Almost Fréchet differentiability of Lipschitz mappings between infinite dimensional Banach spaces

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Abstract

We give several sufficient conditions on a pair of Banach spaces X and Y under which each Lipschitz mapping from a domain in X to Y has, for every $\epsilon > 0$, a point of ϵ-Fréchet differentiability. Most of these conditions are stated in terms of the moduli of asymptotic smoothness and convexity, notions which appeared in the literature under a variety of names. We prove, for example, that for $\infty > r > p \geq 1$, every Lipschitz mapping from a domain in an ℓ_r sum of finite dimensional spaces into an ℓ_p sum of finite dimensional spaces has, for every $\epsilon > 0$, a point of ϵ-Fréchet differentiability, and that every Lipschitz mapping from an asymptotically uniformly smooth space to a finite dimensional space has such points. The latter result improves, with a simpler proof, the result of [16]. We also survey some of the known results on the notions of asymptotic smoothness and convexity, prove some new properties, and present some new proofs of existing results.

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1 Introduction

The purpose of this paper is to prove the existence of points of ε–Fréchet differentiability of general Lipschitz maps between certain classes of Banach spaces X and Y which may be infinite dimensional. These are the first general results of this type in the literature in the context of infinite dimensional Banach spaces.

Let us first recall some basic definitions. A function f from an open set D in a Banach space X into a Banach space Y is said to be Gâteaux differentiable at a point $x_0 \in D$ if there is a bounded linear operator $T: X \to Y$ so that for every $u \in X$ and every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, u) > 0$ for which

$$\left\| \frac{f(x_0 + tu) - f(x_0)}{t} - Tu \right\| < \varepsilon \ , \quad \text{for} \quad 0 < t < \delta \ .$$

The operator T, if it exists, is uniquely determined and is called the Gâteaux derivative of f at x_0. It is denoted by $D^G f(x_0)$. The set of all the points of Gâteaux differentiability of f is denoted here by $\mathrm{diff}^G(f)$.

If one can ensure in the definition above that there is a $\delta(\varepsilon) > 0$ so that $\delta(u, \varepsilon) > \delta(\varepsilon)$ for all u with $\|u\| = 1$, we say that f is Fréchet differentiable at x_0. We then put $D^F f(x_0) = D^G f(x_0)$. The set of all points of Fréchet differentiability of f is denoted by $\mathrm{diff}^F(f)$. Note that f is Fréchet differentiable at x_0 with $D^F f(x_0) = T$ iff $\|f(x_0 + u) - f(x_0) - Tu\| = o(\|u\|)$ as $\|u\| \to 0$.

As is well known (see the discussion below) there are quite strong results ensuring the existence of points of Gâteaux differentiability of Lipschitz functions. Unfortunately, the notion of Gâteaux derivative is often not strong enough so as to be useful in many natural contexts. Fréchet derivatives are very useful in many of these contexts. However, there are till now very few results ensuring the existence of points of Fréchet differentiability of Lipschitz functions.

There is a notion of ε–Fréchet differentiability which is weaker than Fréchet differentiability but can replace Fréchet differentiability in many contexts and for which it is often much easier to prove existence theorems. This notion, which has been used implicitly or explicitly in the literature for quite a long time, is the main subject of the present paper.

Definition 1.1 *A mapping f from an open set D in a Banach space X into a Banach space Y is said to be ε–Fréchet differentiable at $x_0 \in D$ for some $\varepsilon > 0$ if there is a bounded linear operator $T: X \to Y$ and a $\delta > 0$ so that*

$$\|f(x_0 + u) - f(x_0) - Tu\| < \varepsilon\|u\| \quad if \quad 0 < \|u\| < \delta \ . \tag{1}$$

Note that f is Fréchet differentiable at x_0 if and only if it is ε–Fréchet differentiable there for every $\varepsilon > 0$. The set of all points at which f is ε–Fréchet differentiable for a given $\varepsilon > 0$ is denoted by $\mathrm{diff}^{\varepsilon F}(f)$.

For a given $\varepsilon > 0$, the operator T is not determined uniquely by (1). We denote any operator satisfying (1) as $D^{\varepsilon F} f(x_0)$. However, there may be a "preferred" such operator, such as the Gâteaux derivative of f at x_0, if $x_0 \in \mathrm{diff}^G(f)$. (Observe that if $x_0 \in \mathrm{diff}^G(f) \cap \mathrm{diff}^{\varepsilon F}(f)$ then $D^G f(x_0)$ satisfies (1) if ε is changed to 2ε.) One advantage of using $D^G f(x_0)$ is that if f maps a neighborhood of x_0 into a proper closed subspace of Y, then $D^G f(x_0)$ takes values in the subspace. A general operator $D^{\varepsilon F} f(x_0)$ does not have this property.

Note that if there is a Lipschitz map f from $D \subset X$ into Y which does not have a point of ε–Fréchet differentiability for some $\varepsilon > 0$ then there is also such a map for $\varepsilon = 1$ (simply replace f by cf for a suitable c).

Let us now summarize the known results concerning existence of the three types of derivatives mentioned above for Lipschitz functions f.

If $X = R$ then every Lipschitz function from $D \subset X$ into Y has points of Gâteaux differentiability (which is the same as Fréchet differentiability in this context) if and only if Y has the Radon-Nikodým property (RNP). Moreover, if Y fails to have the RNP then there is a Lipschitz function from R to Y which fails to have any point of 1–Fréchet differentiability. This is Theorem 5.21 in [2]. We refer to this book for background material and for references to the original literature. If Y has the RNP and X is separable, then any Lipschitz map f from an open set $D \subset X$ into Y is Gâteaux differentiable outside a Gauss null set in D. This result, due essentially to Aronszajn and Mankiewicz, is Theorem 6.42 in [2]. We refer again to this book for further references and for background material on Gauss null sets. We mention just the following fact concerning Gauss null sets which will be used below. Let $M \subset D$ be a co-null set (i.e., the complement of M with respect to the open set D is Gauss null). Then, whenever u and v are such that the line segment $[u, u + v]$ belongs to D, there is a vector w arbitrarily close to u so that $w + tv \in M$ for almost every $t \in [0, 1]$ (in the usual Lebesgue sense).

If X^* is separable and $Y = R$, then every Lipschitz map f from $D \subset X$ into Y has points of Fréchet differentiability. This theorem is proved in [26]. A simpler proof of this fact (which is still quite involved) is presented in [17]. We do not use this result in the present paper. If X is separable but X^* is not separable, then there is a Lipschitz (even convex) function from X to R which fails to have a point of 1–Fréchet differentiability. This theorem is proved in [15] (see also Proposition 4.12 in [2] and the remark following its proof).

If X^* is separable and $Y = R^n$ with $n > 1$, it is unknown whether every Lipschitz map from X to Y has points of Fréchet differentiability. This is open even in the case where $X = \ell_2$ and $Y = R^2$. On the other hand, it is proved in [16] that if X is superreflexive (e.g., $X = \ell_2$), then any Lipschitz map from X to R^n has for every $\varepsilon > 0$ points of ε–Fréchet differentiability. We do not know whether the assumption that X is superreflexive can be replaced by the (natural) assumption that X^* is separable. One of the main results in this paper is an extension of the result of [16] to a substantially larger class of spaces, namely to spaces which have an equivalent asymptotic uniformly smooth norm (this latter notion will be explained below).

From the results quoted above, it is clear that if we are interested in proving the existence of points of ε-Fréchet differentiability for general Lipschitz maps from a separable X into Y, we must assume that X^* is separable and Y has the RNP. There is, however, another obstruction to the existence of points of ε-Fréchet differentiability of Lipschitz maps. This obstruction is explained in the following simple proposition.

Proposition 1.2 *Assume that either X or Y has an unconditional basis and that there is a non-compact bounded linear operator from X into Y. Then there is a Lipschitz map from X into Y which has no point of 1–Fréchet differentiability.*

Proof: Assume X has an unconditional basis. Renorming X we may assume that X has a 1–unconditional basis $\{e_i\}_{i=1}^{\infty}$. Let A be a bounded non-compact linear operator from X to Y. Then there is a normalized block basis $b_i = \sum_{j=p_i+1}^{p_{i+1}} \alpha_j e_j$ of $\{e_i\}_{i=1}^{\infty}$ so that $\|Ab_i\| > \delta$ for some $\delta > 0$ and all i. We may and shall assume that $\alpha_j \geq 0$ for all j. Consider the map $f : X \to Y$ defined by $f(u) = A|u|$, where for $u = \sum_{i=1}^{\infty} a_i e_i$ one denotes $|u| = \sum_{i=1}^{\infty} |a_i| e_i$. Clearly f is a Lipschitz map whose Lipschitz constant is equal to $\|A\|$. Assume that T is a $\delta/2$–Fréchet derivative of f at $x_0 = \sum_{i=1}^{\infty} \beta_i e_i$. Then

$$\|f(x_0+u) - f(x_0) - Tu\| < \delta/2\|u\|$$

whenever $\|u\| = \gamma_0$ for a suitable $\gamma_0 > 0$. Then let n be such that

$$\left\| \sum_{i=n+1}^{\infty} \beta_i e_i \right\| < \frac{\delta\gamma_0}{4\|A\|}$$

and put

$$x_1 = \sum_{i=1}^{n} \beta_i e_i \ .$$

Then

$$\|f(x_1+u) - f(x_1) - Tu\| < \delta\gamma_0$$

whenever $\|u\| = \gamma_0$. In particular, by taking i such that $p_i > n$ we get the following two inequalities.

$$\|\gamma_0(Ab_i \mp Tb_i)\| = \|f(x_1 \pm \gamma_0 b_i) - f(x_1) \mp \gamma_0 Tb_i\| < \delta\gamma_0 \ .$$

By adding these two inequalities we deduce that $\|Ab_i\| < \delta$ which is a contradiction. The proof when Y has an unconditional basis is similar. ∎

In view of Proposition 1.2, the natural question to consider is the following:

Problem 1.3 *Assume X^* and Y are separable, Y has the RNP and every bounded linear map from X into Y is compact. Does every Lipschitz map from X to Y have points of ε–Fréchet differentiability for every $\varepsilon > 0$?*

Note that the assumptions in Problem 1.3 imply that the space of all bounded linear operators from X to Y has the RNP (see [3]). We do not know the answer to this question (as we mentioned above), even in the case where $\dim Y < \infty$. However, we give in this paper a partial positive answer to Problem 1.3 in many interesting special cases. In particular, we give the first examples of pairs (X, Y) of infinite dimensional Banach spaces so that every Lipschitz mapping from X into Y has a point of 1-Fréchet differentiability (or, in fact, any other reasonable approximation by affine maps).

Several of our results will involve the moduli of asymptotic uniform convexity and smoothness. These moduli were first introduced in [22] under different notations and names.

The modulus of asymptotic uniform smoothness of X is given for $t > 0$ by

$$\bar{\rho}_X(t) = \sup_{\|x\|=1} \inf_{\dim X/Y<\infty} \sup_{y\in Y, \|y\|\le t} \|x+y\| - 1$$

and the modulus of asymptotic uniform convexity of X is given for $t > 0$ by

$$\bar{\delta}_X(t) = \inf_{\|x\|=1} \sup_{\dim X/Y<\infty} \inf_{y\in Y, \|y\|\ge t} \|x+y\| - 1.$$

The space X is said to be asymptotically uniform convex if $\bar{\delta}_X(t) > 0$ for every $t > 0$ and asymptotically uniform smooth if $\bar{\rho}_X(t)/t \to 0$ as $t \to 0$. Clearly a uniformly convex (respectively smooth) space is asymptotically uniformly convex (respectively smooth) but the converse is easily seen to be false.

The moduli of asymptotic uniform convexity and smoothness are easy to compute for subspaces of ℓ_p, $1 \le p < \infty$ or of c_0. If X is a subspace of ℓ_p, then $\bar{\rho}_X(t) = \bar{\delta}_X(t) = (1+t^p)^{1/p} - 1$ while if X is a subspace of c_0, then $\bar{\rho}_X(t) = \bar{\delta}_X(t) = 0$ for $t < 1$. In particular, c_0 is asymptotically uniformly smooth and ℓ_1 is asymptotically uniformly convex.

The next result summarizes some of our main results.

Theorem 1.4 *Let X be separable and Y have the Radon–Nikodým property. Then in each of the following situations every Lipschitz function from a domain in X to Y has, for every $\varepsilon > 0$, a point of ε-Fréchet differentiability.*

(i) *For all $t > 0$ $\bar{\delta}_Y(t) > 0$ and for all $c > 0$ $\bar{\rho}_X(t)/\bar{\delta}_Y(ct) \to 0$ when $t \to 0$.*

(ii) *$\bar{\rho}_X(t) = 0$ for some $t > 0$.*

(iii) *$X = C(\alpha)$ for some countable ordinal α.*

(iv) *$\bar{\rho}_X(t)/t \to 0$ as $t \to 0$ and Y is finite dimensional.*

These results are respectively Theorems 6.1, 5.1, 7.4 and 8.2 (and Remark 8.4) below. The assumption that X^* is separable and that every bounded linear map from X to Y is compact are contained implicitly in the statement of Theorem 1.4 (e.g., it will be shown in the next section that if X is separable and $\bar{\rho}_X(t) < t$ for some $t > 0$, then X^* is also separable).

It follows, for example from (i), that every Lipschitz map from an open set in a subspace of ℓ_p (or c_0) into ℓ_q has for every $\varepsilon > 0$ point of ε–Fréchet differentiability if $1 \leq q < p$. This result is an answer to a question which originally motivated our study.

Part (iv) is the extension of the results of [16] which we already mentioned above. It is the most technically involved part of this paper. Nevertheless, its proof (presented in Section 8) is quite a lot simpler than the proof presented in [16].

From part (iii) we deduce that maps from X into R^n have points of ε–Fréchet differentiability even in cases where X does not admit an equivalent asymptotically uniformly smooth norm (see Section 2.1).

In the statements and proofs of the results of the type stated in Theorem 1.4 we also show, whenever possible, that the points of ε-Fréchet differentiability of the Lipschitz function f can be found in any preassigned co-null set. We do it mostly because one may want to ensure that the point found is also a point of Gâteaux differentiability of f as well as of another Lipschitz function defined on the same domain (but may have a different range space).

For some of the results proved here on the existence of points of ε–Fréchet differentiability (especially for $X = c_0$), it is now known that there actually exist points of Fréchet differentiability. The proofs of these results involve a new type of null sets and are technically more complicated than the proofs presented in this paper. These results will appear in [18].

Section 2.1 of this paper is concerned with the notions of asymptotic uniform convexity and smoothness. We prove there some results which have some novelty but mostly we collect results which appear already in the literature (but sometimes in a disguised form). We present new simplified proofs for some of these results. Only a small portion of Section 2.1 is used later in the paper. Section 2.2 contains some more preliminary material.

Sections 3–7 contain the proofs of parts (i) (ii) and (iii) of Theorem 1.4. Since some of the arguments are a bit technical, we chose to present the proofs in stages. Section 3 is a warm-up section which contains the simplest result of this type. The arguments become more involved as we progress through the sections. Because of this mode of presentation, we had to be, at some places, somewhat repetitive. However, we hope that in this way we made the ideas involved clearer and the results as a whole, more accessible.

2 Preliminaries

2.1 The moduli of asymptotic uniform smoothness and convexity

We begin by reintroducing two parameters of infinite dimensional Banach spaces first considered by Milman [22]. We call them the moduli of asymptotic uniform smoothness and of asymptotic uniform convexity.

Definition 2.1 *For $Y \subset X$, $x \in S_X$ let*

$$\bar{\rho}(t,x,Y) = \bar{\rho}_X(t,x,Y) = \sup_{y \in Y, \|y\| \le t} \|x+y\| - 1$$

and

$$\bar{\rho}(t,x) = \bar{\rho}_X(t,x) = \inf_{\dim X/Y < \infty} \bar{\rho}(t,x,Y).$$

We also put $\bar{\rho}(t) = \bar{\rho}_X(t) = \sup_{\|x\|=1} \bar{\rho}(t,x)$ and we call $\bar{\rho}_X(t)$ the modulus of asymptotic smoothness of X. If $\bar{\rho}_X(t)/t \to 0$ we say that X is asymptotically uniformly smooth.

Definition 2.2 *For $Y \subset X$, $x \in S_X$ let*

$$\bar{\delta}(t,x,Y) = \bar{\delta}_X(t,x,Y) = \inf_{y \in Y, \|y\| \ge t} \|x+y\| - 1$$

and

$$\bar{\delta}(t,x) = \bar{\delta}_X(t,x) = \sup_{\dim X/Y < \infty} \bar{\delta}(t,x,Y).$$

We also put $\bar{\delta}(t) = \bar{\delta}_X(t) = \inf_{\|x\|=1} \bar{\delta}(t,x)$. $\bar{\delta}_X(t)$ is called the modulus of asymptotic convexity of X and we say that X is asymptotically uniformly convex *if for all $0 < t < 1$, $\bar{\delta}(t) > 0$.*

Milman [22] actually considered many different moduli, of which $\bar{\delta}(\cdot)$ and $\bar{\rho}(\cdot)$ are just the most important. In Milman's [22] notation, $\bar{\rho}(t,x) = \delta(t;x,\mathcal{B}^0)$ and $\bar{\delta}(t,x) = \beta(t;x,\mathcal{B}^0)$.

Close relatives to asymptotic uniform convexity and asymptotic uniform smoothness have been investigated quite a lot. The one which has perhaps been most studied is the *uniform Kadec-Klee* (UKK) property of Huff [8]. An asymptotically uniformly convex Banach space X is UKK, and the converse is true if X^* is separable; in fact, if ℓ_1 does not embed into X. Huff also considered *nearly uniformly convex* spaces. A Banach space X is nearly uniformly convex if and only if it is reflexive and asymptotically uniformly convex; equivalently, reflexive and UKK (see [8, Theorem 1]). Prus [27] introduced nearly uniformly smooth spaces and proved that a Banach space is nearly uniformly smooth if and only if its dual is nearly uniformly convex. In our terminology this says that a reflexive space is asymptotically uniformly smooth if and only if its dual is asymptotically uniformly convex. In general, duality theory for asymptotic uniform convexity and smoothness is trickier than for uniform convexity and smoothness. Just as for uniform convexity and smoothness, renorming theory for asymptotic uniform convexity and smoothness is deep and highly developed even though several fundamental questions remain open. Later in this section we describe some of what is known about the duality and renorming theories.

Incidentally, the reason we introduce new terminology for (essentially) known concepts is that our use of "asymptotic" is consistent with the meaning of the word as it is used in contemporary Banach space theory (see [23], [21]). Direct connections between asymptotic uniform convexity/smoothness and asymptotic structure are made in [13].

Of course, uniform convexity implies asymptotic uniform convexity, and uniform smoothness implies asymptotic uniform smoothness; this is made quantitative in Proposition 2.3. Roughly speaking, one expects that properties that hold for finite dimensional spaces and for uniformly convex spaces also hold for at least reflexive asymptotic uniformly convex spaces.

The recent papers [13] and [7] contain implicitly the deepest information about the moduli of asymptotic smoothness and asymptotic convexity as well as references to earlier work. Here we prove only a few facts about the moduli which are connected to our use of them in the later sections or are necessary to get a feel for what the moduli measure.

In the next proposition we gather some properties of the moduli of asymptotic smoothness and asymptotic convexity. This is the only result of Section 2.1 which is specifically used in the sequel.

Proposition 2.3 *(1) $\bar{\rho}(t)$ and $\bar{\delta}(t)$ are non-decreasing Lipschitz functions with Lipschitz constant at most 1. $\bar{\delta}(t) \le \bar{\rho}(t)$ for all t and $\bar{\rho}(t)$ is also convex.*

(2) If $X_0 \subset X$ then $\bar{\rho}_{X_0}(t) \le \bar{\rho}_X(t)$ and $\bar{\delta}_{X_0}(t) \ge \bar{\delta}_X(t)$ for all $0 < t < 1$.

(3) Let ρ_X and δ_X denote the moduli of smoothness and convexity of X. Then for all $0 < t < 1$, $2\rho_X(t) \ge \bar{\rho}_X(t)$ and $\delta_X(t) \le \bar{\delta}_X(t)$.

(4) Let X and Y be two Banach spaces such that for some $0 < t < 1$ $\bar{\rho}_X(t)/\bar{\delta}_Y(t) < 1$. Then every linear operator from X to Y is compact.

Proof: (1) and (2) are simple. To prove (3) recall first the definitions of ρ_X and δ_X.

$$\rho_X(t) = \sup_{\|x\|=1, \|y\| \le t} \frac{\|x+y\| + \|x-y\|}{2} - 1,$$

$$\delta_X(t) = \inf_{\|u\|, \|v\| \le 1, \|u-v\| \ge t} 1 - \frac{\|u+v\|}{2}.$$

Notice that if we define

$$\hat{\rho}_X(t, x, Y) = \sup_{y \in Y, \|y\| \le t} \frac{\|x+y\| + \|x-y\|}{2} - 1$$

then, since $\|x - y\| \ge 1$ if $y \in Y$ and $Y \subseteq \{x;\ x^*(x) = 0\}$, where x^* is a norming functional of x,

$$\frac{1}{2}\bar{\rho}_X(t) \le \sup_{\|x\|=1} \inf_{\dim X/Y < \infty} \hat{\rho}_X(t, x, Y).$$

Hence

$$\sup_{\|x\|=1} \inf_{\dim X/Y < \infty} \hat{\rho}_X(t, x, Y) \le \sup_{\|x\|=1, \|y\| \le t} \frac{\|x+y\| + \|x-y\|}{2} - 1 = \rho_X(t).$$

This proves (3) for ρ_X. To prove (3) for δ_X, fix an $\varepsilon > 0$ and let $x \in S_X$ be such that $\bar{\delta}_X(t, x, Y) < \bar{\delta}_X(t) + \varepsilon$ for Y being the annihilator of a norm one functional, x^*, which norms x. Then there is a $y \in Y$ with $\|y\| \ge t$ such that

$1 \leq \|x+y\| < 1+\bar{\delta}_X(t)+\varepsilon$. Put $u = \frac{x+y}{\|x+y\|}$, $v = u-y = \frac{x}{\|x+y\|} - \left(1 - \frac{1}{\|x+y\|}\right) y$. Then u and v are in the unit ball of X and $\|u-v\| \geq t$. Consequently,

$$\begin{aligned} \delta_X(t) &\leq 1 - \left\|\frac{u+v}{2}\right\| \leq 1 - x^*\left(\frac{u+v}{2}\right) = 1 - \frac{1}{\|x+y\|} \\ &\leq 1 - \frac{1}{1+\bar{\delta}_X(t)+\varepsilon} \leq \bar{\delta}_X(t) + \varepsilon. \end{aligned}$$

Sending ε to zero we get $\delta_X(t) \leq \bar{\delta}_X(t)$.

To prove (4), let $T : X \to Y$ be a norm one operator. By (1), we can choose an $\varepsilon > 0$ such that $\bar{\delta}_Y((1+\varepsilon)^{-1}t) > (1+\varepsilon)^2\bar{\rho}_X(t)$. It is enough to get a subspace $X_0 \subset X$ with $\dim X/X_0 < \infty$ such that $\|T_{|X_0}\| \leq (1+\varepsilon)^{-1}$. Indeed, using (2), we can then iterate and replace $(1+\varepsilon)^{-1}$ by $(1+\varepsilon)^{-n}$ for every n.

Let $e \in S_X$ be such that $\|Te\| > 1-\tau$ (τ will be chosen presently) and Y_0 a finite co-dimensional subspace of Y such that

$$\|Te+y\| - 1 \geq (1+\varepsilon)^{-1}\bar{\delta}_Y((1+\varepsilon)^{-1}t) \text{ if } \|y\| \geq (1+\varepsilon)^{-1}t,\ y \in Y_0.$$

This can be done if τ is small enough. Let X_0 be a finite co-dimensional subspace of X such that $T^*Y_0^\perp \subset X_0^\perp$ (then $TX_0 \subset Y_0$) and $\bar{\rho}_X(t,e,X_0)-1 \leq (1+\varepsilon)\bar{\rho}_X(t)$.

Suppose $\|T_{|X_0}\| > (1+\varepsilon)^{-1}$ and take $x_0 \in X_0$, $\|x_0\| = t$ with $\|Tx_0\| > (1+\varepsilon)^{-1}t$. Then

$$(1+\varepsilon)^{-1}\bar{\delta}_Y((1+\varepsilon)^{-1}t) \leq \|Te+Tx_0\| - 1 \leq \|e+x_0\| - 1 \leq (1+\varepsilon)\bar{\rho}_X(t)$$

which is a contradiction. ■

In [7, Proposition 2.7] it is proved that X^* contains no proper norming subspace (and hence X is Asplund) provided $\bar{\rho}_X(t) < t/2$ for some $0 < t < 1$.

Proposition 2.4 *If $\bar{\rho}_X(t) < t$ for some $0 < t \leq 1$ then X is Asplund; i.e., every separable subspace of X has a separable dual.*

We shall need the following lemma.

Lemma 2.5 *Let X be a separable Banach space with a non-separable dual. Then for all $\tau > 0$ there is a collection $\{x_\alpha^*;\ \alpha < 2^{\aleph_0}\} \subset S_{X^*}$ (α refers to ordinals and $2^{\aleph_0}$ also denotes the first ordinal of cardinality $2^{\aleph_0}$) such that for all finite dimensional $E \subset X^*$*

$$\liminf_\alpha \operatorname{dist}(x_\alpha^*, E) > 1 - \tau. \tag{2}$$

Proof: The density character of X^* equals its cardinality which is $2^{\aleph_0}$. Thus the set of finite dimensional subspaces of X^* also has the same cardinality $2^{\aleph_0}$. Let $\{E_\alpha;\ \alpha < 2^{\aleph_0}\}$ be the set of all finite dimensional subspaces of X^* and for $\alpha < 2^{\aleph_0}$ let G_α be the closed linear span of $\cup_{\beta\le\alpha} E_\beta$. The G_α's are proper subspaces since their density character is at most $|\alpha|\aleph_0 < 2^{\aleph_0}$. Now choose for each $\alpha < 2^{\aleph_0}$ an $x^*_\alpha \in S_{X^*}$ such that $\mathrm{dist}(x^*_\alpha, G_\alpha) > 1 - \tau/2$. ■

Proof of Proposition 2.4: Since $\bar\rho_{X_0}(t) \le \bar\rho_X(t)$ if $X_0 \subset X$, it is enough to assume that X is separable and prove that X^* is also separable. Let $\gamma < 1$ be such that $\bar\rho_X(t) < \gamma t$ for the same t for which the assumption holds and let $\tau > 0$ be such that $\tau + \gamma t < t(1 - \tau/2)$. Assume that X^* is non-separable and use the lemma above to get a set $\{x^*_\alpha;\ \alpha < 2^{\aleph_0}\} \subset S_{X^*}$ satisfying (2). Let $x_0 \in S_X$ be such that $\langle x^*_\alpha, x_0\rangle > 1 - \tau$ for $2^{\aleph_0}$ of the α's. Without loss of generality we may assume that these are all of $\{\alpha < 2^{\aleph_0}\}$.

Let X_0 be a finite co-dimensional subspace of X such that

$$\|x_0 + x\| - 1 < \gamma t \quad \text{if} \quad x \in X_0, \|x\| = t.$$

Then for every $x \in X_0$ with $\|x\| = t$ and for every α

$$\langle x^*_\alpha, x\rangle < \langle x^*_\alpha, x_0 + x\rangle - 1 + \tau \le \|x_0 + x\| - 1 + \tau < \tau + \gamma t < t(1 - \tau/2)$$

so that $\|x^*_{\alpha|X_0}\| \le 1 - \tau/2$. Hence

$$\liminf_\alpha \mathrm{dist}(x^*_\alpha, X_0^\perp) = \liminf_\alpha \|x^*_{\alpha|X_0}\| < 1 - \tau$$

which is in contradiction with (2). ■

Remark: Define $\bar\rho\bar\rho_X(t) = \inf_{\dim X/X_0 < \infty} \bar\rho_{X_0}(t)$. This parameter was considered in [22] and denoted there $\delta^0\delta^0(t; X)$. An immediate corollary to Proposition 2.4 is that if $\bar\rho\bar\rho_X(t) < t$ for some $0 < t \le 1$ then X is Asplund. This improves Theorem 4.17(b) in [22] which asserts that under the same assumption on X every infinite dimensional subspace of X contains a further infinite dimensional subspace with separable dual.

In all the theorems involving ε-Fréchet differentiability of Lipschitz functions, it is necessary that the the target space have the RNP. Since asymptotic uniform convexity is a geometric condition which can replace uniform convexity in some theorems and uniform convexity is much stronger than the RNP, we thought that asymptotic uniform convexity might also imply the RNP.

However, after we wrote preliminary notes on this paper, Girardi [5] pointed out that a lemma of Shachermayer's [28] easily yields that the predual JT_* of the James' tree space is asymptotically uniformly convex, while it is known that JT_* fails the RNP. She also proved that the dual JT^* of the James' tree space is asymptotically uniformly convex (but of course fails the RNP). This is surprising since usually geometric conditions related to the RNP are better behaved in dual spaces than in general spaces. However, Proposition 2.6 (most of which is contained in [4]) shows that asymptotic uniform convexity does imply a geometric condition, the point of continuity property, that appears to be only slightly weaker than the RNP. A Banach space X has the point of continuity property (PCP) provided that every nonempty bounded subset of X contains nonempty relatively weakly open subsets of arbitrarily small diameter. Recall that X has the RNP if and only if every nonempty bounded subset of X contains nonempty slices of arbitrarily small diameter. (A slice of a set is a relatively weakly open subset determined by a single bounded linear functional.)

Proposition 2.6 *If $\bar{\delta}_X(t) > 0$ for all $0 < t \le 1$ then X has the PCP.*

Remarks: 1. The proof below shows that $\bar{\delta}_X(t, x) > 0$ for all $0 < t \le 1$ if and only if x is a weak to norm point of continuity of B_X.
2. Girardi [5] has improved Proposition 2.6 by proving that X has the PCP provided only that $\bar{\delta}_X(1/2) > 0$.

Proof of Proposition 2.6: Suppose C is weakly closed and $\sup_{x\in C} \|x\| = 1$. Given $0 < t < 1$ we find a point x_0 in C, a finite set $F \subset B_{X^*}$ and $\epsilon > 0$ so that if $F(x) = \max_{x^*\in F} |\langle x^*, x\rangle| < \epsilon$ and $\|x_0 + x\| \le 1$ (in particular, if $x_0 + x \in C$) then $\|x\| < t$. From this it is standard to show that X has PCP.

Let $0 < t < 1$ and take $\tau > 0$ so that

$$\tau < \frac{1}{2}\bar{\delta}_X(t)(1-\tau)^2. \tag{3}$$

Take any x_0 in C for which $\|x_0\| \ge 1-\tau$ and get a finite Auerbach set $F \subset S_{X^*}$ so that if $x' \in F_\perp$ and $\|x'\| \ge t$ then $\|(1-\tau)^{-1}x_0 + x'\| \ge 1 + \bar{\delta}_X(t)(1-\tau)$. Hence if $x \in F_\perp$ and $\|x\| \ge (1-\tau)t$ then

$$\|x_0 + x\| \ge (1-\tau) + \bar{\delta}_X(t)(1-\tau)^2 > 1 + \frac{1}{2}\bar{\delta}_X(t)(1-\tau)^2. \tag{4}$$

Suppose now that $\|x_1\| > t$ and $|F(x_1)| < \frac{\tau t}{|F|}$, where $|F|$ denotes the cardinality of F. Since F is Auerbach there is an $x_2 \in X$ so that $\|x_2\| < \tau t$ and $x := x_1 + x_2 \in F_\perp$. Then

$$\|x\| \geq \|x_1\| - \|x_2\| > (1-\tau)t$$

and

$$\begin{array}{rcl} 1 + \frac{1}{2}\bar{\delta}_X(t)(1-\tau)^2 & < & \|x_0 + x\| < \|x_0 + x_1\| + \tau t \\ & < & \|x_0 + x_1\| + \frac{1}{2}\bar{\delta}_X(t)(1-\tau)^2 \end{array}$$

so that $\|x_0 + x_1\| > 1$. ■

Asymptotic uniform convexity and asymptotic uniform smoothness obviously pass to subspaces. Since ℓ_1 is asymptotically uniform convex, asymptotic uniform convexity does not pass to quotients. To see that asymptotic uniform smoothness passes to quotients we need an elementary lemma.

Lemma 2.7 *Let X_0 and Y be subspaces of X with X_0 finite co-dimensional, let $\epsilon > 0$, and let $Q : X \to X/Y$ be the quotient map. Then there is a finite co-dimensional subspace Z_0 of X/Y so that $QB_{X_0} \supset \frac{1}{2+\epsilon} B_{Z_0}$.*

Proof: By replacing X with $X/(X_0 \cap Y)$, X_0 with $X_0/(X_0 \cap Y)$, and Y by its image in $X/(X_0 \cap Y)$ under the quotient map, S, we can assume that Y is finite dimensional and $Y \cap X_0 = \{0\}$ (note that $SB_Y = B_{SY}$ because $Y \supset X_0 \cap Y$).

Choose a finite dimensional subspace F of X^* which $1+\epsilon$–norms Y. Thus if $x \in F_\perp$, then

$$\|Qx\| = d(x, Y) \geq \frac{1}{2+\epsilon}\|x\|.$$

Thus the restriction of Q to the finite co-dimensional subspace $F_\perp \cap X_0$ of X is a $2+\epsilon$–isomorphism and hence $QB_{X_0} \supset \frac{1}{2+\epsilon} B_{Q(F_\perp \cap X_0)}$. ■

As an immediate consequence of Lemma 2.7 we have:

Proposition 2.8 *Let Y be a subspace of X. Then for all $0 < t \leq 1$,*

$$\bar{\rho}_{X/Y}(t) \leq \bar{\rho}_X(2t).$$

Next we show that c_0 and its subspaces are the only separable spaces which have the best modulus of asymptotic uniform smoothness.

Theorem 2.9 *If X is separable and $\bar{\rho}_X(t) = 0$ for some $t > 0$, then X is isomorphic to a subspace of c_0.*

Remarks: 1. The condition that $\bar{\rho}_X(t) = 0$ is equivalent to saying that the norm on X is Lipschitz UKK* in the sense of [6] (see Lemma 2.3 in [6]). Thus Theorem 2.9 is qualitatively equivalent to the extension of a result of Kalton and Werner [12] contained in Theorem 2.2 of [6]. The difficulty of the proof in [6] is greater than that of the proof we present but yields a better isomorphism constant for the embedding of X into c_0.
2. In Theorem 5.5.c in [22], Milman proves that c_0 embeds into X if $\bar{\rho}_X(t) = 0$ for some $t > 0$.

Proof of Theorem 2.9: First assume that X has a shrinking finite dimensional decomposition $\{E_n\}_{n=1}^{\infty}$. For $n < m \leq \infty$ let $E_{n,m}$ be the linear span of $\{E_k : n \leq k < m\}$. We will show that $\{E_n\}_{n=1}^{\infty}$ has a c_0-blocking. The shrinking assumption implies that for each x in X and $0 < t \leq 1$,

$$\bar{\rho}_X(t, x) = \lim_{n \to \infty} \bar{\rho}_X(t, x, E_{n,\infty}). \tag{5}$$

Take $t > 0$ so that $\bar{\rho}_X(t) = 0$ and fix a summable sequence $\{\varepsilon_n\}_{n=1}^{\infty} \downarrow 0$. The convergence in (5) is uniform for x in the unit sphere of a finite dimensional space, so we can choose $1 = n_0 < n_1 < n_2 < \dots$ so that for each $k \geq 1$, if x is in the unit sphere of E_{1,n_k} and y is in $E_{n_{k+1},\infty}$ with $\|y\| \leq t$, then $\|x + y\| \leq 1 + \varepsilon_k$. Define the desired blocking $\{F_n\}_{n=1}^{\infty}$ of $\{E_n\}_{n=1}^{\infty}$ by setting $F_k = E_{n_{k-1},n_k}$ for $k = 1, 2, \dots$. It is enough to check that $\{F_{2n}\}_{n=1}^{\infty}$ and $\{F_{2n-1}\}_{n=1}^{\infty}$ are both c_0-decompositions. To check that, for example, $\{F_{2n}\}_{n=1}^{\infty}$ is a c_0-decomposition, it is sufficient to observe that if x_k is in F_{2k} and $\sup_k \|x_k\| \leq t$, then for each $m = 1, 2, \dots$ for which $\|\sum_{k=1}^{m} x_k\| \geq 1$, the inequality

$$\|\sum_{k=1}^{m+1} x_k\| \leq (1 + \varepsilon_{n_{2m+1}})\|\sum_{k=1}^{m} x_k\| \tag{6}$$

is true. This completes the proof when X has a shrinking finite dimensional decomposition.

In the general case we know from Proposition 2.4 that X^* is separable, hence by [10] (or see [19, 1.g.2]) there is a subspace Y of X so that Y and X/Y both have a shrinking finite dimensional decomposition. In view of Proposition 2.8 we know that $\bar{\rho}_{X/Y}(t) = 0$ for some $t > 0$, hence from the first part of the proof we have that both Y and X/Y embed into c_0, whence since c_0 has the separable extension property we conclude that X also embeds into c_0. ■

We turn now to the duality theory of the moduli. To include nonreflexive spaces in the discussion, we need to introduce on X^* the weak* asymptotic modulus of uniform convexity, which we denote by $\bar{\delta}^*_{X^*}(\cdot)$. This is defined like the asymptotic modulus of uniform convexity, except that the finite co-dimensional subspace Y of X^* appearing in the definition of the asymptotic modulus of uniform convexity is restricted to weak* closed subspaces. The space X^* is said to be weak* asymptotically uniformly convex provided that $\bar{\delta}^*_{X^*}(t) > 0$ for all $t > 0$. The modulus $\bar{\delta}^*_{X^*}(\cdot)$ appears in [22] and weak* asymptotic uniform convexity (at least when X is separable) is called weak* UKK in [4] and UKK* in [14], [7]. In [7] (see [22] for the case where X has a shrinking basis) it is proved that $\bar{\delta}^*_{X^*}(\cdot)$ and $\bar{\rho}_X(\cdot)$ are equivalent to the dual Young's function of each other. (Recall that the dual Young's function, g, of a nonnegative continuous function f on $[0,1]$ which is zero at zero is defined by $g(s) = \sup\{st - f(s) : 0 \le t \le 1\}$.) In particular, X is asymptotically uniformly smooth if and only if X^* is weak* asymptotically uniformly convex.

When X is reflexive, this gives complete information on the duality theory of the moduli of asymptotic convexity and asymptotic smoothness, and the situation exactly parallels the well known duality theory of the moduli of convexity and smoothness, as detailed e.g. in [20, Section 1.e]. When X is nonreflexive, although the duality theory for the modulus of asymptotic smoothness of X is completely understood, it seems that the duality theory for the modulus of asymptotic convexity of X is not. Of course, the case of ℓ_1, which is the "most" asymptotically uniformly convex space, shows that there is no duality theory in general, but perhaps something can be said for asymptotically uniformly convex spaces which do not have subspaces isomorphic to ℓ_1.

Generally one is interested in power type estimates of moduli. Say that $\bar{\delta}_X$ (or $\bar{\delta}^*_{X^*}$) is of power type p provided that there is a constant $K > 0$ so that $\bar{\delta}_X(t) \ge Kt^p$ (or $\bar{\delta}^*_{X^*}(t) \ge Kt^p$) for all $0 \le t \le 1$. Similarly, say that $\bar{\rho}_X(\cdot)$ is of power type p provided that there is a constant $K > 0$ so that $\bar{\rho}_X(t) \le Kt^p$ for all $0 \le t \le 1$. The aforementioned duality implies that $\bar{\rho}_X(t)$ is of power type p if and only if $\bar{\delta}^*_{X^*}(t)$ is of power type q, where $1/p + 1/q = 1$.

To have a good understanding of the renorming theory for the moduli of asymptotic convexity and smoothness, one should discuss indices related to the Szlenk index of the underlying Banach space and analyze certain tree structures in the space. Since this takes us rather far afield, we instead just describe some of the renorming results and refer to [13] and [7] for fuller discussions of the theory. However, we will recall the notion of Szlenk index and mention the most basic connection of it to renorming theory.

Given a weak* compact subset K of the dual X^* of a Banach space X and $\varepsilon > 0$, let $K \setminus K'_\varepsilon$ be those elements of K which have a weak* neighborhood whose intersection with K has diameter less than ε. Evidently K'_ε is a weak* compact subset of K, so one can iterate this process transfinitely (taking intersections at limit ordinals) to form a decreasing sequence of weak* compact sets. If this iteration gives the empty set at some countable ordinal, the ε-Szlenk index $\mathrm{Sz}(\varepsilon, K)$ is defined to be the smallest such ordinal. Otherwise $\mathrm{Sz}(\varepsilon, K)$ is defined to be the first uncountable ordinal ω_1. When K is the unit ball of X^*, $\mathrm{Sz}(\varepsilon, K)$ is also denoted by $\mathrm{Sz}(\varepsilon, X)$. The Szlenk index $\mathrm{Sz}(X)$ of X is defined to be the supremum over $\varepsilon > 0$ of $\mathrm{Sz}(\varepsilon, X)$. Observe that for an infinite dimensional space X, $\mathrm{Sz}(X) = \omega_0$ if and only if $\mathrm{Sz}(\varepsilon, X)$ is finite for all $\varepsilon > 0$, and that the condition $\mathrm{Sz}(X) = \omega_0$ is preserved if the norm on X is replaced by an equivalent norm. Also, notice that if X is asymptotically uniformly smooth, then $\mathrm{Sz}(X) = \omega_0$. When X is separable, a sort of converse is true; that is, if $\mathrm{Sz}(X) = \omega_0$, then X admits an equivalent norm which is asymptotically uniformly smooth and in fact is of power type p for some $p > 1$. Modulo the duality discussed above, this was proved in [13], although partial results appeared earlier, for example in [27] and [14]. Another proof, which gives the best possible value of p, is given in [7].

One reason we wanted to mention the connection between the Szlenk index and renorming theory is that if $\alpha \geq \omega^\omega$ is an ordinal number, then $\mathrm{Sz}C(\alpha) > \omega_0$. So the discussion above implies that the space $C(\alpha)$ is not isomorphic to an asymptotically uniformly smooth Banach space. (A much less direct way to see this is to note that it follows from Theorem 7.4 and Example 7.7.)

There is a close connection between renorming theory and basic sequences. Seminal work was done by Prus [27], although some of the seeds are in [22]. To simplify the discussion, we treat only spaces which have a finite dimensional decomposition (FDD) and refer to [13] for what can be said in the general case.

Recall that if (E_n) is an FDD for a Banach space X, $n_0 < n_1 < \ldots$, and we define F_k by setting $F_k = \sum_{j=n_{k-1}}^{n_k - 1} E_j$, then (F_k) is also an FDD for X and is called a blocking of (E_n). Given $1 \leq p \leq \infty$, an FDD (E_n) is said to have a block upper (respectively, lower) p estimate provided there is a constant $C > 0$ so that if (x_k) is any sequence so that $x_k \in F_k$ for some blocking (F_k) of (E_n), then $\|\sum_{k=1}^\infty x_k\| \leq C\,(\sum_{k=1}^\infty \|x_k\|^p)^{1/p}$ (respectively, $\|\sum_{k=1}^\infty x_k\| \geq C^{-1}\,(\sum_{k=1}^\infty \|x_k\|^p)^{1/p}$) (with the usual interpretation of the right hand sides when $p = \infty$). Then (E_n) is said to have a block upper or lower p estimate with constant C. It is clear that if (E_n) has a block upper p estimate for some $p > 1$ then (E_n) is shrinking, while if (E_n) has a block lower p estimate for some $p < \infty$ then (E_n) is boundedly complete. An FDD has a block upper (respectively, lower) p estimate if and only if its dual FDD has a block lower (respectively, upper) q estimate, where $1/p + 1/q = 1$.

If (E_n) is an FDD for X with a block upper p and block lower r estimate, then there is an equivalent norm on X under which (E_n) has a block upper p and block lower r estimate with constant one. It is easy to check that if (E_n) has a block upper p and block lower r estimate with constant one, then $\bar{\rho}_X(\cdot)$ has power type p and $\bar{\delta}_X(\cdot)$ has power type r. In fact, it is enough that (E_n) has for each $\varepsilon > 0$ a blocking (F_k) which has a skipped block upper p and skipped block lower r estimate with constant $1+\varepsilon$. We say that (E_n) has a skipped block lower r estimate (with constant C) provided that for every blocking (F_k) of (E_n) and vectors $x_k \in F_{2k-1}$, $\|\sum_{k=1}^\infty x_k\| \geq C^{-1}\,(\sum_{k=1}^\infty \|x_k\|^r)^{1/r}$. "Skipped block upper p estimate" is defined analogously. However, this last concept is of limited use because, by the triangle inequality, if (E_n) has a skipped block upper p estimate with constant C, then it has a block upper p estimate with constant $2C$ and "$2C$" can be changed to one by passing to an equivalent norm. There are simple examples of FDDs which have a skipped block lower 1 estimate for which no blocking has a block lower r estimate for any finite r. Moreover, except in the easy case $r = \infty$, it is not known whether the constant for a skipped block lower r estimate can be improved by passing to an equivalent norm. However, if an FDD is boundedly complete and has a skipped block lower r estimate, then there is a blocking of the FDD which has a block lower r estimate. This is proved by use of the "killing the overlap" technique from [9]; deeper related things are in [13]. Putting all of this together, we see that if X has an FDD which has a skipped block lower r estimate and a skipped block upper p estimate, then there is an equivalent norm on X under which $\bar{\rho}_X(\cdot)$ has power type p and $\bar{\delta}_X(\cdot)$ has power type r. On the other hand, it is easy to prove that if there is one equivalent norm on X under which $\bar{\rho}_X(\cdot)$ has power type p and another equivalent norm under which $\bar{\delta}_X(\cdot)$ has power type r, and the FDD for X is shrinking, then there is a blocking of the shrinking FDD which has a block upper p and block lower r estimate. We summarize the main consequence of this discussion in Proposition 2.10, which in large part goes back to Prus [27].

Proposition 2.10 *Let X be a reflexive Banach space with an FDD (E_n) and $1 \leq p \leq r \leq \infty$. The following statements are equivalent.*

(1) (E_n) has a blocking which has a block upper p and block lower r estimate.

(2) There is one equivalent norm on X under which $\bar{\rho}_X(\cdot)$ has power type p and another equivalent norm under which $\bar{\delta}_X(\cdot)$ has power type r.

(3) There is an equivalent norm on X under which $\bar{\rho}_X(\cdot)$ has power type p and $\bar{\delta}_X(\cdot)$ has power type r.

Whether (2) and (3) in Proposition 2.10 are equivalent in all separable reflexive spaces is open. The equivalence of the analogues of (2) and (3) for the moduli of convexity and smoothness is known in two cases. The first occurs when the space X is a Banach lattice. The proof in this case bears some resemblance to the proof of Theorem 2.10 (see [20, Section 1.e]). The second case is that of $p = r$. In this case p must be two and X is then isomorphic to a Hilbert space (see [20, Section 1.f]). There is also a positive result for an analogue of this case for the asymptotic moduli, which we state as Proposition 2.11.

Proposition 2.11 *Let X be a separable reflexive Banach space for which $\bar{\rho}_X(\cdot)$ has power type p, where $1 < p < \infty$, and such that there is an equivalent norm on X for which $\bar{\delta}_X(\cdot)$ has power type p. Then X is isomorphic to a subspace of an ℓ_p sum of finite dimensional spaces.*

Remark: It is more or less obvious that the hypotheses imply that there is a constant $C > 0$ so that every weakly null normalized tree in X has a branch which is C-equivalent to the unit vector basis of ℓ_p, so Proposition 2.11 is a corollary of Theorem 4.1 in [24]. However, the proof of this theorem from [24] is rather complicated. It is based in part on the arguments in [12] which in turn are based on arguments from [11], so we decided to give a relatively simple proof of Proposition 2.11 which is modeled on the proof of Theorem 1 in [11]. We refer to that proof for details which we omit here. Our proof avoids any mention of trees.

Proof: Let $\{x_n, x_n^*\}_{n=1}^\infty$ be a biorthogonal system such that $\{x_n\}_{n=1}^\infty$ has dense linear span in X and $\{x_n^*\}_{n=1}^\infty$ has dense linear span in X^*. Since $\bar{\rho}_X(\cdot)$

has power type p, there is a blocking $E_n = \text{span}\,\{x_j\}_{j=m_n}^{m_{n+1}-1}$, $1 = m_1 < m_2 < \ldots$, so that every skipped blocking of (E_n) is an FDD with basis constant as close to one as we like (say, less than two) and has a block upper p-estimate with some constant C depending only on X. By the duality of the asymptotic moduli for reflexive spaces, there is an equivalent norm on X^* under which $\bar{\rho}_{X^*}(\cdot)$ has power type q, $1/p+1/q=1$. So by passing to a further blocking of the biorthogonal sequence and adjusting C, we can assume that $F_n := \text{span}\,\{x_j^*\}_{j=m_n}^{m_{n+1}-1}$ is such that every skipped blocking of (F_n) is an FDD with basis constant less than two and has a block upper q-estimate with constant C. By passing to one more blocking, we also may assume that for each $n > 1$, the subspace $H_n := F_{n-1}+F_n+F_{n+1}$ of X^* is 3-norming over E_n. Define now $Q : X \to (\sum H_{4n}^*)_p \,(= (\sum H_{4n})_q^*)$ by evaluation; i.e., for $y_n^* \in H_{4n}$ with $\sum \|y_n^*\|^q < \infty$, $\langle Qx, (y_n^*)\rangle = (y_n^*(x))$. The mapping Q is a well-defined bounded linear operator from X into $(\sum H_{4n}^*)_p$ because (H_{4n}) is a skipped blocking of (F_n) and thus has a block upper q estimate. Since each H_{4n} is 3-norming over E_{4n} and (E_{4n}) has a block upper p estimate, we deduce that the restriction of Q to the closed span Y of (E_{4n}) is an isomorphism.

It is simple to check that $Y^\perp$ is the closed linear span of $\{F_k : k \notin (4n)\}$ (see the proof of [11, Theorem 1]). Set $G_n = F_{4n-3} + F_{4n-2} + F_{4n-1}$. Then (G_n) is an FDD for $Y^\perp$ and (G_n) has a block upper q estimate. Thus the dual FDD (G_n^*) for X/Y has a block lower p estimate. But by Proposition 2.8, the asymptotic modulus of smoothness for X/Y has power type p, so by Proposition 2.10 there is a blocking of the FDD (G_n^*) for X/Y which has a block upper p estimate. This implies that X/Y is isomorphic to the ℓ_p sum of a sequence of finite dimensional spaces, and hence so is $Z := (\sum H_{4n}^*)_p \oplus X/Y$. Finally, define $S : X \to Z$ by $Sx = (Qx, x+Y)$. The operator S is an isomorphism because the restriction of Q to Y is an isomorphism. ∎

It follows from [13] that if X is a separable space whose asymptotic modulus of smoothness has power type $p > \infty$, then X is a quotient of a space which has a basis which has a block lower p estimate. This suggests that if X is a reflexive space which satisfies condition (2) in Proposition 2.10, then X embeds into a space with a basis which has a block upper p estimate and a block lower r estimate.

2.2 Other preliminaries

Here we prove two simple lemmas which will be used several times in the sequel.

Lemma 2.12 *Let X be a separable Banach space and Y a Banach space with the Radon–Nikodým property. Let D be a convex domain in X, $f : D \to Y$ a Lipschitz function, M any co-null set and $0 \neq u \in X$. Put*

$$R_u = \Big\{ \frac{f(x+tu) - f(x)}{t} \ ; \ x, x+tu \in D, t > 0 \Big\}$$

and

$$\tilde{R}_u = \Big\{ D^G f(x)u \ ; \ x \in \mathrm{diff}^{\mathrm{G}}(f) \cap M \Big\}.$$

Then $\overline{\mathrm{conv} R_u} = \overline{\mathrm{conv} \tilde{R}_u}$.

Proof: Clearly $\tilde{R}_u \subseteq \overline{R_u}$ and consequently $\overline{\mathrm{conv} \tilde{R}_u} \subseteq \overline{\mathrm{conv} R_u}$. To prove the other inequality we use the Hahn–Banach theorem. Assume that $\{y \ ; \ y^*(y) > a\}$ contains a point of R_u; i.e., there is an $x_0 \in D$ and $t > 0$ so that $x_0 + tu \in D$ and

$$y^*(f(x_0 + tu) - f(x_0)) > ta.$$

This inequality holds also for all x in some neighborhood of x_0. Thus it follows from the assumptions on X and Y and the theorem about existence of Gâteaux derivatives, that for some x in that neighborhood f is Gâteaux differentiable at $x + su$ and $x + su \in M$ for almost all $0 < s < t$. Clearly, for some s in this interval, $y^*(D^G f(x+su)u) > a$. ∎

Lemma 2.13 *Let X be a Banach space and X_0 a finite co-dimensional subspace of X. Then for all $\varepsilon > 0$ there is a finite dimensional subspace E of X such that $B_E + 2B_{X_0} \supset (1-\varepsilon)B_X$.*

Proof: Let $\{y_i\}_{i \in I}$ be a finite $\varepsilon/4$-net in B_{X/X_0} with $\|y_i\| < 1$ for all i. Lift each y_i to an $x_i \in B_X$ (i.e., $Qx_i = y_i$ where $Q : X \to X/X_0$ is the quotient map). Put $E = \mathrm{span}\{x_i\}_{i \in I}$.

If $x \in B_X$ let $i \in I$ be such that $\|y_i - Qx\| < \varepsilon/2$. Then $Q(x_i - x) = y_i - Qx \in X/X_0$, so there is an $x_0 \in X_0$ with $\|x_i - x - x_0\| < \varepsilon/2$. Clearly, $\|x_0\| \leq \|x\| + \|x_i\| + \varepsilon/2 \leq 2 + \varepsilon/2$, so

$$B_X \subset B_E + (2 + \varepsilon/2)B_{X_0} + (\varepsilon/2)B_X.$$

Iterating, we get for each n,

$$B_X \subseteq (1+\varepsilon/2+(\varepsilon/2)^2+\ldots+(\varepsilon/2)^{n-1})(B_E+(2+\varepsilon/2)B_{X_0})+(\varepsilon/2)^n B_X.$$

Hence $B_X \subseteq \frac{1}{1-\varepsilon/2}(B_E+(2+\varepsilon/2)B_{X_0})$ from which the conclusion of the lemma follows. ∎

3 Warmup: maps from c_0 to uniformly convex spaces

Before we get to the main theorems of this paper we thought it is worthwhile to present the simplest possible case of a theorem of the type we consider here. The basic idea of the proofs of the stronger theorems is already present here without many of the technical details needed in the other proofs.

Theorem 3.1 *Let $D \subseteq c_0$ be an open set, Y a uniformly convex space and $f : D \to Y$ a Lipschitz function. Then for every $\varepsilon > 0$ and every co-null set M, $\mathrm{diff}^{\varepsilon\mathrm{F}}(f) \cap M \neq \emptyset$.*

Proof: Assume, as we may that $\mathrm{Lip}(f) = 1$. Let $\varepsilon > 0$ and let $\tau > 0$ be such that $\tau < \varepsilon/2$ and $\delta_Y(a) < 2\tau$ implies $a < \varepsilon/2$. By Lemma 2.12 there is an $x_0 \in D \cap \mathrm{diff}^{\mathrm{G}}(f) \cap M$ for which $\|D^G f(x_0)\| > 1-\tau$. Translating in the domain and in the range, we may and shall assume that $x_0 = 0$ and that $f(0) = 0$. There is a finitely supported $e \in c_0$ such that $\|e\| = 1$ and $\|D^G f(0)e\| > 1-\tau$. Let P be the natural projection onto the support of e. We are going to show that $D^G f(0) \circ P$ is an ε-Fréchet derivative of f at zero.

First note that, since $E = Pc_0$ is finite dimensional, $D^G f(0)_{|E}$ is actually a Fréchet derivative of $f_{|E}$. So there is a $\gamma_0 > 0$ such that for all $x \in E$ with $\|x\| \leq \gamma_0$,

$$\|f(x) - D^G f(0)x\| < \tau\|x\|.$$

Let $z \in c_0$, $\|z\| < \gamma_0/2$. Put $x = Pz$, $y = z - x$. It is clearly enough to show that $\|f(z) - f(x)\| \leq \frac{\varepsilon}{2}\|z\|$. Put $\gamma = \|z\|$.

$$\|f(x+\gamma e) - f(x-\gamma e)\| > 2\gamma\|D^G f(0)e\| - 2\tau\gamma > 2\gamma - 4\tau\gamma, \qquad (7)$$

while

$$\|f(x \pm \gamma e) - f(z)\| \leq \|\pm\gamma e - y\| \leq \gamma \qquad (8)$$

since e and y are disjointly supported and $\|y\| \le \gamma$. This means that $f(z)/\gamma$ is a "4τ-approximate midpoint" of the segment $[f(x-\gamma e)/\gamma, f(x+\gamma e)/\gamma]$. The uniform convexity of Y implies, as we shall presently see, that the distance of $f(z)/\gamma$ from the affine midpoint, $u = \frac{f(x-\gamma e)+f(x+\gamma e)}{2\gamma}$, is smaller than $\delta_Y^{-1}(2\tau)/2 < \varepsilon/4$. The same applies if we take $z = x$ hence

$$\|f(z) - f(x)\| \le \|f(z) - \gamma u\| + \|f(x) - \gamma u\| \le 2(\varepsilon/4)\gamma = \varepsilon\|z\|/2.$$

To estimate the distance between $\frac{f(z)}{\gamma}$ and $\frac{f(x-\gamma e)+f(x+\gamma e)}{2\gamma}$, put $u_\pm = \frac{f(x\pm\gamma e)-f(z)}{\gamma}$. Then (8) and (7) imply

$$\|u_+\|, \|u_-\| \le 1 \quad \text{and} \quad \|u_+ - u_-\| > 2 - 4\tau.$$

Consequently, $\delta_Y(\|u_+ + u_-\|) \le 1 - \frac{\|u_+ - u_-\|}{2} < 2\tau$ and, by the choice of τ, $\frac{\|u_+ + u_-\|}{2} < \varepsilon/4$. ■

4 Maps from ℓ_r to ℓ_p, $r > 2 \ge p > 1$, and the like

The main result of this section is Theorem 4.1 below. To illustrate its use, note that it implies in particular that any Lipschitz map from a domain in a subspace of ℓ_r to ℓ_p, $r > 2 \ge p > 1$, is ε-Fréchet differentiable somewhere, for every $\varepsilon > 0$. In Section 6 we will strengthen this theorem. We found it worthwhile however to prove here this weaker version since its proof, although not so simple, is much simpler than that of Theorem 6.1. Theorem 4.2 below, which is a strengthening of Theorem 4.1 in another direction, is given here mostly because it is more accessible to iterations, as we shall see in Section 7.

Theorem 4.1 *Let X and Y be separable Banach spaces with Y uniformly convex. Assume that for all $c > 0$ $\bar\rho_X(t)/\delta_Y(ct) \to 0$ when $t \to 0$. Then, for every open set $D \subseteq X$, every Lipschitz $f : D \to Y$, every $\varepsilon > 0$ and every co-null set $M \subseteq X$, $\mathrm{diff}^{\varepsilon\mathrm{F}}(f) \cap M \ne \emptyset$.*

Proof: Let $\bar\rho(t)$ be any function satisfying $\bar\rho(t) \ge \bar\rho_X(t)$, $\bar\rho(t) > 0$ for all $t > 0$ and $\bar\rho(t)/\delta_Y(ct) \to 0$ as $t \to 0$ for all $c > 0$. One can take for example $\bar\rho(t) = \bar\rho_X(t) + t\delta_Y(t^2)$. (Actually the case of $\bar\rho_X(t) = 0$ for some

$t > 0$ is dealt with in the next section, we preferred not to exclude this case here.) Assume without loss of generality that $\mathrm{Lip}(f) = 1$. By Lemma 2.12, $\sup\{\|D^G f(x)e\|\ ;\ x \in \mathrm{diff}^G(f) \cap M,\ \|e\| = 1\} = 1$. Let $\varepsilon > 0$ and let $\eta > 0$ be small with respect to ε in a manner to be specified later. Let $x_0 \in \mathrm{diff}^{\mathrm{G}}(f) \cap M$ and $e \in S_X$ be such that

$$\|D^G(f)(x_0)e\| > 1 - \bar{\rho}(\eta). \tag{9}$$

Translating in the domain and in the range, we may assume without loss of generality that $x_0 = 0$ and that $f(0) = 0$.

Since, by Proposition 2.3, every operator from X to Y is compact, we can find a finite co-dimensional subspace X_0 of X such that

$$\|D^G f(0)_{|X_0}\| < \bar{\rho}(\eta) \text{ and } \|e + x\| - 1 < 2\bar{\rho}(\eta) \text{ for } \|x\| \le \eta,\ x \in X_0. \tag{10}$$

Use Lemma 2.13 to find a finite dimensional subspace $E \subset X$ satisfying $e \in E$ and

$$B_E + 2B_{X_0} \supset (1 - \eta)B_X. \tag{11}$$

Since $\dim E < \infty$, there is a $\gamma_0 > 0$ such that

$$\|f(u) - D^G f(0)u\| < \bar{\rho}(\eta)\|u\| \tag{12}$$

for all $u \in E$, $\|u\| < \gamma_0$.

Let $z \in X$ with $\|z\| < \gamma_0/10$. We would like to show that, if η is chosen small enough, then $\|f(z) - D^G f(0)z\| < \varepsilon\|z\|$ and thus that $D^G f(0)$ is an ε-Fréchet derivative of f at 0. Put $\gamma = 3\|z\|/\eta$ and decompose $z = u + v$ with $u \in E$, $v \in X_0$, $\|u\| < 2\gamma\eta/3$, $\|v\| < \gamma\eta$ (which is possible by (11)). Then

$$\|f(z) - D^G f(0)z\| \le \|f(z) - f(u)\| + \|f(u) - D^G f(0)(u)\| + \|D^G f(0)(v)\|$$

and, by (10) and (12) the sum of the last two summands is smaller than $2\bar{\rho}(\eta)\gamma\eta < \frac{\varepsilon}{2}\|z\|$ (if η is small enough). It is thus enough to prove that

$$\|f(z) - f(u)\| < \frac{\varepsilon}{2}\|z\|. \tag{13}$$

This is done by an approximate midpoint argument. Since (10) implies that

$$\|\gamma e \pm v\| = \gamma\|e \pm \frac{v}{\gamma}\| \le \gamma(1 + 2\bar{\rho}(\eta)),$$

we deduce that

$$\|f(u \pm \gamma e) - f(u+v)\| \leq \|\gamma e \mp v\| \leq \gamma(1+2\bar{\rho}(\eta)). \tag{14}$$

Also, by (12),

$$\|f(u+\gamma e) - f(u-\gamma e)\| \geq 2\gamma\|D^G f(0)e\| - 4\bar{\rho}(\eta)\gamma \geq 2\gamma - 6\bar{\rho}(\eta)\gamma, \tag{15}$$

where the second inequality follows from (9). Before continuing the proof we first sketch the midpoint argument. It follows from (14) and (15) that $\frac{1}{\gamma}f(z) = \frac{1}{\gamma}f(u+v)$ is, for some absolute constant c, a $c\bar{\rho}(\eta)$-approximate midpoint of the segment $\frac{1}{\gamma}[f(u+\gamma e), f(u-\gamma e)]$. The uniform convexity of Y and the estimate on its modulus of uniform convexity imply now that the distance of $\frac{1}{\gamma}f(z)$ to $\frac{1}{\gamma}f(u)$ - another $c\bar{\rho}(\eta)$-approximate midpoint - is much smaller than η.

To make this argument precise, put $a = \frac{f(u+\gamma e)-f(u-\gamma e)}{2}$ and $b = f(u+v) - \frac{f(u+\gamma e)+f(u-\gamma e)}{2}$. It is enough to prove that $\|b\| < \frac{\varepsilon}{4}\|z\|$. Indeed, if this is the case for any v as above, then applying it to $v = 0$ we get that also $\|f(u) - \frac{f(u+\gamma e)+f(u-\gamma e)}{2}\| < \frac{\varepsilon}{4}\|z\|$ from which (13) follows.

Now by (14),

$$\|a \pm b\| = \|f(z) - f(u \mp \gamma e)\| \leq \gamma(1+2\bar{\rho}(\eta)).$$

Since $\gamma - 3\bar{\rho}(\eta)\gamma \leq \|a\| \leq \gamma$ it follows from the definition of the modulus of convexity δ that

$$\delta\Big(\frac{\|b\|}{\gamma}\Big) \leq \delta\Big(\frac{2\|b\|}{\gamma(1+2\bar{\rho}(\eta))}\Big) \leq 1 - \frac{\|a\|}{\gamma(1+2\bar{\rho}(\eta))} \leq 5\bar{\rho}(\eta).$$

The assumption that $\bar{\rho}(t)/\delta_Y(ct) \to 0$ now implies that if η is small enough then $\|b\|/\gamma << \eta$. In particular one can choose η so that $\|b\| < \gamma\eta\varepsilon/12 = \frac{\varepsilon}{4}\|z\|$. ■

The proof above gives a bit more than is stated: The ε-Fréchet derivative obtained can also be chosen to have norm arbitrarily close to the maximal possible norm (i.e., to $\mathrm{Lip}(f)$). This follows from (9) above. It turns out that one can achieve even more: For every $e \in S_X$, one can get the ε-Fréchet derivative T to satisfy that $\|Te\|$ is arbitrarily close to the Lipschitz constant restricted to the direction e; i.e., to $\sup\{\|y\| \ ; \ y \in R_e\}$ (recall the definition of R_e in Lemma 2.12). This version of Theorem 4.1 is easier to iterate as we shall see in Section 7 below.

Theorem 4.2 *Let X and Y be separable Banach spaces with Y uniformly convex. Assume that for all $c > 0$, $\bar{\rho}_X(t)/\delta_Y(ct) \to 0$ when $t \to 0$. Let $D \subseteq X$ be a convex domain, $f : D \to Y$ Lipschitz, $\varepsilon > 0$, $e \in X$ and $M \subseteq X$ a co-null set. Then there exists a $x_0 \in \mathrm{diff}^{\varepsilon\mathrm{F}}(f) \cap M$ for which $D^{\mathrm{G}}(f)(x_0)$ is an ε-Fréchet derivative and $\|D^G(f)(x_0)e\| > \sup\{\|y\| \ ; \ y \in R_e\} - \varepsilon$.*

The proof is very similar to that of Theorem 4.1 but involves a new twist.
Proof: Assume as before that $\mathrm{Lip}(f) = 1$ and assume also, as we may, that $\|e\| = 1$. Let $\bar{\rho}$ be as in the proof above. Denote $\alpha = \sup\{\|y\| \ ; \ y \in R_g, \ \|g - e\| \leq \varepsilon/2\}$, and notice that if $\alpha = 0$ then there is nothing to prove (beyond what was already proved), so we assume $\alpha > 0$. Let $\eta > 0$ be small with respect to ε and α in a manner to be specified later and use Lemma 2.12 to find an $x_0 \in \mathrm{diff}^{\mathrm{G}}(f) \cap M$ and a $g \in X$ with $\|g - e\| \leq \varepsilon/2$ such that

$$\|D^G(f)(x_0)g\| > \alpha - \bar{\rho}(\eta). \tag{16}$$

Again we assume that $x_0 = 0$ and that $f(0) = 0$. Use Proposition 2.3 to find a finite co-dimension subspace X_0 of X such that

$$\|D^G f(0)_{|X_0}\| < \bar{\rho}(\eta) \text{ and } \|h + x\| - 1 < 2\bar{\rho}(\eta) \tag{17}$$

for all $\|x\| \leq \eta$ with $x \in X_0$ and all $h \in \mathrm{span}(e, g)$ for which $\|h\| \leq 1$.

Also, use Lemma 2.13 to find a finite dimensional subspace $E \subset X$ satisfying $g \in E$ and

$$B_E + 2B_{X_0} \supset (1 - \eta)B_X \tag{18}$$

and find a $\gamma_0 > 0$ such that

$$\|f(u) - D^G f(0)u\| < \bar{\rho}(\eta)\|u\| \tag{19}$$

for all $u \in E$ for which $\|u\| < \gamma_0$.

Let $z \in X$ with $\|z\| < \gamma_0/10$. Put $\gamma = 6\|z\|/(\eta\varepsilon)$ and decompose $z = u+v$ with $u \in E$, $v \in X_0$, $\|u\| < \gamma\eta\varepsilon/3$, $\|v\| < \gamma\eta\varepsilon/2$. In order to prove that $D^G f(0)$ is an ε-Fréchet derivative of f at zero, it is again enough to prove that

$$\|f(z) - f(u)\| < \frac{\varepsilon}{2}\|z\|. \tag{20}$$

By (19) and (16) (and using $\|g\| < 2$)

$$\|f(u + \gamma g) - f(u - \gamma g)\| \geq 2\gamma\|D^G f(0)g\| - 8\bar{\rho}(\eta)\gamma \geq 2\gamma\alpha - 10\bar{\rho}(\eta)\gamma. \tag{21}$$

We would like to show that

$$\|f(u \pm \gamma g) - f(u+v)\| \leq \gamma\alpha + \gamma\varepsilon\bar{\rho}(\eta). \tag{22}$$

This is where the proof is different from the previous one. Note that by (17)

$$\|e - g \pm v/\gamma\| < \varepsilon/2 + \varepsilon\bar{\rho}(2\|v\|/\gamma\varepsilon) \leq \varepsilon/2 + \varepsilon\bar{\rho}(\eta).$$

Consequently, one can write $e - (g - v/\gamma) = a + b$ where a and b are two (parallel) vectors in X with $\|a\| < \varepsilon/2$ and $\|b\| < \varepsilon\bar{\rho}(\eta)$. Then,

$$\begin{aligned}
f(u+\gamma g) - f(u+v) &= f(u+v+\gamma(g-v/\gamma)) - f(u+v) \\
&= f(u+v+\gamma(g-v/\gamma)) - f(u+v+\gamma(e-a)) \\
&\quad + f(u+v+\gamma(e-a)) - f(u+v)
\end{aligned}$$

and $\mathrm{Lip}(f) = 1$ and the definition of α imply that

$$\|f(u+\gamma g) - f(u+v)\| \leq \gamma\alpha + \gamma\varepsilon\bar{\rho}(\eta).$$

Exactly the same argument works for $f(u-\gamma g) - f(u+v)$ and we thus get (22). Now (21),(22) and a midpoint argument essentially the same as the one in the end of the proof of Theorem 4.1 establish (20) and thus that $D^G f(0)$ is an ε-Fréchet derivative of f at zero. It remains to show that $\|D^G(f)(0)e\| > \sup\{\|y\| \ ; \ y \in R_e\} - \varepsilon$, but this follows from

$$\|D^G(f)(0)e\| \geq \|D^G(f)(0)g\| - \|e-g\| > \alpha - \bar{\rho}(\eta) - \varepsilon/2.$$

■

5 Maps from subspaces of c_0 to spaces with RNP

Here we deal with Lipschitz maps between subspaces of c_0 (or equivalently, by Theorem 2.9, spaces for which $\bar{\rho}_X(t) = 0$ for some $t > 0$) and spaces with the Radon–Nikodým property. A stronger version of the following theorem where one achieves a point of *Fréchet* differentiability in the same situation (but for a different notion of null sets) will appear in [18]. The proof here is much simpler so we thought there is a point to preserve it.

Theorem 5.1 *Let X and Y be separable Banach spaces with Y having the RNP and assume that for some $t > 0$, $\bar{\rho}_X(t) = 0$. Then, for every open set $D \subseteq X$, every Lipschitz $f : D \to Y$, every $\varepsilon > 0$ and every co-null set $M \subseteq X$, $\mathrm{diff}^{\varepsilon\mathrm{F}}(f) \cap M \neq \emptyset$.*

Proof: Assume without loss of generality that D is convex and that $\mathrm{Lip}(f) = 1$. Let $1/4 > \varepsilon > 0$ and let M be a co-null set. Put

$$R = \Big\{\frac{f(x+se) - f(x)}{s} \ ; \ x, x+se \in D, \ s > 0, \ \|e\| = 1\Big\}.$$

Since Y has the Radon–Nikodým property there is a $y_0^* \in S_{Y^*}$ and a $0 < \delta < \varepsilon$ such that the set

$$S(R, y_0^*, \delta) = \{y \ ; \ y \in \overline{\mathrm{conv} R}, \ y_0^*(y) > 1 - 4\delta\}$$

is non-empty and has diameter smaller than ε (see [25, Theorem 5.20]). Let $e \in S_X$ and $s > 0$ be such that for some $x_1 \in D$, $\frac{f(x_1+se)-f(x_1)}{s} \in S(R, y_0^*, \delta)$. Lemma 2.12 implies now that there is an $x_0 \in D \cap \mathrm{diff}^{\mathrm{G}}(f) \cap M$ such that $y_0^*(D^G f(x_0)e) > 1 - \delta$. We are going to show that $D^G f(x_0)$ is also an $(6 + 8t^{-1})\varepsilon$-Fréchet derivative of f at x_0. Translating in the domain and range we can assume without loss of generality that $x_0 = 0$ and $f(0) = 0$.

The assumption that $\bar{\rho}_X(t) = 0$ implies that there is a finite co-dimensional subspace X_0 of X such that $\|e + x\| < 1 + \delta$ for all $x \in X_0$ with $\|x\| \leq t$. Replacing X_0 with its intersection with the annihilator of a functional norming e, we may and shall also assume that $\|e + x\| \geq 1$ for all $x \in X_0$. Finally, since $D^G f(0)$ is compact we can also assume that $\|D^G f(0)_{|X_0}\| < \varepsilon$. By Lemma 2.13 there is a finite dimensional subspace E containing e such that

$$2B_E + 4B_{X_0} \supset B_X. \tag{23}$$

Since E is finite dimensional, there is a $\gamma_0 > 0$ such that for all $u \in E$ with $\|u\| \leq \gamma_0$

$$\|f(u) - D^G f(0)u\| < \delta\|u\| < \varepsilon\|u\|. \tag{24}$$

Let $z \in D$ with $\|z\| = \gamma < t\gamma_0/8$ and put $z = u + v$ with $u \in E$, $v \in X_0$, $\|u\| \leq 2\gamma$, $\|v\| \leq 4\gamma$. Then

$$\|f(z) - D^G f(0)z\| \leq \|f(z) - f(u)\| + \|f(u) - D^G f(0)u\| + \|D^G f(0)v\|. \tag{25}$$

The middle term is smaller than $2\varepsilon\|z\|$ by (24) and the last term is smaller than $4\varepsilon\|z\|$ by the fact that $\|D^G f(0)_{|X_0}\| < \varepsilon$. To evaluate the first term in (25) put $\sigma = 4\gamma/t$ and notice first that by the first inequality in (24)

$$\begin{array}{rcl} y_0^*\Big(\frac{f(u+\sigma e)-f(u)}{\sigma}\Big) & > & \sigma^{-1}(y_0^*(D^G f(0)(u+\sigma e) - D^G f(0)u)) \\ & & -\sigma^{-1}(\delta\|u+\sigma e\| + \delta\|u\|) \\ & \geq & y_0^*(D^G f(0)e) - \sigma^{-1}\delta(\sigma + 2\|u\|) > 1 - 3\delta. \end{array}$$

Assuming for a minute that $y_0^*(f(z)) \leq y_0^*(f(u))$, we get that also

$$y_0^*\Big(\frac{f(u+\sigma e) - f(z)}{\sigma}\Big) > 1 - 3\delta.$$

Since $\|v\| \leq t\sigma$,

$$\sigma \leq \|\sigma e - v\| < \sigma(1+\delta). \tag{26}$$

Consequently,

$$y_0^*\Big(\frac{f(u+\sigma e) - f(z)}{\|\sigma e - v\|}\Big) > \frac{1-3\delta}{1+\delta} > 1 - 4\delta,$$

and we get that both $\frac{f(u+\sigma e)-f(z)}{\|\sigma e - v\|}$ and $\frac{f(u+\sigma e)-f(u)}{\sigma}$ are in $S(R, y_0^*, \delta)$. This implies that

$$\Big\|\frac{f(u+\sigma e) - f(u)}{\sigma} - \frac{f(u+\sigma e) - f(z)}{\|\sigma e - v\|}\Big\| < \varepsilon.$$

Using $\mathrm{Lip}(f) = 1$ and (26) one deduces from this that

$$\begin{array}{rcl} \Big\|\frac{f(u)-f(z)}{\sigma}\Big\| & < & \varepsilon + \|f(u+\sigma e) - f(z)\|\Big(\frac{1}{\sigma} - \frac{1}{\|\sigma e - v\|}\Big) \\ & \leq & \varepsilon + \Big(\frac{\|\sigma e - v\|}{\sigma} - 1\Big) < \varepsilon + \delta \end{array}$$

and thus that

$$\|f(u) - f(z)\| < 2\sigma\varepsilon = 8t^{-1}\varepsilon\|z\|.$$

If $y_0^*(f(z)) > y_0^*(f(u))$ then a similar argument, starting with

$$y_0^*\Big(\frac{f(u) - f(u-\sigma e)}{\sigma}\Big) > 1 - 3\delta,$$

leads to the same estimate.

Combining the estimates for the three terms in (25) we get that

$$\|f(z) - D^G f(0)z\| < (6 + 8t^{-1})\varepsilon\|z\|$$

which concludes the proof. ∎

Like Theorem 4.1, Theorem 5.1 admits a strengthening, giving more information on the ε-Fréchet derivative. This stronger theorem is better suited for iterations and has Theorem 7.4 as an application. We delay its statement and proof to the iteration section 7.

6 Maps from ℓ_r to ℓ_p, $r > p \geq 1$, and the like

Here we strengthen Theorem 4.1 by replacing the assumption of uniform convexity of the range with an assumption of asymptotic uniform convexity and δ_Y with $\bar{\delta}_Y$. The proof however is more complicated and also uses the existence of ε-Fréchet differentiability points for maps from asymptotically uniformly smooth spaces to finite dimensional spaces. This existence was proved in the case of uniformly smooth spaces in [16] and will be proved in its full generality in Section 8.

Theorem 6.1 *Let X be a separable Banach space and let Y be a separable Banach space which has the Radon–Nikodým property. Assume that for all $t > 0$, $\bar{\delta}_Y(t) > 0$ and that for all $c > 0$, $\bar{\rho}_X(t)/\bar{\delta}_Y(ct) \to 0$ when $t \to 0$. Then, for every open set $D \subseteq X$, every Lipschitz $f : D \to Y$, every $\varepsilon > 0$ and every co-null set $M \subseteq X$, $\operatorname{diff}^{\varepsilon \mathrm{F}}(f) \cap M \neq \emptyset$.*

Proof: We shall denote $\bar{\delta} = \bar{\delta}_Y$. Let $\bar{\rho}$ be any function such that for all $t > 0$, $\bar{\rho}(t) > \bar{\rho}_X(t)$ and for all $c > 0$, $\bar{\rho}(t)/\bar{\delta}(ct) \to 0$ as $t \to 0$; for example $\bar{\rho}(t) = \bar{\rho}_X(t) + t\bar{\delta}(t^2)$. (This is done to circumvent the possibility that $\bar{\rho}_X(t) = 0$ for some $t > 0$ which would complicate the estimates below. Of course if this happens we can apply Theorem 5.1, but we prefer a self contained argument.) Assume, without loss of generality, that $\mathrm{Lip}(f) = 1$ and that D is convex. By Lemma 2.12, $\sup\{\|D^G f(x)e\| \,;\, x \in \operatorname{diff}^G(f),\ \|e\| = 1\} = 1$. Put

$$W = \{D^G f(x)e \,;\, x \in \operatorname{diff}^G(f),\ \|e\| = 1\}.$$

Let $\varepsilon > 0$ and let $\eta > 0$ be small with respect to ε in a manner to be specified later. Let $0 < \tau < \frac{1}{2}\bar{\rho}(\eta)$ be such that there exists a functional $y_0^* \in S_{Y^*}$

satisfying $\sup\{\langle y_0^*, w\rangle \; ; \; w \in W\} > 1-\tau/2$ and $\mathrm{diam}(S(W, y_0^*, \tau)) < \frac{1}{2}\bar\rho(\eta)$. Here $S(W, y_0^*, \tau)$ denotes the slice of W of size τ in the direction of y_0^*; i.e., $S(W, y_0^*, \tau) = \{w \in W \; ; \; \langle y_0^*, w\rangle > 1-\tau\}$. Such a slice exists because Y has the RNP; see [25, Theorem 5.20].

Pick any $z_0 \in S(W, y_0^*, \tau)$ and let $Y_0 \subset \ker(y_0^*)$ with $\dim Y/Y_0 < \infty$ be such that $\bar\delta_Y(t, \frac{z_0}{\|z_0\|}, Y_0) > \bar\delta(t)/2$ for all $\varepsilon\eta/6 \le t \le 1$. Let $Q = Q_{Y_0} : Y \to Y/Y_0$ be the quotient map and let u_0^* be the functional on Y/Y_0 induced by y_0^*. Then $u_0^* \circ Q = y_0^*$ and $D^G f(x)e \in S(W, y_0^*, \tau)$ if and only if $D^G(Q\circ f)(x)e \in S(QW, u_0^*, \tau)$. By Corollary 8.3 below (see Remark 8.4), there is an $x_0 \in \mathrm{diff}^{(\bar\rho(\eta)/2)\mathrm{F}}(Q\circ f) \cap \mathrm{diff}^{\mathrm{G}}(f) \cap M$ and an $e \in S_X$ such that

$$D^G(Q\circ f)(x_0)e \in S(QW, u_0^*, \tau).$$

Note that this implies that

$$\Big\|\frac{z_0}{\|z_0\|} - D^G f(x_0)e\Big\| \le \Big\|\frac{z_0}{\|z_0\|} - z_0\Big\| + \|z_0 - D^G f(x_0)e\| < \bar\rho(\eta). \tag{27}$$

Translating in the domain and in the range, we may assume without loss of generality that $x_0 = 0$ and that $f(0) = 0$.

Let $\gamma_0 > 0$ be such that $\|z\| < 10\gamma_0$ implies

$$\|Qf(z) - QD^G f(0)z\| \le \bar\rho(\eta)\|z\|. \tag{28}$$

Since, by Proposition 2.3, every operator from X to Y is compact, we can find a finite co-dimensional subspace X_0 of X such that

$$\|D^G f(0)_{|X_0}\| < \bar\rho(\eta) \text{ and } \|e + x\| - 1 < \bar\rho(\eta) \text{ for } \|x\| \le \eta,\ x \in X_0. \tag{29}$$

By Lemma 2.13, there is a finite dimensional subspace $E \subset X$ satisfying $e \in E$ and

$$B_E + 2B_{X_0} \supset (1-\eta)B_X. \tag{30}$$

Since $\dim E < \infty$, there is a $0 < \gamma_1 < \gamma_0$ such that

$$\|f(u) - D^G f(0)u\| \le \bar\rho(\eta)\|u\| \tag{31}$$

for all $u \in E$ for which $\|u\| < \gamma_1$.

Let $z \in X$ with $\|z\| < \eta\gamma_1/10$. We would like to show that, if η is chosen small enough, then $\|f(z) - D^G f(0)z\| < \varepsilon\|z\|$. Put $\gamma = 6\|z\|/\eta$ and decompose $z = u + v$ with $u \in E$, $v \in X_0$, $\|u\| < 2\gamma\eta/3$, and $\|v\| < \gamma\eta$ (which is possible by (30)). Now,

$$\|f(z) - D^G f(0)z\| \leq \|f(z) - f(u)\| + \|f(u) - D^G f(0)(u)\| + \|D^G f(0)(v)\|$$

and, by (29) and (31) the sum of the last two summands is smaller than $2\bar{\rho}(\eta)\gamma\eta < \frac{\varepsilon}{2}\|z\|$ (if η is small enough). It is thus enough to prove that

$$\|f(z) - f(u)\| < \frac{\varepsilon}{2}\|z\|. \tag{32}$$

By (28) and (29),

$$\begin{aligned} \|Q(f(z) - f(u))\| &\leq \|Q(D^G f(0)(z) - D^G f(0)(u))\| \\ &\quad + \|Q(f(z)) - QD^G f(0)(z)\| \\ &\quad + \|Q(f(u)) - QD^G f(0)(u)\| \\ &\leq 2\bar{\rho}(\eta)\gamma\eta. \end{aligned}$$

So there is a $y_0 \in Y_0$ such that
$\|\gamma^{-1}(f(z) - f(u)) - y_0\| < 4\bar{\rho}(\eta)\eta$. Now

$$\begin{aligned} \|f(\gamma e) - (f(z) - f(u))\| &\leq \|f(\gamma e + u) - f(z)\| \\ &\quad + \|f(\gamma e + u) - f(u) - f(\gamma e)\| \\ &\leq \|f(\gamma e + u) - f(z)\| \\ &\quad + \bar{\rho}(\eta)(\|\gamma e + u\| + \|u\| + \gamma) \end{aligned}$$

by (31). Since

$$\|\gamma e \pm v\| = \gamma\|e \pm \frac{v}{\gamma}\| \leq \gamma(1 + \bar{\rho}(\eta))$$

we get that

$$\|f(\gamma e + u) - f(z)\| \leq \|\gamma e - v\| \leq \gamma(1 + \bar{\rho}(\eta)) \tag{33}$$

and it follows that

$$\begin{aligned} \|f(\gamma e) - (f(z) - f(u))\| &\leq \|f(\gamma e + u) - f(u + v)\| + 3\bar{\rho}(\eta)\gamma \qquad (34) \\ &\leq \gamma(1 + \bar{\rho}(\eta)) + 3\bar{\rho}(\eta)\gamma = \gamma(1 + 4\bar{\rho}(\eta)). \end{aligned}$$

On the other hand,

$$\begin{aligned}\|f(\gamma e)-(f(z)-f(u))\| &\geq \gamma\|D^G f(0)e-\gamma^{-1}(f(z)-f(u))\| \\ &\quad -\|f(\gamma e)-D^G f(0)(\gamma e)\| \\ &\geq \gamma\Big\|\frac{z_0}{\|z_0\|}-y_0\Big\|-6\bar{\rho}(\eta)\gamma \\ &\geq \gamma\Big(1+\frac{1}{2}\bar{\delta}\Big(\frac{\|f(z)-f(u)\|}{\gamma}-4\bar{\rho}(\eta)\eta\Big)\Big)-6\bar{\rho}(\eta)\gamma.\end{aligned}$$

If $\gamma^{-1}\|f(z)-f(u)\| \geq \varepsilon\eta/24+4\bar{\rho}(\eta)\eta$, it would follow from (34) that $\bar{\delta}(\varepsilon\eta/24) \leq 20\bar{\rho}(\eta)$, which would contradict the assumption of the theorem if η is small enough. Thus, $\gamma^{-1}\|f(z)-f(u)\| < \varepsilon\eta/24+4\bar{\rho}(\eta)\eta$ or $\|f(z)-f(u)\| \leq (\frac{\varepsilon}{4}+24\rho(\eta))\|z\| < \frac{\varepsilon}{2}\|z\|$, if η is small enough. This establishes (32) and concludes the proof. ∎

Recall that a map $f : X \to Y$ is a Lipschitz quotient map if it is Lipschitz and the image of any ball contains a ball about the image of the center of radius proportional (with constant of proportionality independent of the ball) to that of the original ball. If such a map exists, Y is said to be a Lipschitz quotient of X. The paper [1] dealt, in large part, with the problem of when is it true that a Lipschitz quotient of a Banach space X is necessarily a linear quotient of it. It was proved there, for example, that any Lipschitz quotient of $L_p(0,1)$, $1<p<\infty$, is a linear quotient of $L_p(0,1)$. The classification of Lipschitz quotients of ℓ_p, $1<p<\infty$, $p \neq 2$, was not completed. It is easy to see that if the Lipschitz quotient map has a point of ε-Fréchet differentiability for ε small enough, then any ε-Fréchet derivative is a linear quotient map of X onto Y. This together with Theorem 6.1 and the result of [1] mentioned above provides some information on this problem: If Y is a Banach space which is a Lipschitz quotient of ℓ_p, $2<p<\infty$, then Y^* is isomorphic to a subspace of L_q, $\frac{1}{p}+\frac{1}{q}=1$, and does not contain an isomorphic image of any ℓ_r, $q<r\leq 2$. Unfortunately this alone does not ensure that Y^* is isomorphic to a subspace of ℓ_q (i.e., that Y is isomorphic to a linear quotient of ℓ_p).

7 Iterations

In this section we show how some of the proofs of the theorems of the previous sections can be iterated to replace the domain space with ℓ_p or c_0 sums of similar spaces. We need a few ad hoc definitions.

Definition 7.1 *Given an $\varepsilon > 0$ and a vector $e \in X$ we say that a Lipschitz function f defined on a domain D in the Banach space X into a Banach space Y is (ε, e)-Fréchet differentiable at x_0 if it has an ε-Fréchet derivative, T, at x_0 satisfying $\|Te\| \geq \sup\{\|y\|;\ y \in R_e(f)\} - \varepsilon$.*

Recall the definition of the set R_e

$$R_e = \Big\{\frac{f(x+te) - f(x)}{t}\ ;\ x, x+te \in D, t > 0\Big\}.$$

Definition 7.2 *We say that the pair of Banach spaces (X, Y) has property P if for every $\varepsilon > 0$, every $e \in X$ and every convex domain D in X, every Lipschitz function f from D to Y is (ε, e)-Fréchet differentiable at some point of D.*

Note that if (X, Y) has property P so does (X', Y) for any X' isomorphic to X. This is the reason it is more accessible to iterations than the mere property of having a point of ε-Fréchet differentiability (or the property of having a point of ε-Fréchet differentiability with close to maximal possible norm of the derivative).

Proposition 7.3 *Let Y be a uniformly convex Banach space and let $2 < r < \infty$ with $t^r/\delta_Y(ct) \to 0$ as $t \to 0$ for all $c > 0$. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of Banach spaces such that for any finite sum, $X = \sum_{i=1}^{n} \oplus X_i$, (X, Y) has property P. Then $((\sum_{i=1}^{\infty} \oplus X_i)_r, Y)$ has property P. The same is true also for the c_0 sum: $((\sum_{i=1}^{\infty} \oplus X_i)_0, Y)$ has property P.*

The proof is very similar to that of Theorem 4.2. We shall only sketch its beginning.

Let D be a convex domain in $X = (\sum_{i=1}^{\infty} \oplus X_i)_r$, $2 < r < \infty$ or $r = 0$, and $f : D \to Y$ a Lipschitz function with $\mathrm{Lip}(f) = 1$. Let $e \in X$, satisfy $\|e\| = 1$ and let $0 < \varepsilon < 1/4$. Put $\alpha = \alpha_e = \sup\{\|y\|\ ;\ y \in R_g,\ \|g - e\| < \varepsilon/2\}$. If $\alpha = 0$ then change e to e' for which $\alpha_{e'} > 1/2$ (which is possible since $\mathrm{Lip}(f) = 1$). Assuming we proved the proposition for e', the point of (ε, e')-Fréchet differentiability found will trivially be also a point of (ε, e)-Fréchet differentiability, so we assume $\alpha = \alpha_e > 0$.

Let $g \in X$ be such that $\|g - e\| < \varepsilon/2$, and for some $x_1 \in D$ and some $0 < s < \infty$, $\left\|\frac{f(x_1+sg)-f(x_1)}{s}\right\| > \alpha - \delta$ (for $\delta > 0$ to be chosen later). We may and shall also assume that $g \in E = \sum_{i=1}^{n} \oplus X_i$ for some finite n and that

$D \cap E \neq \emptyset$. By the assumption and the isomorphic invariance of property P there is a point $x_0 \in D \cap E$ (which without loss of generality we assume is 0, and we also assume $f(0) = 0$) such that $f_{|D\cap E}$ is (ε, e)-Fréchet differentiable at $x_0 = 0$; i.e., there is an operator $T : E \to Y$ and a $\gamma_0 > 0$ such that if $u \in E$ with $\|u\| \leq \gamma_0$, then

$$\|f(u) - Tu\| < \delta\|u\|, \text{ and } \|Tg\| > \alpha - \delta.$$

Let P_E be the natural projection from X onto E and we want to show that $T \circ P_E$ is the required (ε, e)-Fréchet derivative of f at 0. For that it is clearly enough to show that if $\|z\| < \gamma_0/10$, then $\|f(z) - f(P_E z)\| < \frac{\varepsilon}{2}\|z\|$. This is proved by an approximate midpoint argument very similar to that at the end of the proof(s) of Theorem(s) 4.2 (and 4.1), putting $u = P_E z$ and $v = z - u$ and replacing the role of the operator $D^G f(0)$ in these proofs with that of T. ■

Remark: If may look natural to change the definition of property P so as to require that f admits a point of (ϵ, e)-Fréchet differentiability in $D \cap M$ for every co-null set M. However, we do not know how to iterate this stronger property.

Let α be a countable ordinal and consider $[0, \alpha]$ as a compact topological space with the order topology. Recall that $C(\alpha)$ denotes the space of continuous functions on $[0, \alpha]$. Since $C(\alpha^\omega)$ is isomorphic to the c_0 sum of $C(\alpha^n)$, $n = 1, 2, \ldots$, each of which is isomorphic to $C(\alpha)$, we get from Proposition 7.3 by transfinite induction that for α countable and for all uniformly convex Y, $(C(\alpha), Y)$ has property P. In particular, every Lipschitz map from a domain in $C(\alpha)$ to Y has a point of ε-Fréchet differentiability. This can be strengthened to

Theorem 7.4 *Let Y be Banach space with the Radon–Nikodým property and let α be a countable ordinal. Then, for every open set $D \subseteq C(\alpha)$, every Lipschitz $f : D \to Y$, and every $\varepsilon > 0$, $\mathrm{diff}^{\varepsilon\mathrm{F}}(f) \neq \emptyset$.*

Actually a stronger version of the theorem holds: such an f has even (many) points of *Fréchet* differentiability. The proof will appear elsewhere and is much harder so we thought it is worthwhile to preserve the proof below. We need a definition similar to that of property P.

Definition 7.5 *We say that the couple of Banach spaces (X, Y) has property P' if given any $\varepsilon > 0$, $0 \neq e \in X$, $0 \neq y^* \in Y^*$ and any Lipschitz function f from a convex domain D in X to Y, f has an ε-Fréchet derivative, T, at some point $x_0 \in D$ satisfying $y^*(Te) > \sup\{y^*(y)\ ;\ y \in R_e(f)\} - \varepsilon$.*

As with property P, property P' is also invariant under isomorphisms of X: If (X, Y) has property P so does (X', Y) for any X' isomorphic to X. Theorem 7.4 follows immediately from the following proposition.

Proposition 7.6 *Let Y be Banach space with the RNP and let $\{X_i\}_{i=1}^{\infty}$ be a sequence of Banach spaces such that for any finite sum, $E = \sum_{i=1}^{n} \oplus X_i$, (E, Y) has property P'. Then $((\sum_{i=1}^{\infty} \oplus X_i)_0, Y)$ has property P'.*

Proof: Let D be a convex domain in $X = (\sum_{i=1}^{\infty} \oplus X_i)_0$ and $f : D \to Y$ a Lipschitz function with $\mathrm{Lip}(f) = 1$. Let $e \in X$, $y^* \in Y^*$ satisfy $\|e\| = 1$, $\|y^*\| = 1$ and let $0 < \varepsilon < 1/4$. Put $\alpha = \alpha_{e,y^*} = \sup\{y^*(y)\ ;\ y \in R_g,\ \|g - e\| < \varepsilon/2\}$. If $\alpha = 0$ then change e to e' and y^* to y'^* for which $\alpha_{e',y'^*} > 1/2$ (which can be done since $\mathrm{Lip}(f) = 1$). Assuming we proved the theorem for e', y'^*, the same point x_0 will trivially work also for e, y^*, so we assume $\alpha = \alpha_{e,y^*} > 0$. Put $R = \overline{\mathrm{conv}}\{y \in R_g,\ \|g - e\| < \varepsilon/2\}$ and assume first that there is a $\delta < \varepsilon/4$ such that $S = \{y \in R\ ;\ y^*(y) > \alpha - 4\delta\}$ has diameter smaller than $(\varepsilon/4)^2$. (This can always be achieved by changing y^* a little; we shall outline the argument at the end of this proof).

Let $g \in X$ be such that $\|g - e\| < \varepsilon/2$, and for some $x_1 \in D$ and some $0 < s < 1$, $y^*\Big(\frac{f(x_1+sg)-f(x_1)}{s}\Big) > \alpha - \delta$. We may and shall also assume that $g \in E = \sum_{i=1}^{n} \oplus X_i$ for some finite n and that $D \cap E \neq \emptyset$. By the assumption and the isomorphic invariance of property P', there is a point $x_0 \in D \cap E$ such that $f_{|D\cap E}$ is ε-Fréchet differentiable at x_0. That is, there is an operator $T : E \to Y$ and a $\gamma_0 > 0$ such that if $u \in E$ with $\|u\| \leq \gamma_0$, then

$$\|f(x_0 + u) - f(x_0) - Tu\| < \delta\|u\| < \frac{\varepsilon}{4}\|u\|, \text{ and } y^*(Tg) > \alpha - \delta. \qquad (35)$$

As usual we are going to assume that $x_0 = 0$ and that $f(0) = 0$.

Let P_E be the natural projection from X onto E. We are going to show that $T \circ P_E$ is an ε-Fréchet derivative of f at zero and that $y^*(TP_E e) > \alpha - \varepsilon$. Let $z \in X$ with $\|z\| < \varepsilon\gamma_0/8$ and put $\gamma = 4\|z\|/\varepsilon$. Write $z = u + v$ with $u = P_E z$, $v = z - u$. Then by (35),

$$y^*\Big(\frac{f(u + \gamma g) - f(u)}{\gamma}\Big) > y^*(Tg) - 3\delta > \alpha - 4\delta.$$

Now if $y^*(f(z)) \le y^*(f(u))$ then

$$y^*\Big(\frac{f(u+\gamma g)-f(z)}{\gamma}\Big) > \alpha - 4\delta$$

and, since $\|\gamma g - v\| = \gamma$ and $\|g - \frac{v}{\gamma} - e\| \le \max\{\|g-e\|, \frac{\|v\|}{\gamma}\} < \frac{\varepsilon}{2}$,

$$\frac{f(u+\gamma g)-f(z)}{\gamma} \in S.$$

Since the same is true for $\frac{f(u+\gamma g)-f(u)}{\gamma}$, we get that $\|f(z)-f(u)\| < (\varepsilon/4)^2\gamma < \varepsilon\|z\|/2$. A similar argument works if $y^*(f(z)) > y^*(f(u))$. (Start with $y^*\big(\frac{f(z)-f(u-\gamma g)}{\gamma}\big) > y^*\big(\frac{f(u)-f(u-\gamma g)}{\gamma}\big) > \alpha - 4\delta$.) This together with (35) shows that $T \circ P_E$ is an ε-Fréchet derivative of f at zero. Indeed, if $\|z\| < \varepsilon\gamma_0/8$ then

$$\begin{aligned} \|f(z) - TP_E z\| &\le \|f(z) - f(P_E z)\| + \|f(P_E z) - TP_E z\| \\ &< \varepsilon\|z\|/2 + \varepsilon\|P_E z\|/4 < \varepsilon\|z\|. \end{aligned}$$

Finally, $y^*(Te) > y^*(Tg) - (1+\varepsilon)\|g-e\| > \alpha - \delta - (1+\varepsilon)\varepsilon/2 > \alpha - \varepsilon$.

All that was under the assumption that there are slices of the form $\{y \in R\ ;\ y^*(y) > \alpha - 4\delta\}$ of arbitrarily small diameter. If this is not the case, let y_1 be a strongly exposed point of R for which $y^*(y_1) > \alpha - \varepsilon/4$; such a point exists because Y has the RNP (see [25]) .Let y_1^* be a norm one functional which strongly exposes y_1 and let $\delta < \varepsilon$ be such that if $y \in R$ and $y_1^*(y) > y_1^*(y_1) - \delta$ then $\|y - y_1\| < \varepsilon/4$.

By the proof above there is an $x_0 \in D$ and a $\delta/2$-Fréchet derivative, T, of f at x_0 which also satisfies $y_1^*(Te) > y_1^*(y_1) - \delta/2$. Then for t small enough,

$$\Big\|Te - \frac{f(x_0+te)-f(x_0)}{t}\Big\| < \delta/2 \tag{36}$$

and thus

$$y_1^*\Big(\frac{f(x_0+te)-f(x_0)}{t}\Big) > y_1^*(y_1) - \delta.$$

Since $\frac{f(x_0+te)-f(x_0)}{t} \in R$ this implies that

$$\Big\|\frac{f(x_0+te)-f(x_0)}{t} - y_1\Big\| < \varepsilon/4.$$

Since $y^*(y_1) > \alpha - \varepsilon/4$ this in turn implies that

$$y^*\Big(\frac{f(x_0 + te) - f(x_0)}{t}\Big) > \alpha - \varepsilon/2.$$

Finally (36) implies

$$y^*(Te) > \alpha - \varepsilon/2 - \delta/2 > \alpha - \varepsilon.$$

$\blacksquare$

Unlike the results of the previous sections, Theorem 7.4 no longer holds if we replace the domain space $C(\alpha)$ with a general subspace of it.

Example 7.7 *Let S be the Schreier space; i.e., the completion of the space of eventually zero sequences $\{a_i\}_{i=1}^{\infty}$ with respect to the norm*

$$\|\{a_i\}\| = \sup\{\sum_{k=1}^{n} |a_{i_k}| \ ; \ n \in \mathbb{N} \text{ and } n \le i_1 < i_2 < \ldots < i_n\}.$$

Then S is isomorphic to a subspace of $C(\omega^\omega)$ but there is a Lipschitz function $f : S \to \ell_2$ which, for ε small enough, does not have a point of ε-Fréchet differentiability.

Proof: That S is isomorphic to a subspace of $C(\omega^\omega)$ should be clear. Let $f : S \to \ell_2$ be defined by $f(\{a_i\}) = \sum |a_i| e_i$. Then it is easily seen that f is Lipschitz (it is a composition of the "absolute value map" in S with the formal identity $I : S \to \ell_2$. That the later is bounded follows for example from the fact that if $\|\{a_i\}\|_S = 1$ then $\#\{i \ ; \ |a_i| \ge 1/k\} \le 2k$ for all $k \in \mathbb{N}$.) Finally, as was explained in the introduction, a composition of the absolute value map with the formal identity between two spaces with normalized unconditional bases is nowhere ε-Fréchet differentiable, if ε is small enough. $\blacksquare$

8 Maps from AUS spaces to $\mathbb{R}^n$

This section is devoted to proving that given finitely many real valued Lipschitz functions on an asymptotically uniformly smooth separable Banach space, they admit, for every $\varepsilon > 0$, a common point of ε-Fréchet differentiability. Under the stronger assumption of uniform smoothness this was proved in [16]. The proof here is simpler. As in [16] also here the point of ε-Fréchet differentiability can be chosen in any co-null set. In this section we denote, for a function $f : X \to \mathbb{R}$, $f'(x) = D^G(f)(x)$. We begin with a simple lemma.

Lemma 8.1 *Let X and Y be Banach spaces and $D \subset X$ a convex domain. Suppose that $f : D \to Y$ is Lipschitz and $x \in \operatorname{diff}^{\varepsilon \mathrm{F}}(f) \cap \operatorname{diff}^{\mathrm{G}}(f)$. Then there is $\delta > 0$ such that $\|f(x+h+u) - f(x+h) - f'(x)(u)\| \le 4\varepsilon(\|h\| + \|u\|)$ whenever $\|u\| < \delta$.*

Proof: Choose $\delta_1 > 0$ such that $\|f(x+u) - f(x) - f'(x)(u)\| \le 2\varepsilon\|u\|$ for $\|u\| < \delta_1$ and let $\delta = 2\varepsilon\delta_1/\mathrm{Lip}(f)$. Suppose that $\|u\| < \delta$. If $\|h\| + \|u\| \ge \mathrm{Lip}(f)\|u\|/(2\varepsilon)$, the required inequality is obvious. In the opposite case we have $\|h\| + \|u\| < \delta_1$ and so $\|f(x+h+u) - f(x+h) - f'(x)(u)\| \le \|f(x+h+u) - f(x) - f'(x)(h+u)\| + \|f(x+h) - f(x) - f'(x)(h)\| \le 2\varepsilon(\|h\| + \|u\|) + 2\varepsilon\|h\| \le 4\varepsilon(\|h\| + \|u\|)$. ∎

Suppose that $\bar{\rho}_X(t)/t \to 0$ and that $f_1, f_2, \dots$ is a sequence of real-valued Lipschitz functions on X. Let D be a convex domain in X and let S be a co-null set of points of D at which all the f_j-s are Gâteaux differentiable. Denote

$$
\begin{aligned}
\psi_1 &= \sup\{\|f_1'(x)\| : x \in S\}, \\
S_1(t) &= \{x \in S : \|f_1'(x)\| > \psi_1 - t\}, \\
\psi_k &= \lim_{t \searrow 0} \sup\{\|f_k'(x)\| : x \in S_{k-1}(t)\}, \\
S_k(t) &= \{x \in S_{k-1}(t) : \|f_k'(x)\| > \psi_k - t\}.
\end{aligned}
$$

For unit vectors $u_1, \dots, u_k$ we also denote

$$S_k(t; u_1, \dots, u_k) = \{x \in S : f_j'(x)(u_j) > \psi_j - t \text{ for } j = 1, \dots, k\}.$$

Theorem 8.2 *With the notations and assumptions above for every $k = 1, 2, \dots$ the following assertions hold*

(1_k) *For every $\varepsilon > 0$ there is $s > 0$ such that for every $x \in S_k(s)$ there is $\delta > 0$ with the property that*

$$|f_k(x+h+u) - f_k(x+h)| \le \psi_k\|u\| + \varepsilon(\|h\| + \|u\|)$$

whenever $\|u\| < \delta$.

(2_k) *For every $\varepsilon > 0$ there is $s > 0$ such that $S_k(s) \subset \operatorname{diff}^{\varepsilon \mathrm{F}}(f_k)$.*

(3_k) *For every $t, \varepsilon > 0$ there is $s > 0$ such that for every unit vectors $u_1, \dots, u_k$ and every $x \in S_k(s; u_1, \dots, u_k)$ there is $\delta > 0$ such that*

$$\lambda_k\{\sigma \in [0,d]^k : x + h + \sum_{i=1}^{k} \sigma_i u_i \notin S_k(t; u_1, \dots, u_k)\} < \varepsilon(d + \|h\|)d^{k-1}$$

whenever $d < \delta$ and $h \in X$ is such that $x + h + \sum_{i=1}^{k} \sigma_i u_i \in S$ for λ_k almost all $\sigma \in \mathbb{R}^k$. λ_k denotes Lebesgue measure on $\mathbb{R}^k$.

Since (1_1) follows (with any s and δ) from $\psi_1 = \mathrm{Lip}(f_1)$, Theorem 8.2 will be proved by induction once we show that (1_k) $\Rightarrow$ (2_k), (2_1) $\Rightarrow$ (3_1), (3_k) $\Rightarrow$ (1_{k+1}), and (3_k)&(2_{k+1}) $\Rightarrow$ (3_{k+1}).

Proof of (1_k) $\Rightarrow$ (2_k)**:** Choose $\eta > 0$ small compared with ε in a manner to be specified at the end. Find $t > 0$ such that $\bar{\rho}_X(t) < \eta t$. Then choose $\zeta > 0$ whose smallness depends on ε and t, again in a manner to be specified at the end. Finally, use (1_k) to find $0 < s \le \zeta$ such that for every $x \in S_k(s)$ there is $\delta(x) > 0$ with the property that

$$|f_k(x + h + u) - f_k(x + h)| \le \psi_k\|u\| + \zeta(\|h\| + \|u\|) \tag{37}$$

whenever $\|u\| < \delta(x)$.

Fix unit vectors $u_1, \dots, u_k$ such that $S_k(s; u_1, \dots, u_k) \neq \emptyset$. Let $x_0 \in S_k(s; u_1, \dots, u_k)$ and let X_0 be a finite co-dimensional subspace of X such that $f_k'(x_0)_{|X_0} = 0$ and $\|u_k + x\| < 1 + \eta t$ for every $x \in X_0$ with $\|x\| \le t$.

Let E be a finite dimensional subspace of X containing u_k and such that $3B_E + 3B_{X_0} \supset B_X$. Since E is finite dimensional and f_k is Gâteaux differentiable at x_0, it is Fréchet differentiable at x_0 in the direction of E, and hence by Lemma 8.1 there is $0 < \delta < \delta(x_0)/(3(1 + 1/t))$ such that

$$|f_k(x_0 + h + u) - f_k(x_0 + h) - f_k'(x_0)(u)| \le \zeta(\|h\| + \|u\|) \tag{38}$$

whenever $h, u \in E$ and $\|u\| < 6\delta/t$.

Suppose that $\|y\| < \delta$ and write $y = u + x$, where $u \in E$, $x \in X_0$ and $\|u\|, \|x\| \le 3\|y\|$. Denote $r = \|x\|/t$ and

$$\begin{aligned} a &= f_k(x_0 + y) - (f_k(x_0 + u + ru_k) + f_k(x_0 + u - ru_k))/2, \\ b &= (f_k(x_0 + u + ru_k) - f_k(x_0 + u - ru_k))/2. \end{aligned}$$

Since $\|2ru_k\| \le 6\|y\|/t < 6\delta/t$, (38) implies that

$$\begin{aligned} b &\ge f_k'(x_0)(ru_k) - 3\zeta(1+3/t)\|y\| \ge \psi_k r - sr - 3\zeta(1+3/t)\|y\| \\ &\ge \psi_k r - 9\zeta(1+1/t)\|y\|. \end{aligned}$$

Moreover, since $\|x \mp ru_k\| \le \|x\| + r \le 3(1+1/t)\|y\| < 3(1+1/t)\delta < \delta(x_0)$ and the same estimate holds for $\|u \pm ru_k\|$, we infer from (37) that

$$\begin{aligned} |f_k(x_0+u+x) - f_k(x_0+u\pm ru_k)| &\le \psi_k\|x \mp ru_k\| + 6\zeta(1+3/t)\|y\| \\ &= \psi_k r\|u_k \mp tx/\|x\|\| + 6\zeta(1+3/t)\|y\| \\ &\le \psi_k r(1+\eta t) + 6\zeta(1+3/t)\|y\|. \\ &\le \psi_k r + (3\eta\psi_k + 6\zeta(1+3/t))\|y\|. \end{aligned}$$

Using the 'mid-point' inequality $|a| \le \max(|b+a|, |b-a|) - b$ we infer that

$$\begin{aligned} |a| &\le \psi_k r + (3\eta\psi_k + 6\zeta(1+1/t))\|y\| - (\psi_k r - 9\zeta\|y\|(1+1/t)) \\ &= (3\eta\psi_k + 15\zeta(1+1/t))\|y\|. \end{aligned}$$

Hence, using (38) once more, we finally get

$$\begin{aligned} &|f_k(x_0+y) - f_k(x_0) - f_k'(x_0)(y)| \\ &\quad\le |a| + |f_k'(x_0)(x)| + |f_k(x_0+u+ru_k) - f_k(x_0) - f_k'(x_0)(u+ru_k)|/2 \\ &\qquad + |f_k(x_0+u-ru_k) - f_k(x_0) - f_k'(x_0)(u-ru_k)|/2 \\ &\quad\le ((3\eta\psi_k + 15\zeta(1+1/t))\|y\|) + 3\zeta(1+1/t)\|y\| \\ &\quad\le \varepsilon\|y\|, \end{aligned}$$

where the last inequality holds provided that we have chosen $\eta \le \varepsilon/6\psi_k$ and $\zeta \le \varepsilon/(36(1+1/t))$.

Proof of $(2_1) \Rightarrow (3_1)$**:** Denote $\eta = \varepsilon t/5$ and find $0 < s < \eta$ such that $S_1(s) \subset \mathrm{diff}^{\eta\mathrm{F}}(f_1)$.

Suppose that $x \in S_1(s; u_1)$. By Lemma 8.1 there is $\delta > 0$ such that $\|f(x+h+u) - f(x+h) - f'(x)(u)\| \le 4\eta(\|h\| + \|u\|)$ whenever $\|u\| < \delta$. Fix $0 < d < \delta$ and $h \in X$ such that $x + h + \sigma u_1 \in S$ for almost all $\sigma \in \mathbb{R}$, and denote

$$m = \lambda_1\{\sigma_1 \in [0,d] : x + h + \sigma_1 u_1 \notin S_1(t; u_1)\}.$$

Since $\psi_1 = \mathrm{Lip}(f_1)$, we have

$$\begin{aligned} f_1(x+h+du_1) - f_1(x+h) &= \int_0^d f_1'(x+h+\tau u_1)(u_1)d\tau \\ &\leq \psi_1(d-m) + (\psi_1 - t)m = \psi_1 d - tm. \end{aligned}$$

On the other hand, the choice of δ implies

$$\begin{aligned} f_1(x+h+du_1) - f_1(x+h) &\geq f_1'(x)(du_1) - 4\eta(\|h\|+d) \\ &\geq d(\psi_1 - s) - 4\eta(\|h\|+d) \\ &\geq d\psi_1 - 4\eta(\|h\|+d) - \eta d, \end{aligned}$$

and we conclude that

$$m \leq 5\eta(\|h\|+d)/t = \varepsilon(\|h\|+d),$$

as required.

Proof of $(3_k) \Rightarrow (1_{k+1})$**:** Find $t > 0$ such that

$$\|f_{k+1}'(z)\| < \psi_{k+1} + \varepsilon/4 \tag{39}$$

for every $z \in S_k(t)$ and denote

$$\eta = \varepsilon/(4\mathrm{Lip}(f_{k+1})(1 + 4k\mathrm{Lip}(f_{k+1})/\varepsilon)). \tag{40}$$

Use (3_k) to find $0 < s < t$ such that for every unit vectors $u_1, \ldots, u_k$ and every $x \in S_k(s; u_1, \ldots, u_k)$ there is $\delta(x; u_1, \ldots, u_k) > 0$ such that

$$\lambda_k\{\sigma \in [0,d]^k : x + h + \sum_{i=1}^k \sigma_i u_i \notin S_k(t; u_1, \ldots, u_k)\} < \eta(d + \|h\|)d^{k-1}$$

whenever $d < \delta(x; u_1, \ldots, u_k)$ and $h \in X$ is such that $x + h + \sum_{i=1}^k \sigma_i u_i \in S$ for λ_k almost all $\sigma \in \mathbb{R}^k$.

We show that for any $x \in S_{k+1}(s; u_1, \ldots, u_{k+1})$ the inequality

$$|f_{k+1}(x+h+u) - f_{k+1}(x+h)| \leq \psi_k\|u\| + \varepsilon(\|h\| + \|u\|)$$

holds whenever $\|u\| < \delta = 4k\delta(x; u_1, \ldots, u_k)$. It clearly suffices to consider only the case when $\varepsilon(\|h\| + \|u\|) < \mathrm{Lip}(f_{k+1})\|u\|$.

Let E be the linear span of $u_1, \dots, u_k, u$. If necessary, move h slightly so that almost every point of $x + h + E$ belongs to S. Let

$$d = \varepsilon(\|h\| + \|u\|)/(4k\mathrm{Lip}(f_{k+1})). \tag{41}$$

Then $d < \|u\|/(4k) < \delta(x; u_1, \dots, u_k)$ and so for almost every $\tau \in [0, 1]$,

$$\lambda_k\{\sigma \in [0, d]^k : x + h + \tau u + \textstyle\sum_{i=1}^k \sigma_i u_i \notin S_k(t; u_1, \dots, u_k)\}$$
$$< \eta(d + \|h\| + \|u\|)d^{k-1}.$$

Thus

$$\lambda_{k+1}\{(\tau, \sigma) \in [0, 1] \times [0, d]^k : x + h + \tau u + \textstyle\sum_{i=1}^k \sigma_i u_i \notin S_k(t; u_1, \dots, u_k)\}$$
$$< \eta(d + \|h\| + \|u\|)d^{k-1}$$

and hence there is $\sigma \in [0, d]^k$ such that, denoting $y = x + h + \sum_{i=1}^k \sigma_i u_i$, we have by (40) and (41)

$$\begin{aligned} \lambda_1\{\tau \in [0, 1] : y + \tau u \notin S_k(t; u_1, \dots, u_k)\} &< \eta(d + \|h\| + \|u\|)/d \\ &= \varepsilon/(4\mathrm{Lip}(f_{k+1})). \end{aligned}$$

By (39) $|f'_{k+1}(z)(u/\|u\|)| \le \psi_{k+1} + \varepsilon/4$ for every $z \in S_k(t; u_1, \dots, u_k)$, thus we conclude that

$$\lambda_1\{\tau \in [0, 1] : |f'_{k+1}(y + \tau u)(u)| > (\psi_{k+1} + \varepsilon/4)\|u\|\} < \varepsilon/(4\mathrm{Lip}(f_{k+1})).$$

Hence

$$\begin{aligned} |f_{k+1}(y + u) - f_{k+1}(y)| &\le \int_0^1 |f'_{k+1}(y + \tau u)(u)| d\tau \\ &\le (\mathrm{Lip}(f_{k+1})\varepsilon/(4\mathrm{Lip}(f_{k+1})) + (\psi_{k+1} + \varepsilon/4))\|u\| \\ &= \psi_{k+1}\|u\| + \varepsilon\|u\|/2, \end{aligned}$$

which, using (41), gives that

$$\begin{aligned} &|f_{k+1}(x + h + u) - f_{k+1}(x + h)| \\ &\quad\le \psi_{k+1}\|u\| + \varepsilon\|u\|/2 + 2\mathrm{Lip}(f_{k+1})\|\textstyle\sum_{i=1}^k \sigma_i u_i\| \\ &\quad\le \psi_{k+1}\|u\| + \varepsilon\|u\|/2 + 2\mathrm{Lip}(f_{k+1})kd \\ &\quad\le \psi_{k+1}\|u\| + \varepsilon(\|h\| + \|u\|), \end{aligned}$$

as required.

Proof of $(3_k)\&(2_{k+1}) \Rightarrow (3_{k+1})$**:** We first choose $0 < \eta < \varepsilon/4$ such that $\eta < \varepsilon t/(4(\mathrm{Lip}(f_{k+1}) + 4k + 6))$ and find $0 < \hat{t} < t$ such that $\|f'_{k+1}(z)\| \le \psi_{k+1} + \eta$ for all $z \in S_k(\hat{t})$. Then we use (2_{k+1}) and (3_k) to choose $0 < s \le \eta$ such that $S_{k+1}(s) \subset \mathrm{diff}^{\eta\mathrm{F}}(f_{k+1})$ and such that for every unit vectors $u_1, \ldots, u_k$ and every $x \in S_k(s; u_1, \ldots, u_k)$ there is $\delta(x; u_1, \ldots, u_k) > 0$ such that

$$\lambda_k\{\sigma \in [0,d]^k : x + h + \sum_{i=1}^{k} \sigma_i u_i \notin S_k(\hat{t}; u_1, \ldots, u_k)\} < \eta^2(d + \|h\|)d^{k-1}$$

whenever $d < \delta(x; u_1, \ldots, u_k)$ and $h \in X$ is such that for λ_k almost all $\sigma \in \mathbb{R}^k$ $x + h + \sum_{i=1}^{k} \sigma_i u_i \in S$.

Let $x \in S_{k+1}(s; u_1, \ldots, u_{k+1})$. By Lemma 8.1 there is δ in the interval $(0, \delta(x; u_1, \ldots, u_k))$ such that

$$|f_{k+1}(x + h + u) - f_{k+1}(x + h) - f'_{k+1}(x)(u)| \le 4\eta(\|h\| + \|u\|) \tag{42}$$

whenever $\|u\| < \delta$.

The rest of the proof is devoted to showing that the inequality required by (3_{k+1}) holds whenever $d < \delta$ and $h \in X$ is such that $x + h + \sum_{i=1}^{k+1} \sigma_i u_i \in S$ for λ_{k+1} almost all $\sigma \in \mathbb{R}^{k+1}$.

From $d < \delta(x; u_1, \ldots, u_k)$ we infer that for λ_1 almost all $\sigma_{k+1} \in [0, d]$

$$\lambda_k\{\sigma \in [0,d]^k : x + h + \sum_{i=1}^{k+1} \sigma_i u_i \notin S_k(\hat{t}; u_1, \ldots, u_k)\} < \eta^2(2d + \|h\|)d^{k-1},$$

which gives that

$$\lambda_{k+1}\{\sigma \in [0,d]^{k+1} : x + h + \sum_{i=1}^{k+1} \sigma_i u_i \notin S_k(\hat{t}; u_1, \ldots, u_k)\} < \eta^2(2d + \|h\|)d^{k}.$$

Hence, denoting by M the set of those $\sigma \in [0,d]^k$ for which

$$\lambda_1\{\tau \in [0,d] : x + h + \tau u_{k+1} + \sum_{i=1}^{k} \sigma_i u_i \notin S_k(\hat{t}; u_1, \ldots, u_k)\} < \eta d,$$

we infer that

$$\lambda_k([0,d]^k \setminus M) < \eta(2d + \|h\|)d^{k-1}. \tag{43}$$

Fix for a while $\sigma \in M$ and denote

$$\begin{aligned}
y &= x + h + \textstyle\sum_{i=1}^{k} \sigma_i u_i, \\
m &= \lambda_1\{\tau \in [0,d] : y + \tau u_{k+1} \in S_k(\hat{t}; u_1, \dots, u_k) \setminus S_{k+1}(t; u_1, \dots, u_{k+1})\}, \\
m_1 &= \lambda_1\{\tau \in [0,d] : y + \tau u_{k+1} \notin S_k(\hat{t}; u_1, \dots, u_k)\}, \\
m_2 &= \lambda_1\{\tau \in [0,d] : y + \tau u_{k+1} \in S_k(\hat{t}; u_1, \dots, u_k) \cap S_{k+1}(t; u_1, \dots, u_{k+1})\}.
\end{aligned}$$

Observing that $\hat{t} < t$ implies that

$$\begin{aligned}
f'_{k+1}(z)(u_{k+1}) &\le \psi_{k+1} + \eta \text{ for } z \in S_k(\hat{t}; u_1, \dots, u_k), \\
f'_{k+1}(z)(u_{k+1}) &\le \psi_{k+1} - t \text{ for } z \in S_k(\hat{t}; u_1, \dots, u_k) \setminus S_{k+1}(t; u_1, \dots, u_{k+1}),
\end{aligned}$$

and using $m_1 \le \eta d$ and $m_2 \le d - m$, we infer that

$$\begin{aligned}
f_{k+1}(y + d u_{k+1}) - f_{k+1}(y) &= \int_0^d f'_{k+1}(y + \tau u_{k+1})(u_{k+1})\, d\tau \\
&\le \mathrm{Lip}(f_{k+1}) m_1 + (\psi_{k+1} - t) m + (\psi_{k+1} + \eta) m_2 \\
&\le \mathrm{Lip}(f_{k+1}) \eta d + \eta d + d\psi_{k+1} - tm.
\end{aligned}$$

On the other hand, since $d < \delta$, we get from (42) that

$$\begin{aligned}
f_{k+1}(y + d u_{k+1}) - f_{k+1}(y) &\ge f'_{k+1}(x)(d u_{k+1}) - 4\eta(\|h\| + (k+1)d) \\
&\ge d(\psi_{k+1} - s) - 4\eta(\|h\| + (k+1)d) \\
&\ge d\psi_{k+1} - \eta(4\|h\| + (4k+5)d),
\end{aligned}$$

and we conclude that

$$m \le ((\mathrm{Lip}(f_{k+1}) + 4k + 6)\eta d + 4\eta\|h\|)/t \le \varepsilon(\|h\| + d)/4,$$

which gives that

$$\begin{gathered}
\lambda_1\{\sigma_{k+1} \in [0,d] : x + h + \textstyle\sum_{i=1}^{k+1} \sigma_i u_i \notin S_{k+1}(t; u_1, \dots, u_{k+1})\} \\
\le m + m_1 \le \eta d + \varepsilon(\|h\| + d)/4 < \varepsilon(\|h\| + d)/2.
\end{gathered}$$

We are now ready for the final estimate (using also (43))

$$\begin{gathered}
\lambda_{k+1}\{\sigma \in [0,d]^{k+1} : x + h + \textstyle\sum_{i=1}^{k+1} \sigma_i u_i \notin S_{k+1}(t; u_1, \dots, u_{k+1})\} \\
\le \varepsilon(\|h\| + d) d^k/2 + \eta(2d + \|h\|) d^k \le \varepsilon(d + \|h\|) d^k.
\end{gathered}$$

This concludes the proof of Theorem 8.2. ■

The next corollary can be viewed as the main result of this section. It has been used in the proof of Theorem 6.1 above and it follows immediately from the statements (2_k) of Theorem 8.2.

Corollary 8.3 *Suppose that $\bar{\rho}_X(t)/t \to 0$ and that $f_1, f_2, \ldots, f_k$ are real-valued Lipschitz functions on X. Then, for every $\varepsilon > 0$, $f_1, \ldots, f_k$ have a common point of ε-Fréchet differentiability in D.*

Remark 8.4 Note that if X is a separable asymptotically uniformly smooth Banach space and Y is finite dimensional and $f : X \to Y$ is a Lipschitz function then Corollary 8.3 implies that, for every $\varepsilon > 0$, f has a point of ε-Fréchet differentiability. Moreover, recalling the definitions of ψ_1 and $S_1(t)$ and referring back to Theorem 8.2, given any additional Lipschitz function $g : X \to \mathbb{R}$ we can also assure that that this point, x, is also a point of Gâteaux differentiability of g and that $\|D^G g(x)\| > Lip(g) - \varepsilon$. In addition, the point x can be chosen in any co-null set.

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