banach space X has a normalized symmetric basis $(e_n)_n$ and that $\liminf_{n\to\infty} n^{-\frac{1}{2}} \| \sum_{i=1}^{n} e_i \| = 0$. Then X does not coarsely embed into a Hilbert space.*

In [Y] Yu proved that a discrete metric space with bounded geometry must satisfy the coarse geometric Novikov conjecture if it coarsely embeds

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into a Hilbert space, and in [KY] G. Kasparov and Yu proved that to get the same conclusion it is sufficient that the metric space coarsely embeds into a uniformly convex Banach space. Our theorem suggests that the result of [KY] cannot be deduced from the earlier theorem in [Y], but as yet there is no example of a discrete metric space with bounded geometry which coarsely embeds into ℓ_p for some $2 < p < \infty$ but not into ℓ_2. (The reader should be warned that what we called a "coarse embedding" is called a "uniform embedding" is many places, including [DGLY], [KY], and [Y]. Following [N], we use the term coarse embedding to avoid confusion with the closely related notion of uniform embedding as it is used in non linear Banach space theory [BL]; i.e., a bi-uniformly continuous mapping.)

Besides Schoenberg's classical work [S] on positive definite functions, an important tool for proving the theorem is Theorem 5.2 in [AMM], which asserts that the hypothesis on X in the theorem implies that every symmetric continuous positive definite function on X is constant. We present the proof in five steps.

STEP 0: REDUCING TO THE α-HÖLDER CASE

Let $f : X \to H$ be a coarse embedding satisfying

1. $\varphi_1(\|x - y\|) \le \|f(x) - f(y)\| \le \varphi_2(\|x - y\|)$
2. $\varphi_1(t) \to \infty$ as $t \to \infty$.

Our first claim is that we do not lose generality by assuming that $\varphi_2(t) = t^\alpha$ with $0 < \alpha < \frac{1}{2}$.

To prove this claim, note first that $(x, y) \mapsto \|f(x) - f(y)\|^2$ is a negative definite kernel on X. This can be seen by direct computations (see [N, Proposition 3.1]). We refer the reader to [BL, Chapter 8] or [N, Section 2] for the definitions of negative definite kernels and negative definite functions.

So, ([N, Lemma 4.2]) for any $0 < \alpha < 1$, the kernel $N(x, y) = \|f(x) - f(y)\|^{2\alpha}$ is also negative definite and satisfies $N(x, x) = 0$ (such a negative definite kernel is called normalized).

As a result, a theorem of Schoenberg ([S] and [BL, Chapter 8]) allows us to find a Hilbert space H_α and a function $f_\alpha : X \to H_\alpha$ such that $N(x, y) = \|f_\alpha(x) - f_\alpha(y)\|^2$.

On the other hand, since X, being a normed space, is (metrically) convex, the original function $f : X \to H$ is Lipschitz for large distances (see e.g. [BL, proof of Proposition 1.11]). Consequently, without loss of generality, we can assume by rescaling that we have the following for $\|x - y\| \geq 1$:

$$\|f(x) - f(y)\|_{H} \leq \|x - y\|$$

and

$$(\varphi_1(\|x - y\|))^{\alpha} \leq \|f_\alpha(x) - f_\alpha(y)\|_{H_\alpha} \leq \|x - y\|^{\alpha}$$

Now, let $\mathcal{N}$ be a 1-net in X (i.e. $\mathcal{N}$ is a maximal 1-separated subset of X). The restriction of f_α to $\mathcal{N}$ is α-Hölder, so if $0 < \alpha < \frac{1}{2}$, then we can extend f_α to an α-Hölder map $\widetilde{f_\alpha}$ defined on the whole of X (see [WW], last statement of Theorem 19.1):

$$\widetilde{f_\alpha} : X \to H_\alpha$$

$$\forall x \in \mathcal{N}, \quad \widetilde{f_\alpha}(x) = f_\alpha(x)$$

and

$$\forall x, y \in X, \quad \|f_\alpha(x) - f_\alpha(y)\|_{H_\alpha} \leq \|x - y\|^{\alpha}.$$

Now, write $\widetilde{\varphi}_1(t) = \inf\{\|\widetilde{f_\alpha}(x) - \widetilde{f_\alpha}(y)\|, \quad \|x - y\| \geq t\}$, and observe that $\widetilde{\varphi}_1(t) \to \infty$ as $t \to \infty$.

This finishes the proof of our reduction to the case where f is α-Hölder and thus uniformly continuous. So from now on we will assume that our coarse embedding is a map $f : X \to H$ satisfying the following for all $x, y \in X$:

$$\varphi_1(\|x - y\|) \leq \|f(x) - f(y)\| \leq \|x - y\|^{\alpha}$$

where $\varphi_1(t) \to \infty$ as $t \to \infty$.

STEP 1

Set $N(x,y) = \|f(x) - f(y)\|^2$. Then N is a normalized (i.e. $N(x,x) = 0$) negative definite kernel on X, (see [N, Proposition 3.1]). Now if we write $\phi_1(t) = (\varphi_1(t))^2$ and $\phi_2(t) = t^{2\alpha}$, then N satisfies:

$$\begin{cases} \phi_1(\|x-y\|) \leq N(x,y) \leq \phi_2(\|x-y\|), \\ \phi_1(t) \to \infty \text{ as } t \to \infty. \end{cases}$$

STEP 2

The argument in this paragraph comes from [AMM, Lemma 3.5.].

Let μ be an invariant mean on the bounded functions on X (see e.g. [BL] for the definition of invariant means). Define:

$$g(x) = \int_X N(y+x, y)\, d\mu(y)$$

Then we have the following for g:

- g is well-defined because the map $y \mapsto N(y+x, y)$ is bounded for each $x \in X$
- $g(0) = \displaystyle\int_X N(y,y)\, d\mu(y) = 0$
- For scalars $(c_i)_{1 \leq i \leq n}$ satisfying $\displaystyle\sum_{i=1}^n c_i = 0$, we have:

$$
\begin{aligned}
\sum_{i,j=1}^{n} c_i c_j g(x_i - x_j) &= \sum_{i,j} c_i c_j \int_X N(y + x_i - x_j, y)\, d\mu(y) \\
&= \sum_{i,j=1}^{n} c_i c_j \int_X N(y + x_i, y + x_j)\, d\mu(y) \\
&= \int_X \left(\sum_{i,j=1}^{n} c_i c_j N(y + x_i, y + x_j) \right) d\mu(y) \\
&= \int_X (\leq 0)\, d\mu(y) \\
&\leq 0
\end{aligned}
$$

This is because μ is translation invariant, and N is negative definite. This shows that g is a negative definite function on X.

- Finally, since $\int_X d\mu(y) = 1$, we have:

$$\phi_1(\|x\|) \leq g(x) \leq \phi_2(\|x\|).$$

In summary, we have found a negative definite function g on X which satisfies $g(0) = 0$ and $\phi_1(\|x\|) \leq g(x) \leq \phi_2(\|x\|)$, where $\phi_1(t) \to \infty$ as $t \to \infty$.

STEP 3

Let $(e_n)_n$ be the normalized symmetric basis for X. This means that for any choice of signs $(\theta_n)_n \in \{-1, +1\}$ and any choice of permutation $\sigma : \mathbb{N} \to \mathbb{N}$,

$$\|\sum_n \theta_n a_n e_{\sigma(n)}\|_X = \|\sum_n a_n e_n\|_X.$$

The purpose of this paragraph is to show that the negative definite function g we found in the previous paragraph can be chosen to be symmetric, i.e. to satisfy for any choice of signs $(\theta_n)_n \in \{-1,+1\}$ and any choice of permutation $\sigma : \mathbb{N} \to \mathbb{N}$ the equality:

$$g\left(\sum_n \theta_n a_n e_{\sigma(n)}\right) = g\left(\sum_n a_n e_n\right).$$

For $x = \sum_{n=1}^{\infty} x_n e_n \in X$, define $g_m(x)$ to be the average of $g\left(\sum_{n=1}^{\infty} \theta_n x_n e_{\sigma(n)}\right)$ over all choices of signs θ and permutations σ with the restrictions that $\theta_n = 1$ for $n > m$ and $\sigma(n) = n$ for $n > m$.

It follows that for all such θ, σ, and for all $x = \sum_{n=1}^{\infty} x_n e_n \in X$,

$$g_m\left(\sum_{n=1}^{\infty} \theta_n x_n e_{\sigma(n)}\right) = g_m\left(\sum_{n=1}^{\infty} x_n e_n\right).$$

Moreover, we also have

$$\phi_1(\|x\|) \leq g_m(x) \leq \phi_2(\|x\|).$$

Next we show that the sequence $(g_m)_m$ is equicontinuous. To check this, let us first check the continuity of g:

$$
\begin{aligned}
\|g(a)-g(b)\| &\leq \int_X |N(y+a,y)-N(y+b,y)| \, d\mu(y) \\
&= \int_X \left| \|f(y+a)-f(y)\|^2 - \|f(y+b)-f(y)\|^2 \right| \, d\mu(y) \\
&= \int_X (\|f(y+a)-f(y)\| + \|f(y+b)-f(y)\|) \cdot \\
&\qquad |\|f(y+a)-f(y)\| - \|f(y+b)-f(y)\|| \, d\mu(y) \\
&\leq \int_X (\|f(y+a)-f(y)\| + \|f(y+b)-f(y)\|) \cdot \\
&\qquad \|f(y+a)-f(y+b)\| \, d\mu(y) \\
&\leq \int_X (\|a\|^\alpha + \|b\|^\alpha) \;\; \|a-b\|^\alpha \, d\mu(y).
\end{aligned}
$$

So $\|g(a)-g(b)\| \leq \|a-b\|^\alpha (\|a\|^\alpha + \|b\|^\alpha)$ and g is continuous.

Now for the equicontinuity of $(g_m)_m$:

$$
\begin{aligned}
\|g_m(a) - g_m(b)\| &= \|\text{ave}\ \ \left(g\left(\sum \theta_n a_n e_{\sigma(n)}\right) - g\left(\sum \theta_n b_n e_{\sigma(n)}\right)\right)\| \\
&\le \text{ave}\ \ \|g\left(\sum \theta_n a_n e_{\sigma(n)}\right) - g\left(\sum \theta_n b_n e_{\sigma(n)}\right)\| \\
&\le \text{ave}\ \ \Big(\|\sum \theta_n a_n e_{\sigma(n)} - \sum \theta_n b_n e_{\sigma(n)}\|^\alpha \\
&\qquad \cdot \left(\|\sum \theta_n a_n e_{\sigma(n)}\|^\alpha + \|\sum \theta_n b_n e_{\sigma(n)}\|^\alpha\right)\Big) \\
&= \text{ave}\ \ \left(\|a-b\|^\alpha \left(\|a\|^\alpha + \|b\|^\alpha\right)\right) \\
&= \|a-b\|^\alpha \left(\|a\|^\alpha + \|b\|^\alpha\right).
\end{aligned}
$$

So by Ascoli's theorem, there is a subsequence $(g_{m_k})_k$ of $(g_m)_m$ which converges pointwise to a continuous function $\widetilde{g}$. The property of the g_m's implies that $\widetilde{g}$ must necessarily be symmetric. We have that $\widetilde{g}(0) = 0$, and that $\phi_1(\|x\|) \le \widetilde{g}(x) \le \phi_2(\|x\|)$. Finally, as it is easily checked that the g_m's are negative definite functions, it also follows easily that $\widetilde{g}$ is a negative definite function.

STEP 4

There is a relation between negative and positive definite kernels as given by a result of Schoenberg [S]; see also [BL, Chapter 8]. This result states that a kernel N on X is negative definite if and only if e^{-tN} is positive definite for every $t > 0$.

Since $\liminf_{n\to\infty} \dfrac{\|e_1 + e_2 + \cdots + e_n\|}{\sqrt{n}} = 0$, and $\widetilde{f} = e^{-\widetilde{g}}$ is a symmetric continuous positive definite function on X, we conclude by a theorem of Aharoni, Maurey and Mityagin (see [AMM, Theorem 5.2]), that $\widetilde{f}$ is constant.

On the other hand, $\widetilde{f}(0) = e^{-\widetilde{g}(0)} = 1$, while $0 \le \widetilde{f}(x) \le e^{-\phi_1(\|x\|)} \to 0$ as $\|x\| \to \infty$. This gives a contradiction and finishes the proof.

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