Several remarks concerning the local theory of L_p spaces

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Abstract

For the symmetric $X^n_{p,w}$ subspaces of L_p, $p > 2$, we determine the dimension of their approximately Euclidean subspaces and estimate the smallest dimensions of their containing ℓ_p^m spaces. We also show that a diagonal of the canonical basis of ℓ_p, $p > 2$, with an unconditional basic sequence in L_p whose span is complemented, spans a space which is isomorphic to a complemented subspace of L_p.

1 Introduction

One of the main results in [BLM] is that for any n and any $\varepsilon > 0$, every n-dimensional subspace, X, of L_p, $p > 2$, $(1+\varepsilon)$-embeds into ℓ_p^m for $m \le c(p,\varepsilon)n^{p/2}\log n$. A result from [BDGJN] implies that, up to the log factor, this is the best possible result for a general subspace of L_p. Actually the "worst" known space is $X = \ell_2^n$. However, unlike the corresponding result for $1 \le p < 2$ (where the dependence of m on n is linear, up to a logarithmic factor), one may ask for conditions on n-dimensional subspaces of L_p which guarantee a substantially smaller estimate of the dimension of the containing ℓ_p^m space. For a long time the authors speculated that a natural such condition could be that X only contains low dimensional Euclidean subspaces. More precisely, we were hoping that if k is the largest dimension of a subspace of X (with $X \subset L_p$, $p > 2$) which is 2-isomorphic to a Euclidean space, then X well embeds into ℓ_p^m with $m \le Ck^{p/2}$.

The main purpose of this note is to show that this hopeful conjecture fails. The examples are symmetric X_p spaces; see the beginning of section 2 for their definition. We do that by determining, for each $0 < w < 1$, the best (up to universal constants) dimension of an (approximately) Euclidean subspace of $X_{p,w}$ (in Proposition 1), and by giving, for each $0 < w < 1$, a lower bound on the dimension m of an ℓ_p^m containing well isomorphic copy of $X_{p,w}$ (in Proposition 2). The lower bound on m turns out to be best

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possible, up to logarithmic factors, in the range $w > n^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}$. In the complementary range $w < n^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}$ we give, in Proposition 3, an upper estimate on m which does not match the lower bound and which is most probably not the best one. It is however better than what was previously known.

Section 3 is devoted to another observation regarding subspaces of L_p, $p > 2$; this time complemented subspaces with unconditional basis. We show that given such a space with an unconditional basis $\{x_n\}$, any diagonal of the given unconditional basis with the unit vector basis of ℓ_p, $\{x_n \oplus_p \alpha_n e_n\}$, spans a space which is well isomorphic to a well complemented subspace of L_p (although it may not be complemented in the natural embedding in $L_p \oplus_p \ell_p$).

2 Tight embeddings of Euclidean spaces in symmetric X_p spaces and of symmetric X_p spaces in ℓ_p spaces.

Recall that for each $n \in \mathbb{N}$, $0 \le w \le 1$, and $2 < p < \infty$, $X^n_{p,w}$ is $\mathbb{R}^n$ with the norm

$$\|(x_1, x_2, \ldots, x_n)\|_{X^n_{p,w}} = \max\Big\{(\sum_{i=1}^n |x_i|^p)^{1/p},\ w(\sum_{i=1}^n |x_i|^2)^{1/2}\Big\}.$$

For w which is $o(1)$ and such that $wn^{\frac{1}{2}-\frac{1}{p}}$ tends to ∞ as $n \to \infty$, we get spaces whose distances from ℓ_2^n and ℓ_p^n tend to ∞ with n. All these spaces are well isomorphic to well complemented subspaces of L_p [Ro] and up to "well isomorphism" they are all the spaces with good symmetric basis which are well isomorphic to well complemented subspaces of L_p [JMST, Theorem 1.1].

Denote by $\{e_i\}$ the canonical basis of $\mathbb{R}^n$ and let $\{g_i\}$ be independent standard Gaussian random variables. Since $\mathbb{E}\|\sum_{i=1}^n g_i e_i\|_{X^n_{p,w}} \approx wn^{1/2}$ and $\|\sum_{i=1}^n a_i e_i\|_{X^n_{p,w}} \le (\sum_{i=1}^n |a_i|^2)^{1/2}$ for all scalars $\{a_i\}$, it follows from the general theory of Euclidean sections of convex bodies (see, for example, [MS]) that ℓ_2^k $(1+\varepsilon)$-embed into $X^n_{p,w}$ as long as $k \le c(\varepsilon)w^2 n$. We shall now show that this estimate on the dimension is best possible.

Recall that the Khintchine constant B_p, $2 < p < \infty$ is the smallest constant such that

$$\mathrm{Ave}_{\pm}|\sum \pm a_i|^p \le B_p^p(\sum a_i^2)^{p/2} \tag{1}$$

for all sequences $\{a_i\}$ of real numbers. The exact value of B_p is known and in particular $B_p/\sqrt{p}$ is bounded away from 0 and ∞. The idea of the proofs of the next two propositions is taken from [BDGJN].

Proposition 1 *Assume ℓ_2^k K-embeds into $X^n_{p,w}$ and $w \ge n^{\frac{1}{p}-\frac{1}{2}}$. Then $k \le (B_p+1)^2 K^2 w^2 n$.*

Proof: If ℓ_2^k K-embeds into $X_{p,w}^n$, then there are numbers $\{\alpha_{i,j}\}_{i=1,j=1}^{k,\ \ n}$ satisfying

$$(\sum_{i=1}^{k}|a_i|^2)^{p/2} \le \max\Big\{\sum_{j=1}^{n}\Big|\sum_{i=1}^{k}a_i\alpha_{i,j}\Big|^p, w^p\Big(\sum_{j=1}^{n}\Big|\sum_{i=1}^{k}a_i\alpha_{i,j}\Big|^2\Big)^{p/2}\Big\} \le K^p(\sum_{i=1}^{k}|a_i|^2)^{p/2}. \tag{2}$$

For each fixed $1 \le l \le n$, by setting $a_i = \alpha_{i,l}$, we get from the right side of (2)

$$\begin{aligned} K^p(\textstyle\sum_{i=1}^k \alpha_{i,l}^2)^{p/2} &\ge \max\Big\{\textstyle\sum_{j=1}^n |\sum_{i=1}^k \alpha_{i,l}\alpha_{i,j}|^p, w^p\Big(\sum_{j=1}^n\Big(\sum_{i=1}^k \alpha_{i,l}\alpha_{i,j}|\Big)^2\Big)^{p/2}\Big\} \\ &\ge \max\Big\{(\textstyle\sum_{i=1}^k \alpha_{i,l}^2)^p, w^p(\sum_{i=1}^k \alpha_{i,l}^2)^p\Big\} \\ &= (\textstyle\sum_{i=1}^k \alpha_{i,l}^2)^p, \end{aligned}$$

so that for all $1 \le l \le n$,

$$(\sum_{i=1}^{k}\alpha_{i,l}^2)^{p/2} \le K^p. \tag{3}$$

Using the left inequality in (2) for ± 1 coefficients, averaging over these coefficients, and using Khintchine's inequality (1), we deduce that

$$k^{1/2} \le B_p\Big(\sum_{j=1}^{n}(\sum_{i=1}^{k}\alpha_{i,j}^2)^{p/2}\Big)^{1/p} + w(\sum_{j=1}^{n}\sum_{i=1}^{k}\alpha_{i,j}^2)^{1/2}. \tag{4}$$

Using (3) in (4) we get

$$k^{1/2} \le B_pKn^{1/p} + wKn^{1/2}.$$

Since $w \ge n^{\frac{1}{p}-\frac{1}{2}}$, we get that $k \le (B_p+1)^2K^2w^2n$. ■

Next we deal with embeddings of $X_{p,w}^n$ into ℓ_p^m.

Proposition 2 *Assume $X_{p,w}^n$ K-embed into ℓ_p^m. Then $m \ge c\min\{w^{2p}n^{p-1}, n^{p/2}\}$, where the positive constant c depends only on p and K. More precisely, $m \ge (2B_p^pK^p)^{-1}n^{p/2}$ when $w \ge (2B_p^pK^p)^{\frac{1}{2p}}n^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}$ and $m \ge (2B_p^pK^p)^{-2}w^{2p}n^{p-1}$ when $w \le (2B_p^pK^p)^{\frac{1}{2p}}n^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}$.*

Proof: If $X_{p,w}^n$ K-embeds into ℓ_p^m, then there are $\{\alpha_{i,j}\}_{i=1,j=1}^{n,\ \ m}$ such that

$$\sum_{j=1}^{m}|\sum_{i=1}^{n}a_i\alpha_{i,j}|^p \le \max\Big\{\sum_{i=1}^{n}|a_i|^p, w^p(\sum_{i=1}^{n}a_i^2)^{p/2}\Big\} \le K^p\sum_{j=1}^{m}|\sum_{i=1}^{n}a_i\alpha_{i,j}|^p \tag{5}$$

for all scalars $\{a_i\}$.

Fix a subset A of $\{1,2,\ldots,n\}$. It follows from (5) that for every $1 \le l \le n$,

$$\max\Big\{\sum_{i\in A}|\alpha_{i,l}|^p, w^p(\sum_{i\in A}\alpha_{i,l}^2)^{p/2}\Big\} \ge \sum_{j=1}^{m}|\sum_{i\in A}\alpha_{i,l}\alpha_{i,j}|^p \ge (\sum_{i\in A}\alpha_{i,l}^2)^p$$

and it follows that
$$(\sum_{i\in A}\alpha_{i,l}^2)^{p/2}\le \max\Big\{(\sum_{i\in A}|\alpha_{i,l}|^p)^{1/2},w^p\Big\}. \tag{6}$$

Assume A is of cardinality k. Letting $a_i=\pm 1$ for $i\in A$ and zero elsewhere and averaging over the signs, we get from (5)
$$\max\{k,w^pk^{p/2}\}\le B_p^pK^p\sum_{j=1}^m(\sum_{i\in A}\alpha_{i,j}^2)^{p/2}. \tag{7}$$

Using (6), we get from (7) that
$$\begin{aligned}\max\{k,w^pk^{p/2}\} &\le B_p^pK^p\Big(\textstyle\sum_{j=1}^m(\sum_{i\in A}|\alpha_{i,j}|^p)^{1/2}+mw^p\Big)\\ &\le B_p^pK^p\Big(m^{1/2}(\textstyle\sum_{j=1}^m\sum_{i\in A}|\alpha_{i,j}|^p)^{1/2}+mw^p\Big).\end{aligned}$$

Since, by (5), for all i, $\sum_{j=1}^m|\alpha_{i,j}|^p\le 1$,
$$w^pk^{p/2}\le\max\{k,w^pk^{p/2}\}\le B_p^pK^p(m^{1/2}k^{1/2}+mw^p)\le 2B_p^pK^p\max\{m^{1/2}k^{1/2},mw^p\} \tag{8}$$
and this holds for all $1\le k\le n$. Setting $k:=n$ in the extreme sides of (8) yields
$$m\ge\min\{(2B_p^pK^p)^{-2}w^{2p}n^{p-1},(2B_p^pK^p)^{-1}n^{p/2}\}.$$

If $w\ge(2B_p^pK^p)^{\frac{1}{2p}}n^{\frac12(\frac1p-\frac12)}$, we get $m\ge(2B_p^pK^p)^{-1}n^{p/2}$. If $w\le(2B_p^pK^p)^{\frac{1}{2p}}n^{\frac12(\frac1p-\frac12)}$, we get $m\ge(2B_p^pK^p)^{-2}w^{2p}n^{p-1}$. ■

Remark: For some time we thought the following might be true: Given a subspace X of L_p, $2<p<\infty$, let k be the largest dimension of a 2-Euclidean subspace of X. Then the smallest m such X 2-embed into ℓ_p^m is at most a constant depending on p times $k^{p/2}$. That is, the smallest dimension of a containing ℓ_p space of a subspace X of L_p depends only on the dimension of the largest Euclidean subspace of X. Proposition 2 shows that this conjecture is wrong: For example, for $w=n^{\frac12(\frac1p-\frac12)}$, the dimension m of ℓ_p^m which contains a 2-isomorphic copy of $X_{p,w}^n$ is, by Proposition 2, at least of order $m=n^{p/2}$ while, by the discussion preceding Proposition 1, ℓ_2^k $(1+\varepsilon)$-embed into $X_{p,w}^n$ as long as $k\le c(\varepsilon)n^{\frac1p+\frac12}$ (and $(n^{\frac1p+\frac12})^{\frac p2}<<n^{\frac p2}$).

For a fixed K and $w\ge c_pn^{\frac12(\frac1p-\frac12)}$, the result of Proposition 2 is best up to a possible $\log n$ factor; it was proved in [BLM] that any n-dimensional subspace of L_p $(1+\varepsilon)$-embeds into ℓ_p^m for $m\le C(p,\varepsilon)n^{p/2}\log n$. For $w\le c_pn^{\frac12(\frac1p-\frac12)}$, it is not clear if the result obtained here is best possible. In [Sc2] some estimates on the dimension of the containing ℓ_p of an $X_{p,w}^n$ space are given, which in some cases are better than the general estimate of [BLM]. Using the methods of [BLM] one can somewhat improve these results to get the following result, which however still leaves a gap with the lower bound on m in Proposition 2.

Proposition 3 *There is a constant K_p, depending only on $p > 2$, such that, for all $0 < K < \infty$, if $w \le K n^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}$, then $X^n_{p,w}$ K_p-embeds into ℓ_p^m whenever*

$$m \ge C(K,p) n^{1+\frac{(p-2)(p-1)}{p}} (\log n) w^{2(p-2)}.$$

$C(K,p)$ depends only on its two arguments.

Note that, up to logarithmic factors, the estimate on m is better than $n^{p/2}$.

Sketch of proof: As we said above, the proof is a combination of arguments from [Sc2] and [BLM], neither of which is simple, and it does not seem to give the final answer. We thus only sketch the argument.

We first use a specific embedding of $X^n_{p,w}$ in $L_p(0,1)$ as is given in Proposition 11 of [Sc2]: There is (sign and permutation) exchangeable sequence $\{x_i\}_{i=1}^n$ in L_p such that $\|x_i\|_p = 1$ for all i, $(\sum x_i^2)^{1/2} \equiv w n^{1/2}$ and

$$K_p^{-1} \| \sum a_i x_i \|_p \le \max \left\{ (\sum_{i=1}^n |a_i|^p)^{1/p},\ w(\sum_{i=1}^n |a_i|^2)^{1/2} \right\} \le K_p \| \sum a_i x_i \|_p.$$

Moreover, the span of the x_i-s is K_p-complemented in L_p. Here and below K_p denotes a constant depending only on p, not necessarily the same in each instance. Lemma 12 in [Sc2] asserts, in particular, that

$$\|x\|_\infty \le K_p \min\{n^{1/2}, n^{1-1/p} w^2\} \|x\|_p$$

for all x in the span of the x_i-s. (To add to the confusion resulting from the different notations here and in [Sc2], there is a misprint in equation (23) in [Sc2]: $m^{1/q}$ there should be $m^{1/p}$; see the bottom of the same page in [Sc2].) Under our assumption on w, this translates to

$$\|x\|_\infty \le K^2 K_p n^{1-1/p} w^2 \|x\|_p.$$

Theorem 13 of [Sc2] and its proof shows that there is some choice of $\bar{m}$ points of $(0,1)$ with $\bar{m} \le K_p \min\{n^{1+p/2}, n^p w^{2p}\} \le K_p n^p w^{2p}$ and, letting $\bar{x}_i$ be the restriction of x_i to these points, considered as elements of $L_p^{\bar{m}}$ (L_p on the measure space $\{1,\dots,\bar{m}\}$ with the normalized counting measure), we get

$$\frac{1}{2} \| \sum a_i x_i \|_p \le \| \sum a_i \bar{x}_i \|_p \le 2 \| \sum a_i x_i \|_p \tag{9}$$

and

$$\frac{1}{2} (\sum a_i^2)^{1/2} \le \| \sum a_i w^{-1} \bar{x}_i \|_2 \le 2 (\sum a_i^2)^{1/2} \tag{10}$$

for all $\{a_i\}_{i=1}^n$. (Moreover, the span of the $\bar{x}_i$-s is K_p-complemented in $L_p^{\bar{m}}$.) Of course we want to improve the bound on $\bar{m}$, but we need to use the result above (or something similar) as we shall see shortly.

Note that it is still true that

$$(\sum \bar{x}_i^2)^{1/2} \equiv wn^{1/2} \tag{11}$$

and that

$$\|x\|_\infty \le K^2 K_p n^{1-1/p} w^2 \|x\|_p \tag{12}$$

for all x in the span of the $\bar{x}_i$-s.

We would like to apply now the results and techniques of [BLM] and in particular Theorem 7.3 and its proof. We would like to take advantage of the improved L_∞ bound in (12) and this leads to a complication since the entropy bounds in [BLM] are obtained using a "change of density" which, if applied, may destroy the L_∞ bound we have. We thus would like to see that basically the same entropy bounds used in the proof of Theorem 7.3 in [BLM] apply, in our situation, without any change of density.

Let us denote by X_r the span of the $\bar{x}_i$-s in $L_r^{\bar{m}}$. In the notation of [BLM], for any $2 < q < \infty$ and any $t > 0$,

$$E(B_{X_p}, B_{X_q}, t) \le E(B_{X_2}, B_{X_q}, t)$$

where $E(U, V, t)$ denotes the minimal number of translates of tV needed to cover U. Now, by (10), $B_{X_2} \subset 2\{\sum a_i w^{-1}\bar{x}_i;\ \sum a_i^2 \le 1\} =: 2C$ so that

$$E(B_{X_p}, B_{X_q}, t) \le E(C, B_{X_q}, t/2).$$

Using now part of the proof of Proposition 4.6 in [BLM] (the part relaying on Proposition 4.2) we get, for M_{X_q} computed relative to the euclidean structure given by C and using (11) above, that

$$M_{X_q} \le Aq^{1/2} \|(\sum w^{-2}\bar{x}_i^2)^{1/2}\|_{L_q^{\bar{m}}} \le Aq^{1/2} n^{1/2}$$

for A a universal constant. Using Proposition 4.2 in [BLM], we get as in the proof of Proposition 4.6 there that

$$\log E(B_{X_p}, B_{X_q}, t) \le A' nq/t^2$$

for some other universal constant A'. Taking $q = \log \bar{m} \le K_p \log n$ we get

$$\log E(B_{X_p}, B_\infty, t) \le K_p n (\log n)/t^2, \tag{13}$$

so that we recovered the entropy estimate used in the proof of Theorem 7.3 in [BLM]. We now apply this proof to the space $X = X_p$. We can take $\varepsilon = 1/2$ (there is no much point dealing with small ε since we already have a constant K_p in the original embedding of $X_{p,w}^n$ in L_p). Recall that we have an improved estimate on the L_∞ bound (12) which also implies we can take $l = [\log(K^2 K_p n^{1-1/p} w^2)/\log(3/2)] + 1$ in the beginning of the proof of Theorem 7.3 in [BLM]. Using the notation of [BLM], we get from (13) the estimates

$$\log \bar{\bar{\mathcal{A}}}_k, \log \bar{\bar{\mathcal{B}}}_k \le K_p n (\log n)(4/9)^k. \tag{14}$$

Looking now at the end of the proof of Theorem 7.3 in [BLM] and using the parameters above (for ε, l and $\bar{\bar{\mathcal{B}}}_k$), we get the right estimate on m (denoted N in [BLM]).

■

3 Complemented subspaces of L_p with unconditional bases

In this section we prove

Proposition 4 *Let $\{x_n\}$ be a (finite or infinite) C-unconditional basic sequence in L_p, $p>2$, which is K-complemented. Then for all $\{\alpha_n\} \subset \mathbb{R}$, the span of $\{x_n \oplus \alpha_n e_n\}$ in $L_p \oplus \ell_p$ is K'-isomorphic to a K'-complemented subspace of L_p. K' depends only on K, C, and p. $\{e_n\}$ denotes here the unit vector basis of ℓ_p.*

Remark. The case when $\{x_n\}$ is equivalent to the unit vector basis of ℓ_2 is the fundamental result of Rosenthal's [Ro] upon which this entire note rests.

Proof: Denote by $\{h_{n,i}\}_{n=0,i=1}^{\infty,\ 2^n}$ the mean zero L_∞ normalized Haar functions. We first treat the case where $\{x_n\}$ has the special form

$$x_n = \sum_{i=1}^{2^{k_n}} a_{n,i} h_{k_n,i},$$

for some subsequence $\{k_n\} \subset \mathbb{N}$, and the projection onto the span of $\{x_n\}$ has the special form

$$Px = \sum x_n^*(x) x_n$$

where

$$x_n^* = \sum_{i=1}^{2^{k_n}} b_{n,i} h_{k_n,i}.$$

Assume also, as we may, that

$$\|x_n\| = 2^{-k_n/p} \Big(\sum_{i=1}^{2^{k_n}} |a_{n,i}|^p \Big)^{1/p} = 1.$$

Since the Haar system in its natural order is a monotone basis for L_p (see [LT, p. 3]), we have

$$\|x_n^*\| = 2^{-k_n/q} \Big(\sum_{i=1}^{2^{k_n}} |b_{n,i}|^q \Big)^{1/q} \le 2\|P\|.$$

Let $\{e_{n,i}\}_{n=1,i=1}^{\infty,\ n}$ be a rearrangement of the unit vector basis of ℓ_p and put

$$y_n = \alpha_n 2^{-k_n/p} \sum_{i=1}^{2^{k_n}} a_{n,i} e_{k_n,i} \in \ell_p$$

and

$$y_n^* = \alpha_n^{-1} 2^{-k_n/q} \sum_{i=1}^{2^{k_n}} b_{n,i} e_{k_n,i} \in \ell_q.$$

Then

$$y_n^*(y_n) = 1, \quad \|y_n\|_p = \alpha_n, \quad \|y_n^*\|_q \le \alpha_n^{-1} 2\|P\|$$

so that $\bar{P}(x) = \sum y_n^*(x) y_n$ defines a projection of norm at most $2\|P\|$ from ℓ_p onto the closed span of $\{y_n\}$. Consider the basic sequence

$$z_{n,i} = a_{n,i}(h_{k_n,i} \oplus \alpha_n 2^{-k_n/p} e_{k_n,i}), \quad n = 1, 2, \dots, \quad i = 1, 2, \dots, 2^{k_n}$$

in $L_p \oplus \ell_p$ and its closed span Z. By [KS] (see [Mü] for an alternative proof), Z is K'-isomorphic to a K'-complemented subspace of L_p, where K' depends only on K.

Put

$$Q = (P, \bar{P}) : Z \to L_p \oplus \ell_p$$

and notice that

$$Qz_{n,i} = 2^{-k_n} a_{n,i} b_{n,i} x_n \oplus \alpha_n^{-1} 2^{-k_n/q} \alpha_n 2^{-k_n/p} a_{n,i} b_{n,i} y_n = 2^{-k_n} a_{n,i} b_{n,i}(x_n \oplus y_n).$$

In particular, the range of Q is the closed span of $\{x_n \oplus y_n\}$. Also,

$$Q(x_n \oplus y_n) = Q \sum_{i=1}^{2^{k_n}} z_{n,i} = x_n \oplus y_n \ ;$$

that is, Q is a projection (of norm at most $2\|P\|$) from Z onto the closed span of $\{x_n \oplus y_n\}$, and, since Z is well complemented in $L_p \oplus \ell_p$ and the later is isomorphic to L_p (with universal constant), it follows that the closed span of $\{x_n \oplus y_n\}$ is well isomorphic to a well complemented subspace of L_p.

The sequence $\{x_n \oplus y_n\}$ is clearly isometrically equivalent to $\{x_n \oplus \alpha_n e_n\}$, so this completes the proof of the special case. In this case the unconditional constant of $\{x_n\}$ is no larger than the unconditional constant of the Haar basis, so that K' depends only on K and p. The reduction of the general case to the special case just treated follows from (the proof in) [Sc1]. This reduction makes the final constant K' also dependent on C. Here is a sketch of this reduction.

Let $Px = \sum x_n^*(x) x_n$ be the given projection. By a standard perturbation argument we may assume that each of the x_n and x_n^* is a linear combination of indicator functions of dyadic intervals in $[0,1]$. It follows that, for some increasing subsequence $\{k_n\} \subset \mathbb{N}$,

$$x_n = \sum_{i=1}^{2^{k_n}} a_{n,i} |h_{k_n,i}| \quad \text{and} \quad x_n^* = \sum_{i=1}^{2^{k_n}} b_{n,i} |h_{k_n,i}|.$$

Put

$$z_n = \sum_{i=1}^{2^{k_n}} a_{n,i} h_{k_n,i} \quad \text{and} \quad z_n^* = \sum_{i=1}^{2^{k_n}} b_{n,i} h_{k_n,i}.$$

The unconditionality of $\{x_n\}$ (and $\{x_n^*\}$) implies that $\{x_n\}$ is equivalent to $\{z_n\}$, $\{x_n^*\}$ is equivalent to $\{z_n^*\}$ and $Qx = \sum z_n^*(x) z_n$ is a bounded projection. The constants involved depend only on K, C and p.

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