COVERING A COMPACT SET IN A BANACH SPACE BY AN OPERATOR RANGE OF A BANACH SPACE WITH BASIS.

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1. Introduction

A Banach space Y is said to have the *approximation property* (AP) if for every compact $K \subset Y$ and every $\varepsilon > 0$ there exists a linear operator T of finite rank such that $\|Tx - x\| < \varepsilon,\ x \in K$. If in addition, T can be chosen with $\|T\| \leq \lambda$, for some constant λ independent of K and ε, then Y is said to have the *bounded approximation property* (BAP). To complete the line (AP) $\Longleftarrow$ (BAP) denote the property of a Banach space *to have a basis* by (BP). Thus we have

$$(AP) \Longleftarrow (BAP) \Longleftarrow (BP).$$

All these properties are different (see [FJ] and [Sz]). However, the following interesting connection between the BAP and the BP was established in [P1] and [JRZ]

Theorem 1.1. *A separable Banach space E has the BAP iff E is isomorphic to a complemented subspace of a Banach space with basis.*

Actually the space with basis in this theorem may be chosen to be universal for all spaces with the BAP (see [P2] and [Sc]).

In this paper we establish the following connection between the AP and the BP.

Theorem 1.2. *For a Banach space X the following properties are equivalent*
(i) X has the AP.
(ii) For each compact subset K of X there exist a Banach space Y with basis and a one-to-one linear operator $T : Y \to X$ such that $T(Y) \supset K$.
(iii) There exists a reflexive Banach space R with basis and with unconditional finite-dimensional decomposition such that for each compact $K \subset B_X$ and for each $\gamma > 0$ there is a compact one-to-one operator $T : R \to X$ with $T(B_R) \supset K$ and $\|T\| \leq 1+\gamma$.

The condition that T is one-to-one is essential. Indeed, it is not difficult to show that *each compact set in every Banach space* may be covered by an operator range of ℓ_1.

In Section 2 we prove that space R in (iii) may be chosen to be universal for all spaces with AP (see Theorem 2.1).

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Section 3 contains the "converse result" and a characterization of the AP in terms of the existence of an M-basis with special properties. Here we recall that a sequence $\{x_n, x_n^*\}_{n=1}^\infty$ in $X \times X^*$ is *biorthogonal* provided $x_n^*(x_m) = \delta_{nm}$ for all n and m. A biorthogonal sequence $\{x_n, x_n^*\}_{n=1}^\infty$ is *fundamental* provided the linear span of $\{x_n\}_{n=1}^\infty$ is dense in X and is *total* provided $\{x_n^*\}_{n=1}^\infty$ separates the points of X. An *M-basis* is a fundamental and total biorthogonal sequence. A basis can then be thought of as a biorthogonal sequence $\{x_n, x_n^*\}_{n=1}^\infty$ such that for every x in X, the series $\sum x_n^*(x)x_n$ converges to x, so that a basis is an M-basis.

With the help of the results proved in Sections 2 and 3 we get (in Section 3) a factorization theorem for compact operators acting into spaces with the AP (see Theorem 3.3 below).

In Section 4 we deal with $\mathcal{L}_p$-spaces. We characterize these spaces in terms of covering compact sets and in terms of the factorization of compact operators. We also classify those Banach spaces X in which every compact set is contained in the closed, convex, symmetric hull of a basic sequence which converges to zero.

We use standard Banach spaces geometry notation (see [LT]). If X is a Banach space and $A \subset X$ then $[A]$ is the closed linear span of A. Let $H \subset X$ be a closed, convex, symmetric, and bounded subset of a Banach space X. Then it is well-known that the linear submanifold span$H$ becomes a Banach space if we introduce in it the norm with the unit ball H. We denote this later Banach space by E_H.

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2. Covering a compact set in a Banach space with the AP

Theorem 2.1. *There exists a reflexive Banach space R with basis and with unconditional finite-dimensional decomposition such that for each Banach space X with AP, for each compact $K \subset B_X$ and for each $\gamma > 0$ there is a compact one-to-one operator $T : R \to X$ with $T(B_R) \supset K$ and $\|T\| \leq 1 + \gamma$.*

We start with some auxiliary results.

Lemma 2.2. *Let $\{E_i\}$ be a sequence of finite-dimensional subspaces of a Banach space X and $\{C_i\}$ be a sequence of linear operators on X with $C_i(X) \subset E_i$, $i = 1, 2, \ldots$. Let Y be the set of all sequences $y = (x_i)_{i=1}^\infty$, $x_i \in E_i$, such that the series $\sum x_i$ converges, and let Z be the set of all vectors $x \in X$ such that $x = \sum_{i=1}^\infty C_i x$. Then Y is a Banach space with monotone finite-dimensional decomposition (FDD) in the norm $\|y\|' = \sup\{\|\sum_{i=1}^n x_i\| : n = 1, 2, \ldots\}$ and Z is a Banach space in the norm $|||x||| = \sup_n \|\sum_{i=1}^n C_i x\|$, $x = \sum_{i=1}^\infty C_i x$. Moreover the operator $J : Z \to Y$ defined by $Jx = (C_i x)$, $x \in Z$ is an isometry.*

Proof. First we prove that Y is complete. Let $\{y_m\} \subset Y$, $y_m = (x_i^m)$, be such that $\lim_{p,q\to\infty} \|y_p - y_q\|' = 0$. Clearly, for each i the sequence $\{x_i^m\}_{m=1}^\infty$ converges to some $x_i \in E_i$. First we check that the series $\sum x_i$ converges. Fix $\varepsilon > 0$ and find an integer s such that for each $p, q \geq s$ the inequality $\|y_p - y_q\|' < \varepsilon$ holds. Let r be such that $\|\sum_{i=l}^{k} x_i^s\| < \varepsilon$ for each $k > l > r$. We have for $k > l > r$

$$\|\sum_{i=l}^{k} x_i\| = \lim_p \|\sum_{i=l}^{k} x_i^p\| \leq \lim_p \|\sum_{i=l}^{k} x_i^s\| + \lim_p \|\sum_{i=l}^{k}(x_i^p - x_i^s)\| < 3\varepsilon.$$

Thus $y = (x_i) \in Y$. Finally we show that $\lim y_m = y$. Let ε and s be as above. Then for each n and for each $p, q > s$ we have $\|\sum_{i=1}^{n}(x_i^p - x_i^q)\| \leq \varepsilon$. By pushing q to infinity we have $\|\sum_{i=1}^{n}(x_i^p - x_i)\| \leq \varepsilon$, for each n and for each $p > s$. By taking the supremum we have for each $p > s$, $\|y_p - y\|' \leq \varepsilon$.

Next we prove that Z is complete. Put $B_n = \sum_{i=1}^{n} C_i,\ n = 1, 2, ...,\ B_0 = Id_X$. It is well-known that the set W of all vectors $x \in X$ such that the $\lim_n B_n x$ exists, is a Banach space in the norm

$$|||x||| = \sup\{\|B_n x\| :\ n = 0, 1, 2, ...\}.$$

Clearly, $Z \subset W$. We check that Z is closed in W. Let $\{x_m\} \subset Z$, $|||.||| - \lim x_m = x,\ x \in W$ and let $\varepsilon > 0$. Take m so large that $|||x_m - x||| = \sup\{\|B_n x_m - B_n x\| : n = 0, 1, 2, ...\} < \varepsilon$. Clearly, $\|x_m - x\| < \varepsilon$ too. Next by using that $x_m \in Z$, find an n_0 such that for each $n > n_0$ holds $\|x_m - B_n x_m\| < \varepsilon$. We have for $n > n_0$

$$\|x - B_n x\| \leq \|x - x_m\| + \|x_m - B_n x_m\| + \|B_n x_m - B_n x\| < 3\varepsilon$$

which proves that $\lim_n B_n x = x$, i.e. $x \in Z$.

The remaining part of the lemma is clear.

Remark 2.3. If for each i, $\dim E_i = 1$, then the space Y has a monotone basis.

Lemma 2.4. *Let E be a Banach space with monotone FDD and let $K \subset B_E$ be compact. Then for every $\varepsilon > 0$ there is a compact one-to-one operator $A : E \to E$ with $A(B_E) \supset K$ and $\|A\| \leq 1 + \varepsilon$.*

Proof. Let $S_n,\ n = 1, 2, ...$ be the partial sum operators associated with to a given FDD $E = \sum E_i$. Take $\delta > 0$ with $(1 + \delta)^2 < 1 + \varepsilon$ and let a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ be such that $\sum \varepsilon_i < \delta/2$. By using the compactness of K, find an increasing sequence $\{n_i\}$ of integers such that

$$\sup\{\|S_{n_{i+1}} y - S_{n_i} y\| :\ y \in K\} < \varepsilon_i^2,\ \ i = 1, 2, ...$$

Put

$$Q_0 = S_{n_1},\ Q_i = S_{n_{i+1}} - S_{n_i},\ i = 1, 2, ...,\ A_1 = \sum_{i=0}^{\infty} \varepsilon_i Q_i,\ \varepsilon_0 = 1.$$

Clearly, A_1 is a compact one-to-one operator with $\|A_1\| \leq 1 + \delta$. If $y \in K$ then the series $\sum_{i=0}^{\infty}(1/\varepsilon_i) Q_i y$ converges absolutely (say to z). It is not difficult to check that $\|z\| \leq 1 + \delta$ and that $A_1 z = y$. Thus $A_1((1 + \delta)B_E) \supset K$. Put $A = (1 + \delta)A_1$.

Remark 2.5. Let $H = \{y \in B_E :\ y = \sum_{i=0}^{\infty} y_i,\ y_i \in Q_i(E),\ \|y_i\| \leq \varepsilon_i,\ i = 0, 1, ...\}$. It is clear from the proof of the lemma that $A(H) \supset K$.

Lemma 2.6. *Let $L, M \subset Y$ be subspaces of a separable Banach space Y such that $\dim M = \infty$ and $L \cap M = \{0\}$, and let $F \subset Y$ be compact. Then there is an automorphism $D : Y \to Y$ such that $L \cap D(F) \subset \{0\}$ and $D|_M = Id_M$.*

Proof. Let $\{f_i\} \subset B_{M^\perp}$ be a sequence which is total over L. Put $V = \operatorname{cl}\operatorname{co}(B_L \cup \pm F)$. The linear manifold span$V$, being the image of E_V under the injection of E_V into Y, is an operator range, and clearly codim span$V = \infty$. By [Fo1], Lemma 3, there is a minimal (even basic) sequence $\{t_i\} \subset Y$ with $\|t_i\| < 2^{-i-1}$ and such that $\sum a_i t_i \notin$ span$V$ for each non-zero bounded sequence of numbers $\{a_i\}$. Define an operator $C : Y \to Y$ by $Cx = x + \sum f_i(x)t_i$. Clearly, C is an isomorphism of Y onto Y and $C|_M = Id_M$. We check that $C(L) \cap F \subset \{0\}$. Let $u \in C(L) \cap F$. Then for some $z \in L$ we have $u = z + \sum f_i(z)t_i \in F$ and hence $\sum f_i(z)t_i = u - z \in$ span$V$. By the choice of $\{t_i\}$ we have $f_i(z) = 0,\ i = 1, 2, ...,$ and, since $\{f_i\}$ is total over L, it follows that $z = 0$. Thus $u = 0$ which proves that $C(L) \cap F \subset \{0\}$ and hence $L \cap C^{-1}(F) \subset \{0\}$. Put $D = C^{-1}$. This completes the proof.

The following lemma is known [P1]. We give a proof for the sake of completeness.

Lemma 2.7. *For any finite-dimensional Banach space L and for any $\varepsilon > 0$ there is a finite-dimensional Banach space M with monotone basis which contains a $(1+\varepsilon)$-isomorphic copy of L as a $(1 + \varepsilon)$-complemented subspace.*

Proof. Let $\{x_i\}_{i=1}^n$ and $\{x_i^*\}_{i=1}^n$ be an Auerbach system for L, $n = \dim L$. Take an integer p with $n/p < \varepsilon$ and define linear operators U_i in L as follows

$$U_i x = \frac{1}{p} x_r^*(x)x_r,\ i = nl + r,\ 1 \le r \le n,\ l = 0, 1, ..., p - 1.$$

Put $m = pn$. Clearly, $\sum_{j=1}^m U_j = Id_L$. For $i = nl + r$ we have the estimate

$$\|\sum_{j=1}^{i} U_j\| = \|\sum_{j=1}^{nl} U_j + \sum_{j=nl+1}^{nl+r} U_j\| \le l/p + r/p \le (p - 1 + n)/p \le 1 + \varepsilon.$$

Let $E_i = \operatorname{span}\{y_i\}$, $\|y_i\| = 1,\ i = 1, ..., m$, be the (1-dimensional) range of U_i (actually $y_i = x_r$ for $i = nl+r,\ 1 \le r \le n,\ l = 0, 1, ..., p-1$). Define M as the space of all m-dimensional vectors $y = (a_i)_{i=1}^m$ with the norm $\|y\| = \max_{1\le q\le m} \|\sum_{i=1}^q a_i y_i\|$. It is clear that the vectors $e_i = (0, ..., 1, ..., 0)$, (1 stands on the i-th place), $i = 1, ..., m$, form a monotone basis of M. Define an operator $A : L \to M$ by $Ax = (a_i)_{i=1}^m$, where a_i is defined by $U_i x = a_i y_i,\ i = 1, ..., m$. Put $L_1 = A(L)$. From the construction is clear that L is $(1 + \varepsilon)$-isomorphic to L_1. Finally define an operator $P : M \to L_1$ as follows:

$$P(a_i) = A(\sum a_i y_i),\ (a_i) \in M.$$

It is not difficult to see that P is a projection and $\|P\| \le 1 + \varepsilon$.

Proof of Theorem 2.1. Take $\varepsilon > 0$ and a sequence of positive numbers $\{\varepsilon_i\}_{i=0}^\infty$, $\varepsilon_0 = 1$, with $\sum_{i=1}^\infty \varepsilon_i < \varepsilon$.

Let $K \subset B_X$ be a compact subset of a Banach space X with the AP. It is well-known (see [LT], Proposition 1.e.2) that there is a sequence $\{x_i\} \subset X$ with $\lim x_i = 0$ and such that $\operatorname{cl}\operatorname{co}\{x_i\} \supset K$. By a slight modification of the proof given in [LT], Proposition 1.e.2, we can get that $\{x_i\} \subset (1 + \varepsilon)B_X$. Let a sequence of positive numbers $\{\lambda_i\}$ be such that $\lim \lambda_i = \infty$, $\{\lambda_i x_i\} \subset (1 + \varepsilon)B_X$, and $\lim \lambda_i x_i = 0$. Put $K_0 = \operatorname{cl}\operatorname{co}\{\pm\lambda_i x_i\}$.

Since X has the AP there is a sequence $\{B_n\}$ of finite-dimensional operators in X such that

$$\sup\{\|B_n x - x\| : x \in K_0\} < \varepsilon_n, \ n = 1, 2, ...$$

Put $C_1 = B_1, \ C_{n+1} = B_{n+1} - B_n$, $n = 1, 2, ...$ Then for each $x \in K_0, \ x = \sum_{n=1}^{\infty} C_n x$ and

$$\sup_{1 \le n < \infty} \sup\{\| \sum_{i=1}^{n} C_i x\| : x \in K_0\} \le 1 + 2\varepsilon. \tag{2.1}$$

Next for each i put $E_i = C_i(X)$ and define the spaces Y and Z as in Lemma 2.2. In a natural way we can consider K_0 as a subset of Z. Thus, from 2.1 we have

$$\sup\{|||x||| : x \in K_0\} \le 1 + 2\varepsilon.$$

Define a summation operator $B : Y \to X$ as follows:

$$B(x_i) = \sum_{i=1}^{\infty} x_i, \quad (x_i) \in Y.$$

Clearly, $\|B\| = 1$. Put $L = \mathrm{Ker} B$.

Since $\lambda_i \to \infty$ and since K_0 is bounded in Z, it easily follows that K considered as a subset of Z is compact. Let $J : Z \to Y$ be the natural embedding, i.e. $Jx = (V_m x)$. By Lemma 2.2, J is an isometry and, in particular $M = J(Z)$ is a (closed) subspace of Y. A simple verification shows that $L \cap M = \{0\}$.

Now we pass to the space R. We use for R the space constructed in [J]. Recall the construction of R. Let $\mathcal{A}_n$ be the family of all n-dimensional Banach spaces with monotone basis, i.e. an n-dimensional space $M \in \mathcal{A}_n$ iff there is a basis $\{x_i\}_{i=1}^n$ of M such that all partial sum operators have norm 1. Let $\{M_{n,i}\}_{i=1}^{\infty}$ be a dense (in the Banach-Mazur metric) sequence in $\mathcal{A}_n, \ n = 1, 2, ...$ Enumerate the set $\{M_{n,i}\}_{n,i=1}^{\infty}$ into a sequence $\{M_n\}_{n=1}^{\infty}$ and put

$$R = (\sum_{n=1}^{\infty} M_n)_{\ell_2}.$$

Clearly, R has a (monotone) basis and an unconditional finite-dimensional decomposition.

Next, by using Lemma 2.4, find a compact one-to-one operator $A : Y \to Y$, $A = \sum_{i=0}^{\infty} \varepsilon_i Q_i, \ \|A\| \le (1 + 2\varepsilon)^2$, such that $A(B_Y) \supset J(K_0)$. Put $L_j = Q_j(Y), \ j = 0, 1, ...$, and by Lemma 2.7 find for each j a space M_{n_j} such that $M_{n_j} = V_j + W_j$, where V_j is $(1 + \varepsilon_j)$-isomorphic to L_j and $(1 + \varepsilon_j)$-complemented in M_{n_j}. Let $T_j : V_j \to L_j$ be an isomorphism satisfying $\|T_j\| \|T_j^{-1}\| \le 1 + \varepsilon_j, \ j = 0, 1, ...$

Now we define a compact 1-1 operator $S : R \to Y$. We first define S on the subspace $V = (\sum_{j=0}^{\infty} V_j)_{\ell_2} \subset R$ as follows: $S(x_j) = \sum_{j=0}^{\infty} \varepsilon_j T_j x_j, \ x_j \in V_j, \ j = 0, 1, ...$ It is not difficult to see that $S|_V$ is a compact 1-1 operator, and $\|S|_V\| \le 1 + \varepsilon$ (recall that $\varepsilon_0 = 1$). By [Pl]; see also [Fo1]; there is a (closed) infinite-dimensional subspace $E \subset Y$ with $E \cap S(V) = \{0\}$. Next define S on the remaining part of R; i. e., on the subspace $W = (\sum_{n \ne n_j} M_n)_{\ell_2} \oplus (\sum_{j=0}^{\infty} W_j)_{\ell_2}$ in such a way that $S|_W$ is a one-to-one compact operator into E with norm so small that $\|S\| \le 1 + 2\varepsilon$.

Put $F = AS(B_R)$. It is clear from the construction of S and Remark 2.5 that $F \supset K$.

By using Lemma 2.6 find an automorphism $D : Y \to Y$, $\|D\| \le 1+\varepsilon$ such that $L \cap D(F) = \{0\}$ and $D_{|_M} = Id_M$. Put $T = BDAS$. Then $T : R \to X$ is a compact one-to-one operator and $T(B_R) \supset K$. From the construction is clear that $\|T\| \le (1+2\varepsilon)^3$. Clearly, for $\varepsilon > 0$ small enough we get $\|T\| \le 1+\gamma$ which completes the proof.

Remark 2.8. Let U be a Banach space with basis that contains (isomorphically) each Banach space with basis as a complemented subspace (for constructions of such spaces see [P2] and [Sc]). We show that such a space U may be used in Theorem 2.1 instead of R. First, by an obvious modification of the proof of Theorem 2.1, we can get that $T^{-1}(K)$ is compact. Clearly, $R \subset U$. Let $P : U \to R$ be a projection from U onto R and $A_1 : U \to U$ a compact one-to-one operator such that $A_1(B_U) \supset T^{-1}(K)$ (see Lemma 2.4). Put $F_1 = \mathrm{cl}\, A_1(B_U)$, $L_1 = \mathrm{Ker}\, P$ and let $D_1 : U \to U$ be an automorphism such that $L_1 \cap D_1(F_1) = \{0\}$ and $D_1|_R = Id_R$ (see Lemma 2.6). Then the operator $T_1 = TPDA : U \to X$ is a one-to-one compact operator such that $T_1(B_U) \supset K$.

Remark 2.9. Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of compact sets in a Banach space X with the AP. Put $K = \cup_{n=1}^{\infty}(n \max\{\|x\| :\ x \in K_n\})^{-1} K_n$. Clearly, $K \cup \{0\}$ is compact and, by Theorem 2.1, there is a compact operator $T : R \to X$ with $T(B_R) \supset K$. In particular, $T(R) \supset \cup_{n=1}^{\infty} K_n$.

Lemma 2.10. *Let R be a reflexive Banach space with basis and $T : R \to X$ be a 1-1 linear operator from R into a separable Banach space X. Then there are a basis $\{y_i\}$ of R and a norming M-basis $\{x_i\}$ of X such that $\{Ty_i\} \subset \{x_i\}$.*

Proof. Recall that a subspace Y of X^* is called *norming* provided the expression $|||x||| := \sup\{f(x) : f \in Y, \|f\| \le 1\}$ defines an equivalent norm on X. An M-basis $\{x_n, x_n^*\}_{n=1}^{\infty}$ is norming provided $[x_n^*]$ is norming.

First we assume that $[T(R)] = X$. Let $F \subset X^*$ be a countable-dimensional norming subspace of X^*. Since R is reflexive and T is 1-1, it follows that $T^*(F)$ is a countable-dimensional dense subspace of R^*. Clearly, R^* has a basis. Now we use the following result from [T]:

Let E be a Banach space with basis and $L \subset E$ a countable-dimensional dense subspace of E. Then there is a basis $\{z_n\}$ of E with $\mathrm{span}\{z_n\} = L$.

Take $E = R^*$ and $L = T^*(F)$ and let $\{g_n\}$ be a basis of R^* with $\mathrm{span}\{g_n\} = T^*(F)$. Let $\{e_n\}$ be the vectors in R which are biorthogonal to $\{g_n\}$. It is clear that $x_n = Te_n,\ n = 1, 2, ...$, is a norming M-basis of X with the desired properties.

If $X_1 = [T(R)]$ is a proper subspace of X we first construct as above a norming M-basis of X_1 and then extend it (see [Fo2]) to a norming M-basis of X. The proof is complete.

Corollary 2.11. *Let X be a separable Banach space with the AP. Then for each compact set $K \subset X$ there is a norming M-basis $\{x_i\}$ in X with biorthogonal functionals $\{x_i^*\}$ such that for each $x \in K$ we have*

$$x = \sum_{i=1}^{\infty} x_i^*(x) x_i \quad \text{and} \quad \sup_n \sup_{x \in K} \| \sum_{i=1}^{n} x_i^*(x) x_i \| < \infty.$$

Proof. By Theorem 2.1 there is a linear 1-1 operator $T : R \to X$ from a reflexive space R with basis into X such that $T(B_R) \supset K$. Apply Lemma 2.10 and complete the proof.

Remark 2.12. Let μ be a probability measure in a separable Banach space X. An M-basis $\{x_i\}$ for X (with biorthogonal functionals $\{x_i^*\}$) is called *a stochastic basis* for (X, μ) if $\mu \{x \in X : x = \sum x^*(x) x_i\} = 1$, (see [He]). It follows from Remark 2.9 and Corollary 2.11 that if a Banach space X has the AP then for each probability measure μ, (X, μ) has a stochastic basis.

Remark 2.13. In view of Theorem 2.1 the following question is natural: is it possible to substitute for the space R in Theorem 2.1 a space with an unconditional basis? The following example shows that the answer is negative.

By [GL], for each integer n there is a finite-dimensional space E_n such that for each Banach space Y with 1-unconditional basis and for any two operators $A : E_n \to Y$ and $B : Y \to E_n$ with $BA = Id_{E_n}$, the inequality $\|A\| \|B\| \geq n$ holds. Put $X = (\sum E_n)_{\ell_2}$ and let X_1 be a copy of X. Let $T : X \to X_1$ be a linear operator such that $T|_{E_n} = Id_{E_n} / \sqrt{n}$, $n = 1, 2, ...$ Set $K = T(B_X)$. Clearly, K is a compact set. We check that K cannot be covered by a one-to-one operator range of a Banach space with unconditional basis. Suppose the contrary, i.e., for some Banach space Y with unconditional basis and for some operator $B : Y \to X_1$, K is a subset of $B(Y)$. Consider the operator $A = B^{-1} T : X \to Y$. A simple verification shows that A is a closed operator, so A is bounded by the closed graph theorem. Clearly, $T = BA$ and $\sqrt{n} T|_{E_n} = Id|_{E_n} = \sqrt{n} BA|_{E_n}$ for each n. By taking into account the property of the space E_n we deduce that $\sqrt{n} \|A|_{E_n}\| \|B\| \geq n$. In particular, $\|A\| \geq \sqrt{n} / \|B\|$ for each n which contradicts the boundedness of A.

3. The converse result

Theorem 3.1. *Let X be a Banach space such that for each compact $H \subset X$ there are a Banach space Y with the AP and a one-to-one operator $T : Y \to X$ such that $TY \supset H$. Then X has the AP.*

Proof. Fix a compact $K \subset X$ and $\varepsilon > 0$. We shall construct a finite-dimensional operator C in X such that $\sup\{\|Cx - x\| : x \in K\} < \varepsilon$.

By a well-known result (see [LT], Proposition 1.e.2) there is a sequence $\{x_i\} \subset X$ such that $\lim x_i = 0$ and that $\text{cl co}\{x_i\} \supset K$. Take a sequence of numbers $\{\lambda_i\}$ with $\lim \lambda_i = \infty$ and such that $\lim \lambda_i x_i = 0$ and put $H = \text{cl co}\{\pm \lambda_i x_i\}$. Let $A : E_H \to X$ be the natural injection (recall that E_H is the Banach space span$H$ with the unit ball H). Let $T : Y \to X$ be a one-to-one linear operator from a Banach space Y with the AP such that $T(Y) \supset H$. A simple verification shows that the operator $T^{-1}A : E_H \to Y$ is closed and by the closed graph theorem it is bounded. In particular, the set $T^{-1}(H)$ is bounded. From $\lim \lambda_i = \infty$ it easily follows that the set $F = T^{-1}(K)$ is compact. Next by using the AP of Y find a finite-dimensional operator $B : Y \to Y$,

$$By = \sum_{i=1}^{n} f_i(y)e_i, \quad f_i \in Y^*, \ e_i \in Y, \ y \in Y$$

such that $\sup\{\|By - y\| : \ y \in F\} < \varepsilon/2\|T\|$.

Recall that the bw^*-topology on Y^* is just the topology of uniform convergence on the compact subsets of Y. By the Krein-Smulian theorem the bw^*-closure of each convex subset of Y^* coincides with its w^*-closure. In particular, $bw^* - \mathrm{cl}T^*(X^*) = w^* - \mathrm{cl}T^*(X^*)$. Since T is one-to-one it follows that $w^* - \mathrm{cl}T^*(X^*) = Y^*$ and hence $bw^* - \mathrm{cl}T^*(X^*) = Y^*$. The last equality allows us to find for each i, $1 \le i \le n$, a linear functional $g_i \in T^*(X^*)$ such that

$$\sup\{|g_i(y) - f_i(y)| : \ y \in F\} < \varepsilon/(2n\|e_i\|\|T\|).$$

Take $h_i \in X^*$ such that $T^*h_i = g_i$, $i = 1, ..., n$, and define an operator $C : X \to X$ by

$$Cx = \sum_{i=1}^{n} h_i(x)Te_i, \quad x \in X.$$

We check that C is as desired. Take $x \in K$, put $y = T^{-1}x \in F$, and write

$$\|Cx - x\| = \|\sum_{i=1}^{n} h_i(x)Te_i - x\| = \|T(\sum_{i=1}^{n} h_i(x)e_i - T^{-1}x)\| \le \|T\|\|\sum_{i=1}^{n} g_i(y) - y\| =$$

$$\|T\|\|\sum_{i=1}^{n} f_i(y) - y + \sum_{i=1}^{n}(g_i(y) - f_i(y))e_i\| \le \|T\|(\varepsilon/2\|T\| + \sum_{i=1}^{n} |g_i(y) - f_i(y)|\|e_i\|) < \varepsilon.$$

The proof is complete.

Alternate proof. By a theorem of Grothendieck (see Theorem 1.e.4 in [LT]), it is enough to verify that an arbitrary compact operator S from an arbitrary Banach space Z into X is (uniformly) approximable by finite dimensional operators. The operator S factors compactly through some Banach space W (W can even be taken reflexive; see e. g. [F]); that is, there are compact operators $A : Z \to W$, $D : W \to X$ so that $DA = S$. Let T be a one-to-one operator from some space Y which has the AP into X so that TY contains the closure of $D(B_W)$. The linear mapping $T^{-1}D$ is closed, hence is a bounded linear operator. Moreover, $T^{-1}DA : Z \to Y$ is a compact operator into a space with the AP, hence is approximable by finite dimensional operators, whence $S = T(T^{-1}DA)$ is also approximable by finite dimensional operators.

Corollary 3.2. *For a Banach space X the following assertions are equivalent:*
(i) X has the AP.
(ii) Any compact set $K \subset X$ may be covered by a 1-1 operator range of a Banach space with basis.
(iii) For any compact set $K \subset X$ there is an M-basic sequence $\{x_i\}$ in X (with biorthogonal functionals $\{x_i^*\}$) such that $x = \sum x_i^*(x)x_i$ for each $x \in K$.

With the help of the above results we get the following factorization

Theorem 3.3. *For a Banach space X the following assertions are equivalent:*
(i) X has the AP.
(ii) For each Banach space Z and for each compact operator $A : Z \to X$ there are compact operators $B : Z \to R$ and $T : R \to X$ such that T is one-to-one and that $A = TB$. (Here $R = (\sum_{n=1}^{\infty} M_n)_{\ell_2}$ is the space from Theorem 2.1.)
(iii) For each Banach space Z and for each compact operator $A : Z \to X$ there are a Banach space Y with AP and compact operators $B : Z \to Y$ and $T : Y \to X$ such that $A = TB$.

Proof. $(i) \Longrightarrow (ii)$. Put $K = \mathrm{cl}A(B_Z)$ and as it was done in the proofs of Theorems 2.1 and 3.1, find $\{x_i\}_{i=1}^{\infty} \subset X$, $\lim_i x_i = 0$ with $\mathrm{cl}\,\mathrm{co}\{x_i\}_{i=1}^{\infty} \supset K$ and a sequence of numbers $\{\lambda_i\}_{i=1}^{\infty}$, $\lim_i \lambda_i = \infty$ such that $\lim_i \lambda_i x_i = 0$. Put $K_1 = \mathrm{cl}\,\mathrm{co}\{\lambda_i x_i\}_{i=1}^{\infty}$ and by using Theorem 2.1 find a one-to-one compact operator $T : R \to X$ $T(aB_R) \supset K_1$ for some $a > 0$. Put $B = T^{-1}A$ and check that B is a compact operator. Indeed, since $T^{-1}(K_1)$ is bounded it follows that $T^{-1}(K)$ is compact. But $B(B_Z) = T^{-1}A(B_Z) \subset T^{-1}(K)$ which proves that B is a compact operator and completes the proof of $(i) \Longrightarrow (ii)$.
$(ii) \Longrightarrow (iii)$. Obvious.
$(iii) \Longrightarrow (i)$. This is immediate from the result of Grothendieck (Theorem 1.e.4 in [LT]) used in the alternate proof of Theorem 3.1.

4. Covering compact sets in $\mathcal{L}_p$-spaces and factorizing compact operators into $\mathcal{L}_p$-spaces

As we already mentioned in Introduction, any compact subset of any Banach space may be covered by an operator range of ℓ_1. In this section we prove that every compact subset of a Banach space X may be covered by a *one-to-one* range of ℓ_1 if and only if X is an $\mathcal{L}_1$-space (actually we prove a much stronger result, see Theorem 4.7 below). Also we get a characterization of $\mathcal{L}_p$-spaces in terms of covering compact sets and in terms of the factorization of compact operators.

We start with with two auxiliary results.

Lemma 4.1. *For each natural number n there is $m(n) < \infty$ so that if E is an n-dimensional space which is K-isomorphic to a K-complemented subspace of L_p, $1 \le p \le \infty$, and E has a basis with constant K, then $E \oplus \ell_p^{m(n)}$ is $f(K)$-isomorphic to $\ell_p^{n+m(n)}$.*

Proof. By [PR], E is $2K$-isomorphic to a $2K$-complemented subspace of ℓ_p^m, where m depends only on n. Then by [BDGJN], $E \oplus_p \ell_p^{mn}$ is $f(K)$-isomorphic to $\ell_p^{(m+1)n}$.

The next lemma very slightly improves the results in [JRZ] and [NW] that a separable $\mathcal{L}_p$ space has a basis with certain desirable properties.

Lemma 4.2. *Let X be a separable $\mathcal{L}_p$ space, $1 \le p \le \infty$. Then X has a basis $\{x_n\}_{n=1}^\infty$ which satisfies*

$$\sup_{1\le n<m<\infty} d(\ell_p^{m-n}, \operatorname{span}\{x_n, x_{n+1}, \dots, x_{m-1}\}) < \infty.$$

Proof. Note that by Lemma 4.1 it is enough to show that for any sequence $0 = n_1 < n_2 < \dots$, there is a basis $\{x_n\}_{n=1}^\infty$ for X and a constant K so that for all $k = 1, 2, \dots$, $\{x_i\}_{i=n_k+1}^{n_{k+1}}$ is K-equivalent to the unit vector basis for $\ell_p^{n_{k+1}-n_k}$.

First we indicate how to get such a basis when $p < \infty$. In this case X has a complemented subspace which is isomorphic to ℓ_p, and hence X is isomorphic to $X \oplus \ell_p$ since for some Y we have $X \sim Y \oplus \ell_p \sim Y \oplus \ell_p \oplus \ell_p \sim X \oplus \ell_p$. Let $\{y_n\}_{n=1}^\infty$ be a basis for X and let $\{e_n\}_{n=1}^\infty$ be the unit vector basis for ℓ_p. Make a basis for $X \oplus \ell_p$ by interlacing these bases for X and ℓ_p; i. e., use $\{y_1 \oplus 0, 0 \oplus e_1, 0 \oplus e_2, \dots, 0 \oplus e_{m_1}, y_2 \oplus 0, 0 \oplus e_{m_1+1}, 0 \oplus e_{m_1+2}, \dots, 0 \oplus e_{m_2}, \dots\}$. As long as m_k grows fast enough, this will be a basis that has the desired property.

The same argument works when X is a separable $\mathcal{L}_\infty$ space which contains a (necessarily complemented) isomorphic copy of c_0, but to do the general case, one needs to "localize" the construction. Again, let $\{y_n\}_{n=1}^\infty$ be a basis for X. Let $\epsilon > 0$; $\epsilon = 1$ is fine for our purposes. For appropriate m_1, take a sequence $\{e_i^1\}_{i=1}^{m_1}$ which is $1+\epsilon$-equivalent to the unit vector basis of $\ell_\infty^{m_1}$ and which lies in the linear span of $\{y_n\}_{n=1}^\infty$; say, in $\operatorname{span}\{y_1, \dots, y_{s_1}\}$. Let E_1 be the linear span of $\{e_I^1\}_{i=1}^{m_1}$ and write $\operatorname{span}\{y_1, \dots, y_{s_1}\} = E_1 + F_1$, where F_1 is the kernel of a norm $1+\epsilon$ projection from $\operatorname{span}\{y_1, \dots, y_{s_1}\}$ onto E_1. The vectors $e_1^1, \dots, e_{m_1}^1$ form the first m_1 terms of our basis.

For the construction of the next m_2 basis vectors (where m_2 is appropriately large, depending on s_1 as well as the sequence $\{n_k\}_{k=1}^\infty$), take a subspace E_2 of the linear span of $\{y_i\}_{i=s_1+1}^{s_2}$ for some s_2 so that E_2 is $1+\epsilon$-isomorphic to the ℓ_∞ space of its dimension. As long as its dimension is at least m_1, the space $F_1 + E_2$ is isomorphic to the ℓ_∞ space of its dimension (which we take to be m_2) with isomorphism constant depending only on the basis constant of $\{y_n\}_{n=1}^\infty$. Let $\{e_i^2\}_{i=1}^{m_2}$ be a basis for $F_1 + E_2$ which is $d(F_1 + E_2, \ell_\infty^{m_2})$-equivalent to the unit vector basis for $\ell_\infty^{m_2}$; these vectors form the next m_2 terms of the desired basis for X.

Now just iterate this construction. Let F_2 be the kernel of a projection of norm $d(F_1 + E_2, \ell_\infty^{m_2})$ from the linear span of $\{y_i\}_{i=s_1+1}^{s_2}$ onto the linear span of $\{e_i^2\}_{i=1}^{m_2}$. Then select a subspace E_3 of the linear span of $\{y_i\}_{i=s_2+1}^{s_3}$ for some s_3 so that E_3 is $1+\epsilon$-isomorphic to the ℓ_∞ space of its (appropriately large) dimension and continue as in the previous step. It is evident that for any $\{n_k\}_{k=1}^\infty$ this construction can produce a basis $\{x_n\}_{n=1}^\infty$ which has the property mentioned in the first paragraph of the proof.

Theorem 4.3. *The following are equivalent for a Banach space X.*

1. X is a $\mathcal{L}_p$ space.
2. Every compact operator into X factors through some $\mathcal{L}_p$ space.
3. If T is a compact operator from some Banach space Z into X and Y is a separable $\mathcal{L}_p$ space then there are compact operators A from Z into Y and B from Y into X so that $T = BA$ and B is one-to-one.

Remark 4.4. The equivalence of (1) with (2) is known; see [J].

Proof. We show that (1) implies (3). First assume that X is a separable $\mathcal{L}_p$ space and let $T : Z \to X$ be a compact operator; without loss of generality $\|T\| \le 1$. Take a basis $\{x_n\}_{n=1}^\infty$ for X which has the property given in Lemma 4.2. Let $\{S_n\}_{n=1}^\infty$ be the partial sum projections associated with $\{x_n\}_{n=1}^\infty$ and set $S_0 = 0$. Take $0 = n_0 < n_1 < \dots$ so that $\|(S_{n_k} - S_{n_{k-1}})T\| < 4^{-k}$ for each $k > 1$. Now let Y be any separable $\mathcal{L}_p$ space and let $\{y_n\}_{n=1}^\infty$ be a basis for Y which also has the property given in Lemma 4.2. Thus for each k, the spaces $\operatorname{span} y_{n_{k-1}+1}, \dots, y_{n_k}$ and span $\operatorname{span} x_{n_{k-1}+1}, \dots, x_{n_k}$ are isomorphic with isomorphism constant independent of k. Next we use an idea similar to one used in the proof of Lemma 2.4. Take a norm one isomorphism U_k from $y_{n_{k-1}+1}, \dots, y_{n_k}$ onto $\operatorname{span} x_{n_{k-1}+1}, \dots, x_{n_k}$ so that $\sup_k \|U_k\| < \infty$. Define $A : Z \to Y$ by $Az = \sum_{k=1}^{\infty} 2^{k-1} U_k^{-1}(S_{n_k} - S_{n_{k-1}})T$. Let $\{P_n\}_{n=1}^\infty$ be the partial sum projections associated with the basis $\{y_n\}_{n=1}^\infty$. Define $B : Y \to X$ by $By = \sum_{k=1}^{\infty} 2^{-k+1} U_k(S_{n_k} - S_{n_{k-1}})y$. This does it.

If X is a non separable $\mathcal{L}_p$ space, we just replace X with any separable $\mathcal{L}_p$ subspace which contains the range of T and proceed as above.

The implication (3) implies (2) is trivial, so we turn to the implication (2) implies (1). First assume $1 < p < \infty$, so that every separable $\mathcal{L}_p$ space is isomorphic to a complemented subspace of $L_p(0,1)$ by [LP]. Thus condition (2) implies that every compact operator into X factors through $L_p(0,1)$. Now the operators from a space Z into X which factor through $L_p(0,1)$ forms a Banach space under the factorization norm $!T! = \inf \|A\| \cdot \|B\|$, where the infimum is over all factorizations $T = BA$ with $A : Z \to L_p(0,1)$, $B : L_p(0,1) \to X$ (use the fact that $L_p(0,1)$ is isometric to the ℓ_p sum of itself). Thus the uniform boundedness principle yields that there is a constant $K = K_Z$ so that $!T! \le K\|T\|$ for all compact T from Z to X. Applying this with Z the ℓ_1 sum of all finite dimensional subspaces of X, we conclude that there is a constant K so that the inclusion from every finite dimensional subspace of X into X factors through $L_p(0,1)$ with factorization constant at most K. This implies that X is a $\mathcal{L}_p$ space.

This argument does not quite work for $p = 1$ or $p = \infty$ since not every separable $\mathcal{L}_1$ space is isomorphic to a complemented subspace of $L_1(0,1)$ and no infinite dimensional separable $\mathcal{L}_\infty$ space is isomorphic to a complemented subspace of $L_\infty[0,1]$. Here is how to get around that annoyance. The argument in $[LP]$ yields that the injection J_Y from a separable $\mathcal{L}_p$ space into its second dual factors through $L_p(0,1)$ in the extreme case $p = 1$. When $p = \infty$, the same is true by the injectivity of the second dual of a $\mathcal{L}_\infty$ space. The argument in the preceding paragragh then yields that the inclusion from every finite dimensional subspace of X into X^{**} factors through $L_p(0,1)$ with factorization constant at most K. This is enough, by [LR], to guarantee that X is a $\mathcal{L}_p$ space.

Remark 4.5. It is evident that condition (3) in Theorem 4.3 is equivalent to:

4. For every compact subset K of X and every separable $\mathcal{L}_p$ space Y there is a one-to-one compact operator S from Y to X so that $S(B_Y) \supset K$.

That the one-to-oneness is essential in Remark 4.5 is given by the following proposition.

Proposition 4.6. *Let $1 \le p < \infty$. The following are equivalent for a Banach space X.*

5. For every compact subset K of X and every $\mathcal{L}_p$ space Y there is a compact operator S from Y to X so that $S(B_Y) \supset K$.

6. For every compact subset K of X there is a $\mathcal{L}_p$ space Y and an operator S from Y to X so that $S(B_Y) \supset K$.

7. X is isomorphic to a quotient of $L_p(\mu)$ for some measure μ.

Proof. The case $p = 1$ is easy because every Banach space is a quotient of $\ell_1(\Gamma)$ for some set Γ and every $\mathcal{L}_1$ space contains a complemented copy of ℓ_1. So assume that $1 < p < \infty$. If condition (6) holds, then the space Y can always be taken separable and hence Y can be taken to be $L_p(0,1)$ because every separable $\mathcal{L}_p$ space is isomorphic to a complemented subspace of $L_p(0,1)$.

We claim that there is a constant C so that every compact subset of the unit ball of X is covered by $S(B_{L_p(0,1)})$ for some operator S from $L_p(0,1)$ into X for which $\|S\| \le C$. Indeed, otherwise there would be compact subsets $K_n \subset B_X$ so that if $K_n \subset S(B_{L_p(0,1)})$ with $S : L_p(0,1) \to X$, then $\|S\| > n^2$. Set $K = \sup_{n=1}^{\infty} n^{-1} K_n \cup \{0\}$. Then K is a compact subset of B_X and so there is an operator $S : L_p(0,1) \to X$ so that $K \subset S(B_{L_p(0,1)})$. Then $(nS)(B_{L_p(0,1)}) \supset K_n$ and so $\|nS\| \ge n^2$ for all n. This proves the claim.

By the claim, there is a C so that for each finite dimensional subspace E of X there is an operator $S_E : L_p(0,1) \to X$ so that $S_E B_{L_p(0,1)} \supset B_E$ and $\|S_E\| \le C$. Direct the finite dimensional subspaces of X by inclusion. Then any weak-operator cluster point of S_E^* defines an isomorphism from X^* into $L_q(0,1)$, where $1/p + 1/q = 1$. We now know X is reflexive, so that X is a quotient of $L_p(0,1)$. This completes the proof that condition (6) implies condition (7).

Finally we assume that (7) is satisfied. Let Q be a quotient map from $L_p(\mu)$ onto X. If K is a compact subset of X, then it is easy too see that some superspace of K in X is the image of some separable L_p subspace of $L_p(\mu)$ and hence of $L_p(0,1)$. That is, without loss of generality we can assume that X is separable and that $L_p(\mu)$ is $L_p(0,1)$. Now by Michael's selection theorem (or the Bartle-Graves theorem) there is a compact subset K_1 of $L_p(0,1)$ so that $QK_1 = K$. By Lemma 2.4, there is a compact operator T on $L_p(0,1)$ so that $TB_{L_p(0,1)} \supset K_1$. Then $S := QT$ is a compact operator from $L_p(0,1)$ into X for which $SB_{L_p(0,1)} \supset K$. This completes the proof of Proposition 4.6.

We have frequently used the fact that every compact subset of a Banach space is contained in the closed convex symmetric hull of a sequence which converges to zero.

What are the Banach spaces for which the null sequence can always be chosen to have some extra property such as being basic or at least countably linearly independent? It is a good exercise for students to show that Hilbert space does not have one of these stronger properties. A complete answer to the question is given by the next theorem.

Theorem 4.7. *The following are equivalent for a Banach space X.*
1. X is a $\mathcal{L}_1$ space.
2. For each compact subset K of X there is a basic sequence which converges to zero and whose closed convex symmetric hull contains K.
3. For each compact subset K of X there is a sequence $\{y_n\}_{n=1}^\infty$ which converges to zero and whose closed convex symmetric hull contains K and such that $0 \neq \sum\limits_{n=1}^\infty \lambda_n y_n$ whenever $0 < \sum\limits_{n=1}^\infty |\lambda_n| < \infty$.

If X is separable, the preceding conditions are equivalent to
4. For each compact subset K of X there is a basis for X which converges to zero and whose closed convex symmetric hull contains K.

Proof. Assume that (1) holds and K is a compact subset of B_X. Let Y be a separable $\mathcal{L}_1$ subspace of X which contains K and let $\{x_n\}_{n=1}^\infty$ be a basis for X_0 which satisfies the conclusion of Lemma 4.3. Let $\{S_n\}_{n=1}^\infty$ be the partial sum projections associated with $\{x_n\}_{n=1}^\infty$ and set $S_0 = 0$. From Lemma 2.4 we get $0 = n_0 < n_1 < \dots$ so that image of B_X under the operator $T := \sum\limits_{k=1}^\infty 4^{-k+1}(S_{n_k} - S_{n_{k-1}})$ contains K. By the property of the sequence $\{x_n\}_{n=1}^\infty$, for each k there is a basis $z_{n_{k-1}}, \dots, z_{n_k}$ for the linear span E_k of $x_{n_{k-1}}, \dots, x_{n_k}$ so that the unit ball of E_k is contained in the convex symmetric hull of $z_{n_{k-1}}, \dots, z_{n_k}$ and such that $\sup_{1\le n<\infty} \|z_n\| := C < \infty$. Let $C_1 := \sup_k \|S_{n_k} - S_{n_{k-1}}\|$. From the property of the operator T we deduce that K is contained in the closed convex symmetric hull of the sequence y_n, where $y_j = 2CC_1 2^{-k+1} z_j$ if $n_{k-1} < j \le n_k$. This completes the proof that (1) implies (2) and that (1) implies (4) when X is separable. The only remaining non trivial implication is that (3) implies (1). But (3) implies that every compact set in X can be covered by the image of the unit ball of ℓ_1 under some one-to-one operator from ℓ_1 into X, so X is a $\mathcal{L}_1$ space by condition (4) in the remark after Theorem 4.3.

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