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POWER TYPE UNIFORM CONVEXITY OF $\mathfrak{X}$ VIA p-ASYMPTOTIC UNIFORM CONVEXITY OF $L_r(\mathfrak{X})$

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Abstract. We show that if $L_r(\mathfrak{X})$, $1 < r < \infty$, has an asymptotically uniformly convex renorming of power type then $\mathfrak{X}$ admits a uniformly convex norm of power type.

1. Introduction

Early in the 1970s P. Enflo [E] characterized superreflexive Banach spaces as those Banach spaces which admit an equivalent uniformly convex norm. A few years later G. Pisier [P] showed that in fact a uniformly convex renorming of 'power type' could be obtained, thus improving Enflo's result and generalizing the corresponding known result for Banach lattices proved in [FJ]. Here power type means that the modulus $\delta_{\mathfrak{X}}(t)$ of uniform convexity of $\mathfrak{X}$ satisfies $\delta_{\mathfrak{X}}(t) \geq K\, t^p$ for some constants $K > 0$ and $p < \infty$ (in this case we say that $\mathfrak{X}$ is p-convex or that the norm in $\mathfrak{X}$ has power type p).

Pisier's argument shows that there is an equivalent uniformly convex norm of power type p on $\mathfrak{X}$ if and only if for some constant C an inequality of the form

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$$\mathbb{E}\|X_0\|^p + \sum_{n\geq 1} \mathbb{E}\|X_n - X_{n-1}\|^p \leq C^p \sup_n \mathbb{E}\|X_n\|^p \tag{1.1}$$

is satisfied for all $\mathfrak{X}$-valued martingales $(X_n)_{n=0}^{\infty}$, and that it is sufficient for the condition to hold if we consider only the so called Walsh-Paley martingales (see Section 2 for definitions). A duality argument shows that the reverse inequality to (1.1) holds if and only if $\mathfrak{X}$ admits a uniformly smooth renorming of power type p which means that the modulus $\rho_{\mathfrak{X}}(t)$ of uniform smoothness of $\mathfrak{X}$ satisfies $\rho_{\mathfrak{X}}(t) \leq K\, t^p$ for some constant $K > 0$.

In this paper we use a certain flexibility inherent in this martingale approach to show that if $L_r(\mathfrak{X})$, $1 < r < \infty$, admits an asymptotically uniformly convex renorming of power type p then the inequality (1.1) holds for Walsh-Paley martingales when $r = p$. Thus, by Pisier's argument, $\mathfrak{X}$ has a uniformly convex norm of power type p. If $r < p$ the same approach is used to obtain Maurey's inequality in page 278 of [B] to show that $\mathfrak{X}$ has a uniformly convex renorming of power type p' for every $p' > p$.

Using Theorem 4 from [DGK], Girardi [G, Observation 8] verified a qualitative version of the previous claim, which we state here as the following Remark:

Remark. ([G]) Let $1 < r < \infty$. For a Banach space $\mathfrak{X}$, the following are equivalent.

(a) $\mathfrak{X}$ is uniformly convexifiable.
(b) $L_r(\mathfrak{X})$ is uniformly convexifiable.
(c) $L_r(\mathfrak{X})$ is asymptotically uniformly convexifiable.
(d) $L_r(\mathfrak{X})$ is asymptotically uniformly smoothable.

If the modulus of uniform convexity in (a) above is assumed to be of power type p then the corresponding modulus in (b) is also of power type p when $r \leq p$ [F].

2. Preliminaries

The notions of asymptotic uniform convexity and smoothness, among others, were studied more than thirty years ago by V. D. Milman, and a number of researchers investigated related concepts in the intervening years. The 'asymptotic' terminology was coined in [JLPS], where the moduli of asymptotic uniform convexity and smoothness are defined as follows:

Definition 1. For $\mathcal{Y}$ a subspace of $\mathfrak{X}$ and $x \in S_{\mathfrak{X}}$ let

$$\overline{\delta}_{\mathfrak{X}}(t, x, \mathcal{Y}) = \inf_{y \in \mathcal{Y}, \|y\| \geq t} \|x + y\| - 1$$

and

$$\overline{\delta}_{\mathfrak{X}}(t, x) = \sup_{\dim \mathfrak{X}/\mathcal{Y} < \infty} \overline{\delta}_{\mathfrak{X}}(t, x, \mathcal{Y}).$$

We also put $\overline{\delta}_{\mathfrak{X}}(t) = \inf_{\|x\|=1} \overline{\delta}_{\mathfrak{X}}(t, x)$. $\overline{\delta}_{\mathfrak{X}}(t)$ is called the modulus of asymptotic uniform convexity of $\mathfrak{X}$. We say that $\mathfrak{X}$ is *asymptotically uniformly convex* (AUC, in short) if for all $0 < t \leq 1$, $\overline{\delta}_{\mathfrak{X}}(t) > 0$.

Definition 2. For $\mathcal{Y}$ a subspace of $\mathfrak{X}$ and $x \in S_{\mathfrak{X}}$ let

$$\overline{\rho}_{\mathfrak{X}}(t, x, \mathcal{Y}) = \sup_{y \in \mathcal{Y}, \|y\| \leq t} \|x + y\| - 1$$

and

$$\overline{\rho}_{\mathfrak{X}}(t, x) = \inf_{\dim \mathfrak{X}/\mathcal{Y} < \infty} \overline{\rho}_{\mathfrak{X}}(t, x, \mathcal{Y}).$$

Set $\overline{\rho}_{\mathfrak{X}}(t) = \sup_{\|x\|=1} \overline{\rho}_{\mathfrak{X}}(t, x)$. We call $\overline{\rho}_{\mathfrak{X}}(t)$ the modulus of asymptotic uniform smoothness of $\mathfrak{X}$. When $\overline{\rho}_{\mathfrak{X}}(t)/t \to 0$, we say that $\mathfrak{X}$ is *asymptotically uniformly smooth* (AUS, in short).

Basic properties of $\overline{\delta}_{\mathfrak{X}}$ and $\overline{\rho}_{\mathfrak{X}}$ are described in [JLPS]. We list a few of them which will be useful in our context. Here $\delta_{\mathfrak{X}}(t)$ and $\rho_{\mathfrak{X}}(t)$ are the moduli of uniform convexity and uniform smoothness of $\mathfrak{X}$ respectively.

Proposition 2.1. (1) *$\overline{\rho}_{\mathfrak{X}}(t)$ and $\overline{\delta}_{\mathfrak{X}}(t)$ are non-decreasing Lipschitz functions and their Lipschitz constant are at most 1. $\overline{\delta}_{\mathfrak{X}}(t) \leq \overline{\rho}_{\mathfrak{X}}(t)$ for all t and $\overline{\rho}_{\mathfrak{X}}(t)$ is also convex.*

(2) *If $\mathfrak{X}_0 \subset \mathfrak{X}$ then $\bar{\rho}_{\mathfrak{X}_0}(t) \leq \overline{\rho}_{\mathfrak{X}}(t)$ and $\bar{\delta}_{\mathfrak{X}_0}(t) \geq \overline{\delta}_{\mathfrak{X}}(t)$ for all $0 < t < 1$.*

(3) *Let $\rho_{\mathfrak{X}}$ and $\delta_{\mathfrak{X}}$ denote the moduli of smoothness and convexity of $\mathfrak{X}$. Then, $2\rho_{\mathfrak{X}}(t) \geq \overline{\rho}_{\mathfrak{X}}(t)$ and $\delta_{\mathfrak{X}}(t) \leq \overline{\delta}_{\mathfrak{X}}(t)$ for all $0 < t < 1$.*

Note that (3) says in particular that a uniformly convex (smooth) space is asymptotically uniformly convex (smooth).

We say that $\overline{\delta}_{\mathfrak{X}}$ is of *power type* p, $1 \leq p \leq \infty$, if there is a constant $K > 0$ such that $\overline{\delta}_{\mathfrak{X}}(t) \geq Kt^p$ for all $0 \leq t \leq 1$; and that $\overline{\rho}_{\mathfrak{X}}$ is of *power type* q, $1 \leq q \leq \infty$, if there is a constant $K > 0$ such that $\overline{\rho}_{\mathfrak{X}}(t) \leq Kt^q$ for all $0 \leq t \leq 1$. In both cases t^∞ is interpreted to be zero. Subspaces of ℓ_p, for instance, have both asymptotic moduli of power type p (in fact $\overline{\delta}_{\ell_p}(t) = \overline{\rho}_{\ell_p}(t) = (1+t^p)^{1/p} - 1$) while for c_0 both moduli are identically zero for $t < 1$.

We turn now to $\mathfrak{X}$-valued martingales. Consider the probability space $(\Omega, \mathcal{A}, \mathbf{P})$, where $\Omega = \{-1,1\}^{\mathbb{N}}$, $\mathcal{A}$ is the Borel σ-algebra and, $\mathbf{P}$ is the Haar measure; i.e., on $\{-1,1\}$ we define $\mathbf{P}_n\{-1\} = 1/2 = \mathbf{P}_n\{1\}$ and set $\mathbf{P} = \prod_{n=1}^\infty \mathbf{P}_n$. Let $\mathcal{A}_0 = \{\emptyset, \Omega\}$, and, for $n \geq 1$, let $\mathcal{A}_n$ be the σ-algebra generated by the first n coordinates of Ω.

A sequence $(X_n)_{n\geq 0}$ is called a *Walsh-Paley martingale* if each X_n is an $\mathcal{A}_n$-measurable $\mathfrak{X}$-valued Bochner integrable function and

$$\int_A X_n \, d\mathbf{P} = \int_A X_{n+1} \, d\mathbf{P}$$

for each A in $\mathcal{A}_n$. The sequence of martingale differences is defined as $d_0 = X_0$ and $d_n = dX_n = X_n - X_{n-1}$.

Because of the particular nature of the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and the filtration $(\mathcal{A}_n)_{n\geq 0}$, the Walsh-Paley martingales admit a simple description (see [B, Part 4] for a more thorough explanation). Essentially, one obtains a subset of $\mathfrak{X}$ of the form

$$\{x_{\epsilon_1\ldots\epsilon_k} : 1 \leq k \leq n, \epsilon_i = \pm 1\}$$

where the $x_{\epsilon_1\ldots\epsilon_k}$ satisfy

$$x_{\epsilon_1\ldots\epsilon_k} = \frac{1}{2}(x_{\epsilon_1\ldots\epsilon_k 1} + x_{\epsilon_1\ldots\epsilon_k -1}) \tag{2.1}$$

and the martingale sequence is defined by

$$X_0 = \frac{1}{2}(x_1 + x_{-1}) \quad \text{ and } \quad X_k(\epsilon_1, \epsilon_2, \ldots) = x_{\epsilon_1\ldots\epsilon_k} \tag{2.2}$$

for $k = 1, \dots, n$, and $X_m = X_n$ for all $m > n$ (we consider stationary martingales only).

In what follows $L_r(\mathfrak{X})$ is the space of $\mathcal{A}$-measurable, $\mathfrak{X}$-valued Bochner integrable functions with $(\Omega, \mathcal{A}, \mathbf{P})$ as before and equipped with its usual norm.

3. p-CONVEX RENORMING OF $\mathfrak{X}$

Our aim in this section is to prove the following

Theorem 3.1. *If $L_r(\mathfrak{X})$, $1 < r < \infty$, has an asymptotically uniformly convex renorming of power type p $(p \geq r)$ then $\mathfrak{X}$ has a p-convex renorming when $r = p$ and a p'-convex renorming, for every $p' > p$, when $p > r$.*

As we mentioned in the introduction, by Pisier's theorem [P, Theorem 3.1] it is enough to show that for any Walsh-Paley martingale $(X_n)_{n\geq 0}$ in $L_r(\mathfrak{X})$ the inequality (1.1) holds for the equivalent p-asymptotically uniformly convex norm in $L_r(\mathfrak{X})$.

Note that if $(X_n)_{n\geq 0}$ is a Walsh-Paley martingale then it follows from (2.1) and (2.2) that

$$\mathbf{P}[X_k = x_{\epsilon_1 \dots \epsilon_k}] = \frac{1}{2^k} \tag{3.1}$$

and

$$[X_k = x_{\epsilon_1 \dots \epsilon_k}] = [X_{k+1} = x_{\epsilon_1 \dots \epsilon_k 1}] \cup [X_{k+1} = x_{\epsilon_1 \dots \epsilon_k -1}]. \tag{3.2}$$

As a difference to Pisier's argument [P], where Jame's p-estimates for basic sequences in $L_r(\mathfrak{X})$ (in the form of martingale differences) are used, here, because of the definition of asymptotic uniform convexity, we need to have the martingale differences inside or 'close to' certain finite codimensional subspaces of $L_r(\mathfrak{X})$. The next lemma is a step in that direction.

Lemma 3.2. *Suppose $(X_n)_{n\geq 0}$ is a martingale satisfying (3.1) and (3.2) above for every $k \geq 0$. Then for each k there exists a sequence $(X_{k,j})_{j\geq k+1}$ such that $X_{k,j} \to X_k$ weakly in $L_r(\mathfrak{X})$ as $j \to \infty$ and for every j, $(X_0, \dots, X_{k+1})$ has the same joint distribution as $(X_0, \dots, X_k, X_{k,j})$.*

Proof. Let r_j be the j^{th} coordinate projection in $\{-1,1\}^{\mathbb{N}}$ (which corresponds to the Rademacher function r_j in $L_r([0,1])$ via the canonical isometry between $L_r(\Omega, \mathbf{P})$ with $L_r([0,1])$). Denote $\epsilon = \epsilon_1 \dots \epsilon_k$. Then, on $[X_k = x_\epsilon]$ define $X_{k,j}$ for $j \geq k+1$ by

$$X_{k,j} = x_{\epsilon 1}\, 1_{[r_j=1]} + x_{\epsilon -1}\, 1_{[r_j=-1]} \tag{3.3}$$

where 1_A is the indicator function of A.

To show that $X_{k,j}$ converges weakly to X_k, let $\xi \in L_r(\mathfrak{X})^*$. We assume for simplicity of the argument that $L_r(\mathfrak{X})^* = L_{r'}(\mathfrak{X}^*)$, where $\frac{1}{r} + \frac{1}{r'} = 1$. This is OK because the sequence $(X_{k,j})_{j \geq k+1}$ takes values in a finite dimensional subspace of $\mathfrak{X}$. Thus we can assume, without lost of generality, that $\xi = x^* \otimes f$ for some $f \in L_{r'}(\Omega)$ and $x^* \in \mathfrak{X}^*$, because the linear span of such functionals is dense in $L_{r'}(\mathfrak{X}^*)$. Then,

$$\begin{aligned}\langle X_{k,j}, \xi \rangle &= \langle x_{\epsilon 1}\, 1_{[r_j=1]}, x^* \otimes f \rangle + \langle x_{\epsilon -1}\, 1_{[r_j=-1]}, x^* \otimes f \rangle \\ &= \left(\int_{[r_j=1]} f \, d\mathbf{P} \right) x^*(x_{\epsilon 1}) + \left(\int_{[r_j=-1]} f \, d\mathbf{P} \right) x^*(x_{\epsilon -1})\end{aligned}$$

As $j \to \infty$ the last expression converges to

$$\left(\frac{1}{2} \int_\Omega f \, d\mathbf{P} \right) x^*(x_{\epsilon 1}) + \left(\frac{1}{2} \int_\Omega f \, d\mathbf{P} \right) x^*(x_{\epsilon -1})$$

which is, by (2.1), equal to

$$\left(\int_\Omega f \, d\mathbf{P} \right) x^*(x_\epsilon) = \langle X_k, \xi \rangle.$$

The final statement is obvious. □

Next lemma shows how to obtain power type lower estimates when the values of t, as in the definition of AUC, are large.

Lemma 3.3. *Let $\overline{\delta}_{\mathfrak{X}}$ be of power type p. Then there exists a constant $c = c(p)$ so that for each $x \in S(\mathfrak{X})$ and $t > 0$ there exists a finite codimensional subspace $\mathcal{Y} = \mathcal{Y}(t, x)$ of $\mathfrak{X}$ so that if $y \in \mathcal{Y}$ with $\|y\| \geq t$ then*

$$\|x+y\|^p > 1 + ct^p.$$

PROOF. Suppose $\overline{\delta}_{\mathfrak{X}}$ is of power type p. Then for $\|x\| = 1$, $0 < t \leq 1$, and $0 < \alpha < 1$, there exists a finite codimensional subspace $\mathcal{Y} = \mathcal{Y}(t)$ of $\mathfrak{X}$ such that $\overline{\delta}_{\mathfrak{X}}(t, x, \mathcal{Y}) > (1-\alpha)Kt^p$. It follows that for $y \in \mathcal{Y}$ with $\|y\| \geq t$ we have

$$\|x+y\|^p > 1 + (1-\alpha)Kt^p.$$

A similar estimate holds if t has a big value:

Indeed, let $b > 2$ be fixed. If $t > b$ and $y \in \mathcal{Y}$ is such that $\|y\| \geq t$, then

$$\|x+y\|^p > (1-t)^p \geq 1 + ct^p$$

for some constant c since $t > b > 2$. Here $\mathcal{Y}(t, x) = \mathfrak{X}$.

When $1 \leq t \leq b$ an estimate as above is also obtained since $\overline{\delta}_{\mathfrak{X}}(1, x, \mathcal{Y}(1, x)) \leq \overline{\delta}_{\mathfrak{X}}(t, x, \mathcal{Y}(1, x))$. □

We now prove Theorem 3.1.

PROOF. Denote by $\|\cdot\|_r$ the usual norm in $L_r(\mathfrak{X})$ and by $\|\cdot\|$ the equivalent norm in $L_r(\mathfrak{X})$ (i.e., $\frac{1}{\kappa}\|\cdot\|_r \leq \|\cdot\| \leq \kappa\|\cdot\|_r$ for some $\kappa \geq 1$) which satisfies $\overline{\delta}_{L_r(\mathfrak{X})}(t) \geq Kt^p$ for all $0 < t < 1$. Let $(\tilde{X}_n)_{n\geq 0}$ be a Walsh-Paley martingale as in (2.2).

Let $X_0 = \tilde{X}_0$ and $X_1 = \tilde{X}_1$. Now, assume that $X_0, \ldots, X_k$ have been defined and consider $\frac{X_k}{\|X_k\|}$. If $t = \frac{\|\tilde{X}_{k+1} - \tilde{X}_k\|}{\kappa^2\|X_k\|}$ then by lemma 3.3 there exists a finite codimensional subspace $\mathcal{Y} = \mathcal{Y}(t, X_k/\|X_k\|)$ of $L_r(\mathfrak{X})$ such that for $y \in \mathcal{Y}$ with $\|y\| \geq t$ we have

$$\left\| \frac{X_k}{\|X_k\|} + y \right\|^p > 1 + ct^p \tag{3.4}$$

for some constant c.

Consider a sequence $X_{k,j}$ weakly convergent to X_k as in lemma 3.2. Choose j_0 large enough so that the distance from $\frac{1}{\|X_k\|}(X_{k,j_0} - X_k)$ to $\mathcal{Y}$ (with respect to $\|\cdot\|$) is less than some $\varepsilon > 0$ so that the left hand side of (3.4) is still bigger than the right hand side of (3.4) plus ε. Set $X_{k+1} = X_{k,j_0}$.

Observe that by the final part of lemma 3.2, the norms $\|\tilde{X}_{k+1} - \tilde{X}_k\|_r$ and $\|X_{k+1} - X_k\|_r$ coincide; hence, for $y = \frac{X_{k+1}-X_k}{\|X_k\|}$ we have $\|y\| \geq t$. Then from (3.4) we have

$$\left\| \frac{X_k}{\|X_k\|} + \frac{X_{k+1} - X_k}{\|X_k\|} \right\|^p > 1 + \frac{c}{\kappa^{4p}} \cdot \frac{\|X_{k+1} - X_k\|^p}{\|X_k\|^p} \tag{3.5}$$

which implies

$$\|X_{k+1}\|^p > \|X_k\|^p + \frac{c}{\kappa^{4p}} \|X_{k+1} - X_k\|^p. \tag{3.6}$$

Now, if n is the stationary point of the martingale then from (3.6) we have

$$\|X_n\|^p \geq \|X_{n-1}\|^p + \frac{c}{\kappa^{4p}} \|X_n - X_{n-1}\|^p.$$

A recursive computation shows that

$$\|X_n\|^p \geq \|X_0\|^p + \frac{c}{\kappa^{4p}} \sum_{k=1}^{n} \|X_k - X_{k-1}\|^p$$

hence

$$\|X_n\|^p \geq \frac{c}{\kappa^{4p}} \left(\|X_0\|^p + \sum_{k=1}^{n} \|X_k - X_{k-1}\|^p \right)$$

changing back to the $\|\cdot\|_r$ norm gives

$$\sup_{n\geq 0} \|X_n\|_r^p \geq \frac{c}{\kappa^{6p}} \left(\sum_{k=1}^{n} \|X_k - X_{k-1}\|_r^p + \|X_0\|_r^p \right).$$

Thus for $r = p$ and $C = \kappa^6/c^{1/p}$ we have Pisier's inequality (1.1) as desired.

When $r < p$ we follow the argument in Part 4, Chapter IV, Section 2 of [B]. First notice that the steps in our previous argument show that the inequality

$$\|X_n - X_0\|_r^p \geq \frac{c}{\kappa^{6p}} \sum_{k=1}^{n} \|X_k - X_{k-1}\|_r^p$$

also holds.

This is the equivalent inequality for Walsh-Paley martingales to the inequality in line 3, page 278 of [B]. Now, by Hölder's inequality we have

$$\left(\sum_{k=1}^{n} \|X_k - X_{k-1}\|_r^r \right)^{1/r} \leq n^{\frac{1}{r} - \frac{1}{p}} \left(\sum_{k=1}^{n} \|X_k - X_{k-1}\|_r^p \right)^{1/p}.$$

Thus we get

$$\left(\sum_{k=1}^{n} \|X_k - X_{k-1}\|_r^r \right)^{1/r} \leq \frac{\kappa^6}{c^{1/p}} \cdot n^{\frac{1}{r} - \frac{1}{p}} \cdot \|X_n - X_0\|_r \tag{3.7}$$

which is equation (2) on page 278 of [B]. There it is shown how to obtain from (3.7) a uniformly convex renorming of $\mathfrak{X}$ of power type p' for every $p' > p$. □

4. Final Remarks

1. We conjecture that Theorem 3.1 can have a stronger statement, namely, that the p-convex renorming of $\mathfrak{X}$ can be obtained even in the case $r < p$ from the p-asymptotic uniform convexifiability of $L_r(\mathfrak{X})$.

2. It is interesting to observe that although ℓ_1 is the most asymptotically uniformly convex space, in the sense that $\overline{\delta}_{\ell_1}(t) = t$, the space L_1, hence also, $L_1(\mathfrak{X})$, cannot be given an asymptotically uniformly convex renorming since L_1 fails the point of continuity property (PCP) (see, e.g., [EW] for more information on the PCP) and this property is implied by being asymptotically uniformly convex [JLPS, Proposition 2.6].

3. The analogue of Theorem 3.1 for q-smoothness assuming q-asymptotic uniform smoothness of $L_r(\mathfrak{X})$ holds as well via our approach in the case $r = q$ since there is a reverse inequality to inequality (1.1).

Indeed, from [P] we have that a Banach space $\mathfrak{X}$ is q-smooth if for some constant C the following inequality is satisfied for all Walsh-Paley martingales,

$$\sup_n \mathbb{E}\|X_n\|^q \leq C^q \left(\mathbb{E}\|X_0\|^q + \sum_{n\geq 1} \mathbb{E}\|dX_n\|^q \right)$$

First, the corresponding lemma to lemma 3.3 holds as the inequality $(1+s^q)^q \leq 1+2qs^q$ holds for small values of s. Note that for $t \geq 1$ we have a similar estimate since in general for $\|x\| = 1$ and $t \geq \|y\| \geq 1$, $\|x+y\|^q \leq 1+2^q t^q$.

Then, following the argument of Theorem 3.1 with the modulus of asymptotic uniform smoothness instead of the modulus of asymptotic uniform convexity, we obtain an inequality similar to (3.6); i.e., for some constant $\tilde{c}$, $\|X_k\|^q < \|X_{k-1}\|^q + \tilde{c}\,\|X_k - X_{k-1}\|^q$, from which the result follows.

For the case $r > q$ one can proceed by duality since it is known that $\overline{\delta}_{\mathfrak{X}^*}(\cdot)$ and $\overline{\rho}_{\mathfrak{X}}(\cdot)$ are equivalent to the dual Young's function of each other in the reflexive case (see, e.g., [GKL] but note that there a different language is used for the notions of AUC and AUS). Note that we can assume that $\mathfrak{X}$ is reflexive since it is known that the inequality (3.6) and the correspondent inequality for smoothness are equivalent to $\mathfrak{X}$ being superreflexive (see, e.g., [W, Theorem 3.2]). Thus from the equivalent q-asymptotically uniformly smooth norm of $L_r(\mathfrak{X})$ one obtains the p-asymptotic uniform convexity of $L_{r'}(\mathfrak{X}^*)$, $\frac{1}{r} + \frac{1}{r'} = 1$. Hence by Theorem 3.1 there is for any $p' > p$, an equivalent uniformly convex norm on $\mathfrak{X}^*$ of power type p' and by the Lindenstrauss duality formula $\mathfrak{X}$ is q'-uniformly smooth, where q' is the conjugate exponent of p'.

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