# Diamond graphs and super-reflexivity *

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**Abstract**

The main result is that a Banach space $X$ is not super-reflexive if and only if the diamond graphs $D_n$ Lipschitz embed into $X$ with distortions independent of $n$. One of the consequences of that and previously known results is that dimension reduction a-la Johnson–Lindenstrauss fails in any non super reflexive space with non trivial type.

We also introduce the concept of Lipschitz $(p, r)$-summing map and prove that every Lipschitz mapping is Lipschitz $(p, r)$-summing for every $1 \leq r < p$.

## 1 Introduction

In [BC] Brinkman and Charikar proved that the diamond graphs $D_n$, which were known ([GNRS]) to Lipschitz embed into $\ell_1$ with distortion independent of $n$, can be used to give a lower bound on dimension reduction in $\ell_1$; precisely, for each $n$ there are subsets $A_n$ of $\ell_1$ of cardinality $\overline{A_n} = n$ so that if $\alpha > 0$ and $f_n : A_n \to \ell_1^{\lceil n^\alpha \rceil}$, then $\mathrm{Lip}(f_n)\mathrm{Lip}(f_n^{-1}) \geq c\alpha^{-1}$ for some universal $c > 0$. Lee and Naor [LN] substantially simplified the proof of this fact. In Corollary 2 we give an even simpler proof, modelled on the proof of [LN] but substituting an easy inductive argument for the magical telescoping series inequality there.

Recall that a Banach space $X$ is super reflexive iff whenever a Banach space $Y$ is finitely represented in $X$ (which means that for every $\varepsilon > 0$, every finite dimensional subspace $E$ of $Y$ linearly embeds into $X$ with distortion less than $1 + \varepsilon$), then $Y$ is reflexive. Enflo [E] (see also [P]) proved that $X$ is super reflexive iff $X$ has an equivalent uniformly convex norm.

Our main focus in this paper is to show (in Theorem 1) that the diamond graphs give a metric characterization of super reflexivity; that is, a Banach space $X$ is not super reflexive if and only if the diamond graphs $D_n$ Lipschitz embed into $X$ with distortion

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independent of $n$. The proof is, in our opinion, simpler than Bourgain's proof [BOU2] that a Banach space $X$ is not super reflexive if and only if the diadic trees $T_n$ Lipschitz embed into $X$ with distortion independent of $n$. As far as we see neither of these non linear characterizations of super reflexivity directly implies the other.

As a by-product we get that dimension reduction as in Hilbert space (embedding any $n$ points into a $O(\log n)$-dimensional Euclidean space, [JL]) is not possible in any non super reflexive space with non trivial type.

In [FJ] an appropriate definition of Lipschitz $p$-summing map was introduced and there it was shown that the (linear) $p$-summing norm of a linear operator is the same as its Lipschitz $p$-summing norm. In section 5 we give an analogous definition of Lipschitz $(p, r)$-summing map (so that, e.g., the Lipschitz $(p, p)$-summing norm of a Lipschitz map is the same as its Lipschitz $p$-summing norm). However, it turns out that the (linear) $(p, r)$-summing norm of a linear operator can be much larger than its Lipschitz $(p, r)$-summing norm. In fact, the Lipschitz $(p, r)$-summing norm of any Lipschitz map is finite for every $1 \le r < p$! An equivalent formulation of this is that every metric space satisfies a family of inequalities which look to us non trivial. This will be explained in detail in Section 5. As we shall also explain there we were led to this family of inequalities by an interpretation of Theorem 1 coming from the linear theory of Banach spaces.

In Section 6 we indicate the validity of the analog statement to Theorem 1 applied to the sequence of Laakso graphs. This is a family of graphs very similar to the diamond graphs but with the additional property that they are uniformly doubling.

# 2 Preliminaries

We begin by describing the inductive construction of the diamond graph $D_n$ whose vertex set will be a subset of $\{0, 1\}^{2^n}$ and the edge set will be a subset of the edge set of the Hamming cube $\{0, 1\}^{2^n}$. The exposition follows closely that in [BC] but there will be an important change in the way we index the vertices of the graph. $D_0$ is the graph with two vertices, labelled 0 and 1 and one edge connecting them. Giving $D_{i-1}$ whose vertex set is a subset of $\{0, 1\}^{2^{i-1}}$ we replace each vertex $(a_1, a_2, \dots, a_{2^{i-1}})$ with $(a_1, a_1, a_2, a_2, \dots, a_{2^{i-1}}, a_{2^{i-1}}) \in \{0, 1\}^{2^i}$ (this is where the indexing differs from that in [BC]; there the replacement would be $(a_1, a_2, \dots, a_{2^{i-1}}, a_1, a_2, \dots, a_{2^{i-1}})$). For each two new vertices whose Hamming distance is 2 (i.e, coming from an edge in $D_{i-1}$) we also add the two points in $\{0, 1\}^{2^i}$ which are of Hamming distance 1 from each of these two vertices. If $x' = (x_1, \dots, x_{2^i})$ is a vertex of $D_i$ we say that $x = (x_1, \dots, x_1, x_2, \dots, x_2, \dots, x_{2^i}, \dots, x_{2^i}) \in D_n$, where each $x_j$ repeats $2^{n-i}$ times, is *the vertex developed from $x'$ by $n - i$ doubling operations.* If $(x_1, \dots, x_{2^i})$ is one of the new vertices of $D_i$ (as opposed to one coming from doubling a vertex of $D_{i-1}$) we say that $x = (x_1, \dots, x_1, x_2, \dots, x_2, \dots, x_{2^i}, \dots, x_{2^i}) \in D_n$ (the vertex developed from $x'$ by $n - i$ doubling operations) is *a vertex of level $i$*.

Suppose that $x'$ and $y'$ are two vertices of $D_i$ connected by an edge and $x$, $y$ are the two vertices of $D_n$ developed from them by $n - i$ doubling operations, then (unless $i = 0$) exactly one of $x$ or $y$ is a vertex of level $i$. $x$ and $y$ differ exactly in one interval,

$(j-1)2^{n-i}, j2^{n-i}$, where one of them has all 1-s and the other all 0-s (these facts are easy to prove by induction). The set of vertices of $D_n$ lying pointwise between $x$ and $y$ will be called a *sub-diamond of level* $i$. The one of $x$ and $y$ that has all ones on the said interval,i.e., $x \vee y$, will be called the *top vertex* of this sub-diamond and the other the *bottom vertex*. The vertex which has ones in the left half of the said interval and zeros in the right half will be called the *leftmost* vertex of this sub-diamond and the vertex which has ones in the right half of the said interval and zeros in the left half will be called the *rightmost vertex* of this sub-diamond. The vertices of the sub-diamond whose distance to the bottom vertex is at most their distance to the top one will said to lie *below the diagonal* and the rest *above the diagonal*. The vertices of the sub-diamond whose distance to the leftmost vertex is at most their distance to the right most one will said to be *on the left* of the sub-diamond and the rest *on the right*.

Note that $D_n$ has $4^n$ edges and $2+2\sum_{i=0}^{n-1}4^i = 2+2\frac{4^n-1}{3}$ vertices.

It is easy to prove, by induction, that, given two vertices of $D_n$, either both lie on a path connecting the top and bottom vertices of $D_n$, or one is on the left side and one is on the right side of some sub-diamond (possibly $D_n$ itself).

# 3 Embedding the diamonds in non super reflexive spaces

**Proposition 1** *Let $x_1, x_2, \ldots, x_{2^n}$ be a sequence of norm one points in a Banach space $X$ with the properties:*

*(i) $\|\sum_{i\in A} x_i\| \geq a|A|$, for any subset $A \subset \{1,2,\ldots,2^n\}$*

*(ii) $\|\sum_{i\in A} a_i x_i\| \leq b\|\sum_{i=1}^{2^n} a_i x_i\|$, for any interval $A \subset \{1,2,\ldots,2^n\}$ and any real coefficients $a_1,\ldots,a_{2^n}$.*

*Define $f : D_n \to X$ by $f(a_1,\ldots,a_{2^n}) = \sum_{i=1}^{2^n} a_i x_i$. Then $f$ has distortion at most $4a^{-1}b$.*

**Proof:** If $u$ and $v$ lie on a path between the (images of the) original two vertices, $(0,0,\ldots,0)$ and $(1,1,\ldots,1)$ (such a path is a monotone sequence in $\{0,1\}^{2^n}$) then $f(u)-f(v) = \pm\sum_{i\in A} x_i$ where $|A| = d_{D_n}(u,v)$. By $(i)$, $ad_{D_n}(u,v) \leq \|f(u)-f(v)\| \leq d_{D_n}(u,v)$.

If $u$ and $v$ do not lie on such a path then there is some sub-diamond, say of level $n-k$, such that $u$ is on the left side of it and $v$ on the right (or vice versa). Denote the bottom and top vertices of this sub-diamond by $v_B$ and $v_T$. This means that for some subinterval $A \subset \{1,2,\ldots,2^n\}$ of size $2^k$, $v_T$ restricted to $A$ is the all 1-s vector, $v_B$ restricted to $A$ is the all 0-s vector, and $v_T$ and $v_B$ are identical on the complement of $A$. If $u = (u_1,\ldots,u_{2^n})$ and $v = (v_1,\ldots,v_{2^n})$ lie on the same side of the diagonal of this sub-diamond then their distance in $D_n$ is $\min\{d_{D_n}(u,v_B)+d_{D_n}(v_B,v), d_{D_n}(u,v_T)+d_{D_n}(v_T,v)\}$. Let us assume that they both lie on or below the diagonal (the treatment of the other case is similar). Write $A = B\cup C$ where $B$ (resp. $C$) is the left (resp. right) half of $A$. Then $u$ restricted to $C$ is the all zero vector and $v$ restricted to $B$ is the all zero vector. $u$ and $v$ are equal (and equal to $v_T$ and $v_B$) on the complement of $A$. Consequently, since the $x_i$-s are all norm one,

$$\begin{aligned}\|f(u)-f(v)\| &= \|\sum_{i\in B} u_i x_i - \sum_{i\in C} v_i x_i\| \le \|\sum_{i\in B} u_i x_i\| + \|\sum_{i\in C} v_i x_i\| \\ &\le d_{D_n}(u, v_B) + d_{D_n}(v_B, v).\end{aligned}$$

On the other hand, by $(ii)$ then $(i)$,

$$\begin{aligned}\|f(u)-f(v)\| &\ge b^{-1}\max\{\|\sum_{i\in B} u_i x_i\|, \|\sum_{i\in C} v_i x_i\|\} = ab^{-1}\max\{d_{D_n}(u, v_B), d_{D_n}(v_B, v)\} \\ &\ge \frac{a}{2b}(d_{D_n}(u, v_B) + d_{D_n}(v_B, v)).\end{aligned}$$

If $u$ and $v$ are not on the same side of the diagonal of the sub-diamond determined by $v_T$ and $v_B$, let us assume for example that $u$ is above the diagonal (and on the left) and $v$ below the diagonal (and on the right). In this case it is easy to see that the distance between $u$ and $v$ in $D_n$ is between $2^{k-1}$ and $2^k$. Also $u$ restricted to $B$ is the all 1-s vector (while $v$ restricted to $B$ is zero), so

$$\begin{aligned}2^k \ge \|f(u)-f(v)\| &= \|\sum_{i\in B} u_i x_i - \sum_{i\in C} v_i x_i\| \ge b^{-1}\|\sum_{i\in B} x_i\| \\ &\ge ab^{-1}2^{k-1}.\end{aligned}$$

■

**Remark 1** Note that we have only used assumption $(ii)$ for $\pm 1, 0$ coefficients.

One of the characterization of super-reflexivity says that any non super reflexive Banach space contains for every $a < 1$ and $b > 4$ and any $n$ a sequence as in Proposition 1. By the definition of super-reflexivity, it is enough to have this statement for any non-reflexive space. It is not easy to trace the origin of this statement. It explicitly appear in [Ja1, Theorem 8] and partially attributed there to Pełczyński. Vitali Milman told us that Ptak and Pełczyński are probably the origin and Milman and Milman also proved the same a bit earlier than James' paper.

**Corollary 1** *For each $\lambda > 1$, each $n \in \mathbb{N}$ and each non-super-reflexive space $X$, $D_n$ Lipschitz embeds into $X$ with distortion at most $16\lambda$.*

# 4 Non embedability of the diamonds in uniformly convex spaces

Recall that the modulus of uniform convexity of a normed space $X$ is the function $\delta : (0,2) \to [0,1]$ defined by

$$\delta(\varepsilon) = \delta_X(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| \ ; \ \|x\|, \|y\| \le 1, \ \|x-y\| \ge \varepsilon\}.$$

**Lemma 1** *Let $X$ be a normed space and $f : D_1 \to X$ with* $\mathrm{Lip}(f^{-1}) \le 1$ *and* $\mathrm{Lip}(f) \le M$*. Then* $\|f(1,1) - f(0,0)\| \le 2M(1 - \delta(\frac{2}{M}))$

**Proof:** Without loss of generality we may assume $f(0,0) = 0$. Denote $x = f(1,1)$ and $x_1 = x - f(1,0)$, $x_2 = f(1,0)$, $x_3 = x - f(0,1)$, $x_4 = f(0,1)$. Then, $1 \le \|x_i\| \le M$ for $i = 1,2,3,4$. Since $\|\frac{x_2}{M} - \frac{x_4}{M}\| \ge \frac{2}{M}$, we get that

$$1 - \frac{\|x_2 + x_4\|}{2M} \ge \delta(\frac{2}{M}).$$

Similarly,

$$1 - \frac{\|x_1 + x_3\|}{2M} \ge \delta(\frac{2}{M}).$$

Consequently,

$$2(1 - \delta(\frac{2}{M})) \ge \frac{\|x_1 + x_2 + x_3 + x_4\|}{2M} = \frac{\|x\|}{M}.$$

■

Applying the lemma we get that if $M_n$ is the best constant $M$ such that there is an embedding $f$ of $D_n$ into $X$ with $d_{D_n}(x,y) \le \|f(x) - f(y)\| \le M d_{D_n}(x,y)$, then $M_{n-1} \le M_n(1 - \delta_X(\frac{2}{M_n}))$. Indeed, by the Lemma the distance between the images of the top and the bottom vertices of any sub diamond of level $n-1$ is at most $2M_n(1-\delta_X(\frac{2}{M_n}))$ (and at least 2). The collection of all top and bottom vertices of level $n-1$ of $D_n$ is isometric to $D_{n-1}$: The distance between any two points of that subset of $D_n$ is exactly twice the distance of the vertices of $D_{n-1}$ they developed from. This implies that $M_{n-1} \le M_n(1 - \delta_X(\frac{2}{M_n}))$.

Passing to the limit as $n \to \infty$ we easily get that the $D_n$ do not embed with a uniform distortion in any uniformly convex space. This fact was basically known, although we could not find an explicit reference. Its analog for the Laakso graphs (see Section 6) was proved in [La] (see also [Ty]). Also, [LN] (and [LMN] for the Laakso graphs) contains the result under the additional assumption that the modulus of convexity dominates $c\varepsilon^2$ for some $c > 0$. However, for some of the applications we shall need a finer estimate of the distortion. Notice that the exact dependence on the constant $c$ will be important for us later. [MN] essentially contains this result with a different, more involved proof.

**Proposition 2** *Let $X$ be a normed space whose modulus of uniform convexity is of power type $p$, for some $2 \leq p < \infty$; i.e, $\delta_X(\varepsilon) \geq c\varepsilon^p$. Let $M_n$ denote the best distortion of an embedding of $D_n$ into $X$. Then $M_n \geq 2c^{1/p}n^{1/p}$.*

**Proof:** By the Lemma and the remark following it, for all $k \geq 1$, $M_k - M_{k-1} \geq M_k c(\frac{2}{M_k})^p = \frac{c2^p}{M_k^{p-1}}$. So,

$$M_n \geq c2^p \sum_{k=1}^{n} M_k^{-p+1} + M_0 \geq c2^p n M_n^{-p+1},$$

implying $M_n \geq 2c^{1/p}n^{1/p}$. ∎

Since a Banach space is super reflexive if and only if it has an equivalent norm which is uniformly convex with modulus of uniform convexity of power type $p$ for some $2 \leq p < \infty$ (see [P]), we get as a corollary of the last proposition and of Corollary 1 the following main theorem of this paper.

**Theorem 1** *A Banach space $X$ is not super reflexive if and only if the diamond graphs $D_n$ Lipschitz embed into $X$ with distortions independent of $n$.*

We now state some more corollaries of Proposition 2. The first is a proof of the main result of [BC] which combined with the above is even simpler than the simple proof in [LN]. The deduction of the result from Proposition 2 is identical to that in [LN].

**Corollary 2 (BC)** *If $D_n$ Lipschitz embeds into $\ell_1^k$ with distortion $K$ then $k \geq \overline{D_n}^{\beta/K^2}$, for a universal $\beta > 0$.*

**Proof:** For any $p > 1$, the (Lipschitz) distance between $\ell_1^k$ and $\ell_p^k$ is at most $k^{(p-1)/p}$. $\delta_{L_p}(\varepsilon) \geq \alpha(p-1)\varepsilon^2$, for all $1 < p < 2$ and for a universal $\alpha > 0$ (see e.g. [Fi] or [BCL]). It thus follows from Lemma 1 that $K \geq 2k^{-(p-1)/p}\alpha^{1/2}(p-1)^{1/2}n^{1/2}$. Choosing $p = 1 + \frac{1}{\log k}$ we get $\log k \geq \beta n/K^2$ for a universal $\beta > 0$. ∎

There are non super reflexive spaces that have non trivial type ([Ja2]; see also [PX]). These provide interesting examples of spaces in which the Johnson–Lindenstrauss dimension reduction result cannot occur.

**Corollary 3** *If $X$ is a non super reflexive space with non trivial type $p$ then*

*(i) The $D_n$-s uniformly Lipschitz embed in $X$, but*

*(ii) for any sequence $k_n$ with $k_n = O(n)$, if $X_n$ are subspaces of $X$ of dimensions $k_n$ then the distortion of embedding $D_n$ into $X_n$ tends to infinity.*

**Proof:** $(i)$ follows from Corollary 1. Since $X$ has non trivial type, by [MW], $d(Y, \ell_2^{\dim Y}) = o((\dim Y)^{1/2})$ for every subspace $Y$ of $X$. If $\dim Y = k_n$ with $k_n = O(n)$, and $D_n$ embeds into $Y$ with distortion $D$, then it embeds into $\ell_2^{k_n}$ with distortion $Do(k_n^{1/2}) = Do(n^{1/2})$. By the previous proposition, since $\delta_{\ell_2}(\varepsilon) > \varepsilon^2/8$, $D \geq 2^{-1/2} n^{1/2}/o(k_n^{1/2}) \to \infty$. ■

**Remark 2** There are also non super reflexive spaces $X$ in which $d(Y, \ell_2^{\dim Y}) = O(\log(\dim Y))$ for every subspace $Y$ of $X$ [PX]. In a similar way to the proof of the previous corollary we get that if $D_n$ Lipschitz embeds into a $k_n$-dimensional subspace of $X$ then $k_n \geq 2^{c(\log \overline{D_n})^{1/2}/D}$, for a universal $c > 0$.

**Remark 3** The summing operator $S : \ell_1 \to \ell_\infty$ is defined to be the continuous linear extension of the mapping $e_n \mapsto \sum_{i=1}^n e_i$, where $\{e_n\}_{n=1}^\infty$ is the unit vector basis. Lindenstrauss and Pełczyński [LiP] proved that a Banach space $X$ is non reflexive iff $S$ factors through $X$, which means that there are bounded linear operators $A : \ell_1 \to X$ and $B : X \to \ell_\infty$ so that $S = BA$. Moreover, if $X$ is non reflexive then $A$ and $B$ can be chosen so that $\|A\| \cdot \|B\| < C$ for some absolute constant $C$. Now $\{e_i\}_{i=1}^{2^n}$ in $\ell_1$ and $\{Se_i\}_{i=1}^{2^n}$ in $\ell_\infty$ both satisfy the conditions on $\{x_i\}_{i=1}^{2^n}$ in Proposition 1 (with constants $a$, $b$ independent of $n$), so there are mappings $F_n$ of $D_n$ into $\ell_1$ with uniformly bounded distortion so that $SF_n$ have uniformly bounded distortion. This implies Theorem 1, in view of the result in [LiP].

# 5 $(p, r)$-summing Lipschitz maps

We begin with explaining our motivation for the main result of this section, Proposition 4. This motivation depends on some non trivial facts from the linear theory which the (unmotivated) reader may prefer to skip. Such a reader should skip to Definition 1 below.

Remark 3 says that the summing operator, $S$, preserves with bounded distortion a sequence of metric spaces that are fairly complicated. This led us to wonder whether or not $S$ is *finitely strictly singular*, which means that $S$ does not preserve with bounded distortion any sequence of finite dimensional subspaces of $\ell_1$ whose dimensions tend to infinity. It turned out that $S$ is finitely strictly singular; in fact,

**Proposition 3** *There exists a constant $C$ such that if $E$ is an $n$-dimensional subspace of $\ell_1$, then*

$$\alpha(S_E) := \inf_{x \in E;\, \|x\|_1 = 1} \|Sx\|_\infty \leq C n^{-1/2}. \tag{1}$$

**Proof:** This proposition is not really needed to understand the main result in this section, so we use without further reference tools that are standard in Banach space theory. The interested non expert can find everything we use in [MS] or [DJT].

Since there is a constant $\delta > 0$ so that every $n$-dimensional subspace of $\ell_1$ contains a further subspace that has dimension at least $\delta n$ and is 2-isomorphic to a Euclidean space, it is enough to verify (1) when $E$ is 2-isomorphic to $\ell_2^n$. Let $Y$ be a non reflexive space that has type 2 (see [Ja2] or [PX]) and let $A : \ell_1 \to Y$, $B : Y \to \ell_\infty$ be bounded linear operators so that $S = BA$ and $\|B\| = 1$. Clearly $\alpha(A_E) \geq \alpha(S_E)$. That is, letting $U$ be the inverse of the restriction of $A$ to $E$, we have that $\|U\| \leq \alpha(S_E)^{-1}$. Since $U$ is mapping into a space that is 2-isomorphic to a Hilbert space, we have that $U$ has an extension to a linear operator $\tilde{U} : X \to E$ with $\|U\| \leq 2T_2(X)\alpha(S_E)^{-1}$ (here $T_2(X)$ is the (gausian) type 2 constant of $X$). Thus $\tilde{U}A$ is a projection from $\ell_1$ onto a 2-isomorph of $\ell_2^n$ and hence

$$(1/8)\sqrt{n} \leq \|\tilde{U}A\| \leq 2T_2(X)\alpha(S_E)^{-1}.$$

■

After we discovered Proposition 3, Pełczyński reminded us that he and Kwapień [KP] proved that $S$ is $(p, r)$-summing for every $p > r \geq 1$. This immediately implies that $S$ is finitely strictly singular (but without the precise estimate given by Proposition 3). Now, one can modify the definition of $(p, r)$-summing to define a concept of Lipschitz $(p, r)$-summing in the same way that the definition of $p$-summing was modified in [Fa] to define the concept of Lipschitz $p$-summing. Since the summing operator preserves copies of of the diamond graphs, it follows that the identity operators on the diamond graphs $D_n$ are Lipschitz $(p, r)$-summing, with constants independent of $n$, for all $p > r \geq 1$. This means that the diamond graphs satisfy a family of inequalities that look non trivial. We were surprised when further investigation yielded that all metric spaces satisfy these inequalities.

We begin with the definition of Lipschitz $(p, r)$-summing function.

**Definition 1** *A function $F$ from a metric spaces $(M, d)$ into a metric space $(N, \rho)$ is said to be Lipschitz $(p, r)$-summing with constant at most $K$ if for all finite sequences of pairs of points $\{x_i, y_i\}_{i=1}^n$ in $M$ and all positive numbers $\{c_i\}_{i=1}^n$*

$$(\sum_{i=1}^n c_i^p \rho(F(x_i), F(y_i))^p)^{1/p} \leq K \sup(\sum_{i=1}^n c_i^r |f(x_i) - f(y_i)|^r)^{1/r}, \tag{2}$$

*where the* sup *is taken over all real Lipschitz functions $f$ on $M$ of Lipschitz constant* 1. *The smallest possible $K$ is denoted $\pi_{p,r}^L(F)$. (The superscript L stands for Lipschitz.)*

For a linear operator $F$, the definition of $(p, r)$-summing is the same except that the supremum in the right side of (2) is taken over all linear functionals $f$ whose norm is at most one. Of course, linearity of $F$ allows the definition in the linear case to be written more succinctly.

The definition of Lipschitz $(p, p)$-summing is the same as the definition of Lipschitz $p$-summing given in J. Farmer's 1992 dissertation [Fa] and repeated in [FJ]. $\pi_{p,p}^L(F)$ is the same as the quantity $\pi_p^L(F)$ defined there.

It is quite easy to show that if $p \geq r$, $q \geq s$ and $\frac{1}{s} - \frac{1}{q} \leq \frac{1}{r} - \frac{1}{p}$ then any $(q, s)$-summing function $F$ is also $(p, r)$-summing and $\pi_{p,r}^L(F) \leq \pi_{q,s}^L(F)$. (The proof is very similar to the simple proof in the linear setting; see e.g. page 198 in [DJT].)

We now state the main result in this section.

**Proposition 4** *For any $p > 1$, any Lipschitz function is $(p,1)$-summing with constant at most its Lipschitz constant times a constant $C_p$ which is $O((p-1)^{-1})$ for $p \to 1$ and O(1) for $p \to \infty$ . Consequently, it is also $(p,r)$-summing for any $p > r > 1$ with constant at most its Lipschitz constant times a constant depending only on $p$ and $r$.*

**Proof:** It is enough to prove the proposition for the identity map on $(M,d)$. Let $\{x_i, y_i\}_{i=1}^n$ be any points in the metric space $(M,d)$ and let $\{c_i\}_{i=1}^n$ be any positive numbers with $\sum_{i=1}^n c_i^p d(x_i,y_i)^p = 1$. Put $\sigma_j = \{i; (c_i d(x_i,y_i))^{p-1} \in (2^{-j}, 2^{-j+1}]\}$, $j = 1, 2, \ldots$.

As was mentioned in [FJ] (or, in view of [FJ, Theorem 1], in an equivalent form in [MR]), Bourgain [BOU1] really proved that $\pi_1^L(I_Z) \le C \log n$ for all $n$ point metric spaces $Z$. This implies that for some universal $C$ and for each $j$ there is a real Lipschitz function $f_j$ on $M$ with Lipschitz constant 1 satisfying

$$\begin{aligned} \sum_{i\in\sigma_j} c_i^p d(x_i,y_i)^p &\le C \log \overline{\sigma_j} \sum_{i\in\sigma_j} (c_i d(x_i,y_i))^{p-1} c_i |f_j(x_i) - f_j(y_i)| \\ &\le C 2^{1-j} \log \overline{\sigma_j} \sum_{i\in\sigma_j} c_i |f_j(x_i) - f_j(y_i)|. \end{aligned} \tag{3}$$

Given a sequence $\varepsilon$ of $\varepsilon_j = \pm 1$, let $f_\varepsilon = \sum_j \varepsilon_j 2^{1-j} \log \overline{\sigma_j} f_j$.

Note that for each $\varepsilon$ the Lipschitz constant of $f_\varepsilon$ is at most $\sum_j 2^{1-j} \log \overline{\sigma_j}$. To evaluate this sum note that $\overline{\sigma_j} 2^{-jp/(p-1)} \le 1$ so that

$$\sum_j 2^{1-j} \log \overline{\sigma_j} \le \frac{p}{p-1} \sum_{j=1}^{\infty} j 2^{1-j} \le C_p.$$

Where $C_p = O((p-1)^{-1})$ when $p$ approaches 1 and O(1) when $p$ approaches infinity.

Note also that for each $i$ and $j$ such that $i \in \sigma_j$,

$$\mathbb{E}|f_\varepsilon(x_i) - f_\varepsilon(y_i)| \ge 2^{1-j} \log \overline{\sigma_j} |f_j(x_i) - f_j(y_i)|,$$

where the expectation $\mathbb{E}$ is the average over all $\varepsilon \in \{-1,1\}^n$. Consequently, using (3),

$$\begin{aligned} 1 = \sum_{i=1}^n c_i^p d(x_i,y_i)^p &= \sum_j \sum_{i\in\sigma_j} c_i^p d(x_i,y_i)^p \\ &\le C \sum_j \sum_{i\in\sigma_j} c_i 2^{1-j} \log \overline{\sigma_j} |f_j(x_i) - f_j(y_i)| \\ &\le C \mathbb{E} \sum_{i=1}^n c_i |f_\varepsilon(x_i) - f_\varepsilon(y_i)|. \end{aligned} \tag{4}$$

■

**Remark 4** In [FJ] it was proved that $\pi_p(T) = \pi_p^L(T)$, $1 \le p < \infty$, for every linear operator $T$. Proposition 4 shows that there is no analogous result for the $(p,r)$-summing norms.

**Remark 5** The order $O((p-1)^{-1})$ for the constant $C_p$ when $p$ approaches 1 is best possible. This can be seen by considering the examples of $n$ point metric spaces whose best embedding into a Hilbert space have distortion of order $\log n$ (see [LLR], expanders with a fix degree can serve as such examples). If $(M,d)$ is such an $n$-point metric space then the identity on $M$ satisfies $\pi_1^L(I_M) \ge c \log n$, for a universal $c > 0$ (see [FJ]). Also, in the definition of $\pi_{p,1}^L(I_M)$ it is clearly enough to consider at most $n^2$ couples of points $(x_i, y_i) \in M^2$ (no repetitions are needed). Consequently, for all such couples of points and all coefficients $c_i$,

$$\sum c_i d(x_i,y_i) \le n^{2(1-1/p)} (\sum c_i^p d(x_i,y_i)^p)^{1/p} \le n^{2(1-1/p)} \pi_{p,1}^L(I_M) \sup \sum c_i |f(x_i) - f(y_i)|.$$

We thus get $n^{2(1-1/p)} \pi_{p,1}^L(I_M) \ge c \log n$. Given $1 < p < 2$ choose $n$ so that $\log n$ is of order $(p-1)^{-1}$ to get that $\pi_{p,1}^L(I_M)$ is at least of order $(p-1)^{-1}$.

Clearly, also the order $O(1)$ for the constant $C_p$ when $p$ approaches $\infty$ is best possible.

We conclude this section with a possible measure of the complexity of a metric space. Saying that $\pi_1^L(I_X) \le K$ is the same as saying that you can situate $X$ isometrically in $L_\infty(\mu)$ for some probability $\mu$ so that the $L_\infty(\mu)$ and $L_1(\mu)$ norms are $K$-equivalent on $X$ (see [FJ]). In some sense, then, $\pi_1^L(I_X) < \infty$ says that $X$ is reasonably simple, and the size of $\pi_1^L(I_X)$ for, say, a finite metric space $X$ provides one measure of the simplicity of $X$. Perhaps here it is worth mentioning that $\pi_1^L(I_{\ell_2^n})$ is of order $\sqrt{n}$ and that if $X$ is an infinite metric space such that the distance between any two distinct points is one, then $\pi_1^L(I_X) = 2$ (see [FJ]). When $\pi_1^L(I_X) = \infty$, a related measure of the complexity of $X$ is provided by the asymptotics of $\pi_{p,1}^L(I_X)$ as $p \downarrow 1$.

# 6 The Laakso graphs

In this final section we indicate how to prove a theorem similar to Theorem 1 for another interesting sequence of graphs. These are graphs introduced by Laakso [La]. They have similar properties to those of the diamond graphs but in addition are doubling: every ball in each of these graphs can be covered by a union of 6 balls of half the radius of the original ball. See [LaP] for a simple proof of this fact, originally proved (with a different constant) in [La]. We prefer to introduce these graphs as subgraphs of the Hamming cubes, with all edges of length 1; consequently our distance function is a multiple of the ones used in [La], [LiP] and [LMN]. As for the diamond graphs, the actual embedding into the Hamming cubes will be important for us.

The vertex set of the graph $L_n$ will be a subset of $\{0,1\}^{4^n}$ and the edge set will be a subset of the edge set of the Hamming cube $\{0,1\}^{4^n}$. $L_0$ is the graph with two vertices, labelled 0 and 1 and one edge connecting them. Given $L_{i-1}$ whose vertex set is a subset of $\{0,1\}^{4^{i-1}}$ we replace each vertex $(a_1, a_2, \dots, a_{2^{i-1}})$ with

$$(a_1, a_1, a_1, a_1, a_2, a_2, a_2, a_2, \dots, a_{2^{i-1}}, a_{2^{i-1}}, a_{2^{i-1}}, a_{2^{i-1}}) \in \{0,1\}^{4^i}.$$

For each two new vertices whose Hamming distance is 4 (i.e, coming from an edge in $L_{i-1}$) we now add four new vertices in the following way: the two new vertices differ only in four consecutive coordinates in which one of them has all ones and the other all zeros, the four additional new vertices will be identical to these two in all coordinates except these four, and the restriction of these four new vertices to these four coordinates will be: $((1,1,0,1),(1,1,0,0),(0,1,0,1)$ and $(0,1,0,0)$. So, for example, $L_1$ will be

$$\begin{array}{c} (1,1,1,1) \\[1em] (1,1,0,1) \\ (1,1,0,0) \qquad (0,1,0,1) \\ (0,1,0,0) \\ (0,0,0,0) \end{array}$$

with an edge connecting any vertex to the one(s) immediately below it. We shall term these six vertices as: very top, top, leftmost, rightmost, bottom, and very bottom. In analogy to the notations of the diamond graphs, we can also talk about sub-Laakso graphs of level i, on top, very top, etc in sub-Laasko graphs and so on.

In a very similar way to the proofs presented, we can prove

**Theorem 2** *A Banach space $X$ is not super reflexive if and only if the Laakso graphs $L_n$ Lipschitz embed into $X$ with distortions independent of $n$.*

We only indicate briefly the outline of the proof. As we already indicated the fact that the Laakso graphs do not uniformly Lipschitz embed in a uniformly convex space was known ([La], [Ty] and [LMN] for a somewhat more restrictive statement). We shall however indicate the simple proof as it gives a somewhat more quantitative statement (the analog of Proposition 2). So to prove that $L_n$ do not uniformly embed in a uniformly convex Banach space we first prove

**Lemma 2** *Let $X$ be a normed space and $f : L_1 \to X$ with $\mathrm{Lip}(f^{-1}) \le 1$ and $\mathrm{Lip}(f) \le M$. Then $\|f(1,1,1,1) - f(0,0,0,0)\| \le 4M(1 - \delta(\frac{2}{M}))$*

The proof of the lemma is almost identical with the proof of Lemma 1: Start with the assumption that $f(0,0,0,0) = 0$ and put $x = f(1,1,1,1)$, $x_1 = x - f(1,1,0,0)$, $x_2 = f(1,1,0,0)$, $x_3 = x - f(0,1,0,1)$, $x_4 = f(0,1,0,1)$. Now continue as in the proof of Lemma 1 to get exactly the same statement as in Proposition 2 (for $L_n$ instead of $D_n$).

To prove that the $L_n$-s uniformly embed in any non super reflexive space use an embedding similar to the one in the statement of Proposition 1: Let $x_1, x_2, \dots, x_{4^n}$ be a sequence satisfying the assumptions of Proposition 1 (with $4^n$ replacing $2^n$) and let $f : L_n \to X$ be defined by $f(a_1, \dots, a_{4^n}) = \sum_{i=1}^{4^n} a_i x_i$. If $u, v \in L_n$ lie on a path from the very top to the very bottom then $f$ acts as an isometry on them. If not, we look at the smallest sub-Laakso graph $u$ and $v$ belong to and then necessarily they both lie between the top and bottom (as opposed to the very top and the very bottom) vertices of this sub graph and they lie on opposite sides there. The rest of the argument is very similar to the one in the proof of Proposition 1 and we omit it.

**Remark 6** In light of [BKL], It might very well be that Theorem 1 extends to any series parallel graph. We didn't try to prove this.

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