AFFINE APPROXIMATION OF LIPSCHITZ FUNCTIONS AND NONLINEAR QUOTIENTS

S. Bates[*+], W.B. Johnson[†‡], J. Lindenstrauss[‡+],
D. Preiss and G. Schechtman[‡+]

Abstract. New concepts related to approximating a Lipschitz function between Banach spaces by affine functions are introduced. Results which clarify when such approximations are possible are proved and in some cases a complete characterization of the spaces X, Y for which any Lipschitz function from X to Y can be so approximated is obtained. This is applied to the study of Lipschitz and uniform quotient mappings between Banach spaces. It is proved, in particular, that any Banach space which is a uniform quotient of L_p, $1 < p < \infty$, is already isomorphic to a linear quotient of L_p.

1. Introduction

In the framework of geometric nonlinear functional analysis there is by now a quite developed theory of bi-uniform and bi-Lipschitz homeomorphisms between Banach spaces as well as bi-uniform and bi-Lipschitz embeddings. The existence of a bi-uniform or bi-Lipschitz homeomorphism between X and Y often (although not always) implies the existence of a linear isomorphism between these two spaces (see e.g. [Rib1], [HM], [JLS] and for a complete survey of the available information the forthcoming book [BL]). The situation is similar for bi-Lipschitz embeddings, while for bi-uniform embeddings the situation is different (see e.g. [AMM] for bi-uniform embeddings into Hilbert spaces).

The original purpose of the research reported in this paper was to initiate the study of nonlinear quotient mappings in the same context. In the process, we encountered the need to develop new notions of approximating Lipschitz functions by

1991 *Mathematics Subject Classification.* 46B20, 54Hxx.

Key words and phrases. Banach spaces, Uniform quotient, Lipschitz quotient, ultraproducts, Fréchet derivative, approximation by affine property.

[*]Supported by an NSF Postdoctoral Fellowship in Mathematics

[†]Supported in part by NSF DMS-9623260

[‡]Supported in part by the U.S.-Israel Binational Science Foundation

[+]Participant, Workshop in Linear Analysis and Probability, Texas A&M University

affine ones which go beyond derivatives. We believe these may prove to be fundamental notions whose interest goes beyond the particular applications to nonlinear quotient mappings we have here. Readers who are interested mainly in these notions can concentrate on the second half of this introduction and on section 2.

We consider two related notions of nonlinear quotient mappings. A uniformly continuous mapping F from a metric space X onto a metric space Y is a *uniform quotient mapping* if for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ so that for every $x \in X$, $F(\mathrm{B}_\epsilon(x)) \supset \mathrm{B}_\delta(Fx)$. If in addition the mapping is Lipschitz and δ can be chosen to be linear in ϵ, F is called a *Lipschitz quotient mapping*. A map $F : X \to Y$ is a uniform quotient map if and only if $F \times F : X \times X \to Y \times Y$ maps the uniform neighborhoods of the diagonal in $X \times X$ onto the set of uniform neighborhoods of the diagonal in $Y \times Y$.

Linear quotient mappings between Banach spaces are Lipschitz quotient mappings and bi-Lipschitz (resp. bi-uniform) homeomorphisms are Lipschitz (resp. uniform) quotient mappings. The class of Lipschitz or uniform quotients is larger than the class of maps that can be obtained as compositions of these two obvious classes of examples. Indeed, for mappings which are composition of maps from the classes above the inverse image of a point is always connected while, for example, the map $f(re^{i\theta}) = re^{i2\theta}$ from $\mathbb{R}^2$ onto itself is a Lipschitz quotient mapping such that the inverse image of any nonzero point consists of exactly two points. Based on this simple example one can also build infinite dimensional examples.

As we shall see below (in sections 3 and 4), it is sometimes quite delicate to check that a given mapping is a Lipschitz or uniform quotient mapping.

One of the first questions one would like to study about these new notions of nonlinear quotients is to what extent they can be "linearized". The simplest question in this direction is under which conditions on X or Y does the existence of a Lipschitz or uniform quotient mapping from X onto Y imply the existence of a linear quotient mapping.

It turns out that existing examples concerning bi-uniform homeomorphisms (in the separable setting) and bi-Lipschitz homeomorphisms (in the non-separable, non-reflexive setting) show that linearization is, in general, impossible for quotient mappings. On the other hand, we show that at least some of the positive results concerning homeomorphisms can be carried over to the quotient setting.

In the study of bi-uniform homeomorphisms between Banach spaces, the first step towards linearization is usually to pass to a bi-Lipschitz homeomorphisms between ultrapowers of the two Banach spaces. This step carries over easily to the quotient setting (see Proposition 3.4 below). We are thus reduced to the question of linearizing Lipschitz quotient mappings (and of passing back from ultrapowers to the original spaces). The most natural way to pass from Lipschitz mappings to linear ones is via differentiation. Recall that a mapping f defined on open set G in a Banach space X into a Banach space Y is called *Gâteaux differentiable* at $x_0 \in G$ if for every $u \in X$

$$\lim_{t \to 0} (f(x_0 + tu) - f(x_0))/t = D_f(x_0)u$$

exists and $D_f(x_0)$ (= the differential of f at x_0) is a bounded linear operator from X to Y.

The map f is said to be *Fréchet differentiable* at x_0 if the limit above exists uniformly with respect to u in the unit sphere of X.

Fréchet derivatives are very good for linearization purposes when they are available, but their existence is rarely ensured. Rather general existence theorems are known for Gâteaux derivatives and these are very useful in the theory of bi-Lipschitz embeddings and bi-Lipschitz homeomorphisms. Unfortunately, the existence of a Gâteaux derivative does not seem to help in the setting of Lipschitz quotients. As we shall see in Proposition 3.11, the Gâteaux derivative of a Lipschitz quotient mapping can be identically zero at some points. It still maybe possible to use Gâteaux derivatives in this theory, but Proposition 3.11 shows that one should take care of choosing carefully a specific point of differentiability (or maybe a generic one). We do not know how to do it.

As we mentioned above, Fréchet derivatives work very nicely when they exist, which is rare. Even some weaker versions of Fréchet derivatives would suffice for most purposes in our context. One such version is the ϵ-Fréchet derivative. A mapping f defined on open set G in a Banach space X into a Banach space Y is said to be *ϵ-Fréchet differentiable* at x_0 if there is a bounded linear operator T from X to Y and a $\delta > 0$ so that

$$\|f(x_0 + u) - f(x_0) - Tu\| \leq \epsilon \|u\| \quad \text{for} \quad \|u\| \leq \delta.$$

Unfortunately, there is no known existence theorem even for points of ϵ-Fréchet differentiability for Lipschitz mappings in situations that are most relevant to us (e.g., Y is infinite dimensional and isomorphic to a linear quotient of X). There are such theorems if Y is finite dimensional ([LP]) and also in some situations where Y is infinite dimensional but every linear operator from X to Y is compact ([JLPS2]). The theorem in [LP] enables one to prove a result describing the finite dimensional (linear) quotients of uniform quotient spaces which is analogous to a result of Ribe ([Rib1]) on bi-uniform homeomorphisms. Fortunately, it turned out that in our context the same can be achieved by a linearization theorem for Lipschitz mappings into finite dimensional spaces whose proof is considerably simpler than that of the theorem from [LP]. We refer to Theorem 3.5 for the specific result we proved.

Section 2 is devoted to the examination of two new notions of approximating Lipschitz functions by affine functions. Given two Banach spaces X and Y, we denote by $\mathrm{Lip}\,(X,Y)$ the set of Lipschitz functions from a domain in X to Y and we say that $\mathrm{Lip}\,(X,Y)$ has the *approximation by affine property* (AAP) if for every Lipschitz function f from the unit ball B of X into Y and every $\epsilon > 0$ there is a ball $B_1 \subset B$ of radius r, say, and an affine function $L : X \to Y$ so that

$$||f(x) - Lx|| \leq \epsilon r, \quad x \in \mathrm{B}_1.$$

This inequality is clearly satisfied if f has an ϵ-Fréchet derivative at the center of B_1. AAP is however a definitely weaker requirement than ϵ-Fréchet differentiability. The second property we examine in section 2 is a uniform version of this property. We say that $\mathrm{Lip}\,(X,Y)$ has the *uniform approximation by affine property* (UAAP) if the radius $r = r(\epsilon, f)$ of B_1 above can be chosen to satisfy $r(\epsilon, f) \geq c(\epsilon) > 0$ simultaneously for all functions f of Lipschitz constant ≤ 1 ($c(\epsilon)$ of course depends also on X and Y). The existence of Fréchet derivatives does not entail any such uniform estimate, and it is easy to see that in some situations the affine approximant L in the definition of the uniform approximation by affine property cannot be a Fréchet derivative at any point, even if such derivatives exist almost everywhere.

We remark in passing that the notions above make sense also when dealing with a single function $f : G \to Y$ defined on a domain G in X. We suggest to say that f admits *affine localization* if for every $\epsilon > 0$ and every ball $B \subset G$ there is a ball $B_1 \subset B$ of radius r, say, and an affine function $L : X \to Y$ so that

$$||f(x) - Lx|| \leq \epsilon r, \quad x \in \mathrm{B}_1.$$

Uniform affine localization can be defined similarly. We shall not deal with single functions here and the notions of (uniform) affine localization are only suggestions for appropriate names.

The main result of section 2, Theorem 2.7, gives a complete characterization of the pairs of spaces for which $\mathrm{Lip}\,(X,Y)$ has the uniform approximation by affine property: $\mathrm{Lip}\,(X,Y)$ has the UAAP if and only if one of the two spaces is finite dimensional and the other is super-reflexive (that is, has an equivalent uniformly convex norm). For the study of nonlinear quotients we only need part of this characterization, Theorem 2.3, which states that $\mathrm{Lip}\,(X,Y)$ has the uniform approximation by affine property if X is super-reflexive and Y is finite dimensional. Even the approximation by affine property under these conditions would suffice for our application.

In Proposition 2.8 (respectively, Proposition 2.9) we also characterize the spaces X for which $\mathrm{Lip}\,(X,\mathbb{R})$ (respectively, $\mathrm{Lip}\,(\mathbb{R},X)$) has the approximation by affine property. One interesting feature of these characterizations is the following: $\mathrm{Lip}\,(X,\mathbb{R})$ has the approximation by affine property if (and only if) every Lipschitz $f : X \to \mathbb{R}$ has a point of Fréchet differentiability. On the other hand $\mathrm{Lip}\,(\mathbb{R},X)$

may have the approximation by affine property also in a situation where there is a Lipschitz mapping from $\mathbb{R}$ to X which fails to have, for some $\epsilon > 0$, even a point of ϵ-Fréchet differentiability.

In section 3 we apply Theorem 2.3 to get that if Y is a uniform quotient of a superreflexive X, then Y is linearly isomorphic to a linear quotient of an ultrapower of X. This yields, for example, that a uniform quotient of a Hilbert space is isomorphic to a Hilbert space. While most of the positive results in section 3 are of "local" nature in the sense that they describe the finite dimensional (linear) quotients of Y and consequences of such descriptions, the section ends with a result on the structure of Lipschitz quotients which is of "global" nature and is new even for bi-Lipschitz homeomorphisms (Theorem 3.18): A Lipschitz quotient of an Asplund space is Asplund.

Recall that the *Gorelik principle* [JLS] says that a bi-uniform homeomorphism from X onto Y cannot carry the unit ball in a finite codimensional subspace of X into a "small" neighborhood of an infinite codimensional subspace of Y. Moreover, the same is true for the composition of a bi-uniform homeomorphism with a linear quotient mapping. This raises the natural question of whether the same holds for general uniform quotient mappings. It turns out that in general this is not the case and section 4 is devoted to several examples related to this. It follows from Proposition 4.1 that there is a uniform quotient mapping from ℓ_2 onto itself which sends a ball in a hyperplane to zero. We also show, in Proposition 4.2 that for every separable Y there is a Lipschitz quotient mapping from ℓ_1 onto Y which sends a hyperplane to zero.

We do not know for example if a Lipschitz quotient mapping between superreflexive spaces can map a finite codimensional subspace to a point. Questions of this type are of interest also in the finite dimensional setting. Lipschitz quotient mappings from $\mathbb{R}^n$ to itself which have positive Jacobian almost everywhere are special cases of quasiregular mappings. See [Ric] for a recent book on this topic. The topic of quasiregular mappings is quite developed. One deep theorem of interest to us, due to Reshetnyak (see [Ric, p. 16]), says that the level sets of a quasiregular mapping are discrete. We conjecture that there is a result of a similar nature for Lipschitz quotient mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$. In Proposition 4.4 we give an elementary proof that the level sets of a Lipschitz quotient mapping from $\mathbb{R}^2$ onto $\mathbb{R}^2$ are discrete (in [JLPS1] it is shown that they are even finite).

It is also of interest to study uniform quotient mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$. In Proposition 4.3 we give an example of a uniform quotient mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ which carries a 2-dimensional disk to zero.

A forthcoming paper [JLPS1] contains a detailed study of uniform quotient mappings from $\mathbb{R}^2$ to itself. It is shown there, for example, that such a map may carry an interval to 0, but that if the modulus of continuity $\Omega(\cdot)$ of the uniform quotient map satisfies $\Omega(t) = o(\sqrt{t})$ as $t \to 0$, then all level sets of the map are finite. The results herein and in [JLPS1] indicate that the subject of nonlinear quotient mappings between Euclidean spaces is a promising research area for geometric topology and geometric measure theory.

Unexplained background can be found in [JLS] and the book [LT].

2. Linearizing Lipschitz Mappings

We begin with a definition:

Definition 2.1. *Given two Banach spaces X,Y,* $\mathrm{Lip}\,(X,Y)$ *is said to have the approximation by affine property (AAP, in short) provided that for each ball B in X, every Lipschitz mapping $f : B \to Y$, and each $\epsilon > 0$, there is a ball $B_1 \subset B$ and an affine mapping $g : B_1 \to Y$ so that*

$$\sup_{x \in B_1} ||f(x) - g(x)|| \leq \epsilon r \mathrm{Lip}\,(f),$$

where r is the radius of B_1. If there is a constant $c = c(\epsilon) > 0$ so that B_1 can always be chosen so that its radius is at least c times the radius of B, we say that $\mathrm{Lip}\,(X,Y)$ *has the uniform approximation by affine property (UAAP, in short).*

In Theorem 2.7 we show that a pair of Banach spaces has the uniform approximation by affine property if and only if one of the spaces is super-reflexive and the other is finite dimensional. For applications to nonlinear quotients, the most important fact (Theorem 2.3) is that if X is a uniformly smooth Banach space and Y is a finite dimensional space, then $\mathrm{Lip}\,(X,Y)$ has the uniform approximation by affine property. Notice that the affine approximant one obtains to the Lipschitz mapping from this result cannot always be obtained by differentiation even if the domain of the Lipschitz mapping is the real line (consider the mapping from $\mathbb{R}$ to $\mathbb{R}$ which is $(-1)^n$ at the integer n and linear on each interval $[n, n+1]$).

Note that if every Lipschitz mapping from a domain in X into Y has for each $\epsilon > 0$ a point of ϵ-Fréchet differentiability, then $\mathrm{Lip}\,(X,Y)$ has the approximation by affine property. Consequently, Theorem 11 in [LP] implies that $\mathrm{Lip}\,(X,Y)$ has the approximation by affine property whenever X is uniformly smooth and Y is finite dimensional. So we could avoid Theorem 2.3 in the sequel. On the other hand, the uniform approximation by affine property seems to be an interesting property in itself and is more tractable than the approximation by affine property. Moreover, Theorem 2.3 is much easier to prove than Theorem 11 in [LP].

Proposition 2.2. *If* $\mathrm{Lip}\,(X,\mathbb{R})$ *has the uniform approximation by affine property, then* $\mathrm{Lip}\,(X,Y)$ *has the uniform approximation by affine property for every finite dimensional space Y.*

Proof. We show first that if $\{f_i\}_{i=1}^n$ are real valued Lipschitz functions with $\mathrm{Lip}\,(f_i) \le 1$ for $1 \le i \le n$ on a ball $B = \mathrm{B}_r(x_0)$ in X and if $\epsilon > 0$, then there are a constant $\tilde{c} = \tilde{c}(\epsilon, n, X)$, a ball $\widetilde{B} \subset B$ of radius $s \ge \tilde{c}r$ and affine $g_i : X \to \mathbb{R}$ so that for x in $\widetilde{B}$, $|f_i(x) - g_i(x)| \le \epsilon s$ for all i. For $n = 1$ this is just the definition of UAAP. The general case is proved by induction; we just do the case $n = 2$ from which the general case will be clear.

Let $c(\epsilon)$ be as in Definition 2.1 and find a ball $B_1 \subset B$ of radius $r_1 \ge c(\epsilon c(\epsilon))r$ and an affine g_1 so that for x in B_1, $|f_1(x) - g_1(x)| \le \epsilon c(\epsilon) r_1$. Applying again Definition 2.1 we find a ball $B_2 \subset B_1$ of radius $r_2 \ge c(\epsilon)r_1$ and an affine g_2 so that for x in B_2, $|f_2(x) - g_2(x)| \le \epsilon r_2$. Clearly on B_2 (as on B_1), $|f_1(x) - g_1(x)| \le \epsilon c(\epsilon) r_1 \le \epsilon r_2$.

Now suppose that F is a mapping from a ball $B = \mathrm{B}_r(x_0)$ in X into a normed space Y of dimension n with $\mathrm{Lip}\,(F) \le 1$. Let $\{y_i, y_i^*\}_{i=1}^n$ be an Auerbach basis for Y; that is, $y_i^*(y_j) = \delta_{ij}$ and $||y_i|| = 1 = ||y_i^*||$. By what we proved above there are real valued affine $\{g_i\}_{i=1}^n$ on X and a ball $\widetilde{B} \subset B$ of radius $s \ge \tilde{c}r$ so that $|y_i^* F(x) - g_i(x)| \le \frac{\epsilon s}{n}$ for x in $\widetilde{B}$ and $1 \le i \le n$. The affine function $G : X \to Y$ defined by $Gx = \sum_{i=1}^n g_i(x) y_i$ satisfies for x in $\widetilde{B}$ $||Fx - Gx|| \le \sum_{i=1}^n |y_i^*(Fx) - g_i(x)| \le \epsilon s$. ■

Theorem 2.3. *Suppose X is a uniformly smooth Banach space. Then* $\mathrm{Lip}\,(X, Y)$ *has the uniform approximation by affine property for every finite dimensional space Y.*

Proof. By Proposition 2.2, it is enough to check that $\mathrm{Lip}\,(X, \mathbb{R})$ has the UAAP. So let f be a mapping from some ball $B = \mathrm{B}_r(x_0)$ in X into $\mathbb{R}$ with $\mathrm{Lip}\,(f) \le 1$.

By rescaling we can assume, without loss of generality, that the radius of B is one. Now if $|f(x) - f(x_0)| \le 2\epsilon$ for all x in B, then we can well-approximate f on all of B by the function which is constantly $f(x_0)$, so assume that this is not the case. In particular, we assume that $\mathrm{Lip}\,(f; 1) \ge \epsilon$, where for $t > 0$

$$\mathrm{Lip}\,(f; t) := \sup_{||x-y|| \ge t} \frac{||f(x) - f(y)||}{||x - y||}.$$

Since X is uniformly smooth, we can choose $1/2 > \delta > 0$ so that if $||x|| = 1$ and $||y|| \le \delta$, then

$$||x + y|| = 1 + x^*(y) + r(y), \tag{2.1}$$

with $|r(y)| \le \epsilon ||y||$, and where x^* is the unique norming functional for x; that is, $||x^*|| = 1 = x^*(x)$. (The remainder function $r(\cdot)$ of course depends on x.) Define $k = k(\epsilon)$ by

$$(1 + \delta\epsilon)^{k-1} \epsilon \le 1 < (1 + \delta\epsilon)^k \epsilon$$

and observe that there exists d with $2 \geq 2d \geq 4^{-k}$ so that

$$\text{(2.2)} \qquad \operatorname{Lip}(f; d/2) < (1+\delta\epsilon) \operatorname{Lip}(f; 2d),$$

for otherwise iteration would yield

$$1 \geq \operatorname{Lip}(f; 4^{-k}) \geq (1+\delta\epsilon)^k \operatorname{Lip}(f; 1) \geq (1+\delta\epsilon)^k \epsilon > 1.$$

Let d' be the supremum of all d, $2 \geq 2d \geq 4^{-k}$, which satisfy (2.2). By making a translation of the domain of f, we can assume, in view of (2.2), that for some z, $-z$ in B, where $||z|| \equiv d \leq d'$ with $d' - d$ as small as we like, we have:

$$\operatorname{Lip}(f; d/2) < (1+\delta\epsilon) \frac{f(z) - f(-z)}{2d}.$$

By adding a constant to f, we can assume that $f(z) = dL = -f(-z)$ for some $0 < L \leq \operatorname{Lip}(f) \leq 1$.

Let z^* be the norming functional for $\frac{z}{||z||}$. Suppose that $||w|| \leq \delta d$ with w in B, and note that the distance from w to both z and $-z$ is at least $d/2$. We use (2.1) to estimate $|f(w) - Lz^*(w)|$. First,

$$\begin{aligned} f(w) &= f(w) - f(-z) + f(-z) \leq (1+\delta\epsilon) L \, \|z + w\| - dL \\ &= (1+\delta\epsilon) Ld \left\| \frac{z}{d} + \frac{w}{d} \right\| - dL \leq (1+\delta\epsilon) Ld \left[1 + z^* \left(\frac{w}{d} \right) + r \left(\frac{w}{d} \right) \right] - dL \\ &\leq (1+\delta\epsilon) Ld \left[1 + \frac{z^*(w)}{d} + \frac{\epsilon ||w||}{d} \right] - dL \leq Lz^*(w) + \epsilon(2 + \delta + \delta\epsilon)\delta Ld \\ &\leq Lz^*(w) + 3\epsilon\delta d. \end{aligned}$$

Similarly, starting from

$$-f(w) = f(z) - f(w) - f(z),$$

we see that $f(w) \geq Lz^*(w) - 3\epsilon\delta d$, so that

$$|f(w) - Lz^*(w)| \leq 3\epsilon\delta d$$

whenever $\|w\| \leq \delta d$ and w is in B. This completes the proof since $\mathrm{B}_{\delta d}(0) \cap B$ contains a ball of radius $\frac{\delta}{2} d$ and d is bounded below by a constant depending only on ϵ and δ. ■

It is clear that if $\operatorname{Lip}(X, Y)$ has the UAAP (or AAP) then so does any pair (X_1, Y_1) with X isomorphic to X_1 and Y isomorphic to Y_1. Thus we could have stated Theorem 2.3 for super-reflexive X instead of uniformly smooth X; we chose the latter because uniform smoothness arises naturally in the context in which we are working. Moreover, the proportional size of the ball on which the affine approximation is obtained is uniform over all spaces X having a common modulus of smoothness.

Next we show that the isomorphic version of Theorem 2.3 classifies those spaces X for which $\operatorname{Lip}(X, \mathbb{R})$ has the uniform approximation by affine property.

Proposition 2.4. Lip $(X, \mathbb{R})$ *has the uniform approximation by affine property if and only if X is super-reflexive.*

Proof. The "if" direction follows from Theorem 2.3. For the other direction, suppose for contradiction that X, and hence also X^*, is not super-reflexive. Given any $\epsilon > 0$ and natural number N we get from [JS] a sequence $\{x_i^*\}_{i=1}^{2^N}$ of unit vectors in X^* so that for every $m = 1, 2, ..., 2^N$ and $a_i \geq 0$,

$$||\sum_{i=1}^{m} a_i x_i^* - \sum_{i=m+1}^{2^N} a_i x_i^*|| \geq \left(1 - \frac{\epsilon}{2}\right) \sum_{i=1}^{2^N} a_i. \tag{2.3}$$

Define a dyadic tree as follows: For $k = 0, 1, ..., N$ and $j = 0, 1, ..., 2^k - 1$, set

$$x_{j,k}^* = 2^{k-N} \sum_{i=j \cdot 2^{N-k}+1}^{(j+1)2^{N-k}} x_i^*. \tag{2.4}$$

By (2.3), $(1 - \epsilon/2) \leq ||x_{j,k}^*||$, by (2.4),

$$x_{j,k}^* = \frac{x_{2j,k+1}^* + x_{2j+1,k+1}^*}{2} \tag{2.5}$$

and by (2.3),

$$||x_{2j,k+1}^* - x_{2j+1,k+1}^*|| > 2(1 - \epsilon). \tag{2.6}$$

Define $0 < \lambda < 1$ by $\lambda^N = 1/3$ and define f on X by

$$f(x) = \max_{j,k} \lambda^k \left(||x|| + 2|\langle x_{j,k}^*, x\rangle|\right). \tag{2.7}$$

So f is an equivalent norm on X and Lip $(f) = 3$.

Suppose r satisfies

$$\epsilon r > 18(1 - \lambda) \quad \text{and} \quad \mathrm{B}_r(x) \subset \mathrm{B}_1(0). \tag{2.8}$$

We show that the restriction of f to $\mathrm{B}_r(x)$ is not close to an affine mapping. Choose j and k with k as small as possible so that

$$f(x) = \lambda^k \left(||x|| + 2|\langle x_{j,k}^*, x\rangle|\right), \tag{2.9}$$

and assume for definiteness that $\langle x^*_{j,k}, x\rangle \geq 0$. Since $\lambda^N = 1/3$ and k is minimal, we see that $k < N$. By (2.7) and (2.9),

$$\lambda^{k+1}\left(||x|| + 2\left[|\langle x^*_{2j,k+1}, x\rangle| \vee |\langle x^*_{2j+1,k+1}, x\rangle|\right]\right) \leq f(x)$$

so that

$$|\langle x^*_{2j,k+1}, x\rangle| \vee |\langle x^*_{2j+1,k+1}, x\rangle| \leq \lambda^{-1}\left(\langle x^*_{j,k}, x\rangle + 2^{-1}(1-\lambda)||x||\right).$$

Therefore, for $i = 0, 1$ we have

$$\langle x^*_{2j+i,k+1}, x\rangle \geq (2-\lambda^{-1})\langle x^*_{j,k}, x\rangle - (2\lambda)^{-1}(1-\lambda)||x||. \tag{2.10}$$

By (2.3) there is y in Y with $||y|| = 1$ and $\langle x^*_{2j,k+1} - x^*_{2j+1,k+1}, y\rangle > 2(1-\epsilon)$, and hence

$$\langle x^*_{2j,k+1}, y\rangle > 1 - 2\epsilon \quad \text{and} \quad \langle x^*_{2j+1,k+1}, y\rangle < -1 + 2\epsilon. \tag{2.11}$$

Therefore, by using (2.10) and (2.11) in the second inequality below, we get

$$\begin{aligned} f(x+ry) &\geq \lambda^{k+1}\left(||x+ry|| + 2\langle x^*_{2j,k+1}, x+ry\rangle\right) \\ &\geq \lambda^{k+1}\left(||x|| - r + 2(2-\lambda^{-1})\langle x^*_{j,k}, x\rangle - \lambda^{-1}(1-\lambda)||x|| + 2(1-2\epsilon)r\right) \\ &\geq \lambda f(x) + \lambda^{k+1}(1-4\epsilon)r - 3\lambda^k(1-\lambda) \qquad \text{by} \quad (2.9) \\ &\geq f(x) + 3^{-1}(1-4\epsilon)r - 6(1-\lambda) \quad \text{since} \quad |f(x)| \leq 3 \text{ and } \lambda^{k+1} \geq 3^{-1} \\ &\geq f(x) + 3^{-1}(1-5\epsilon)r. \qquad \text{by} \quad (2.8) \end{aligned}$$

Similarly, $f(x-ry) \geq f(x) + 3^{-1}(1-5\epsilon)r$, and hence

$$\sup_{y \in \mathrm{B}_r(x)} |f(y) - L(y)| \geq 6^{-1}(1-5\epsilon)r$$

for every affine function L. ■

If we reverse the order of $\mathbb{R}$ and X we get the same characterization of the uniform approximation by affine property for the pair of spaces.

Proposition 2.5. *A Banach space X is super-reflexive if and only if* $\mathrm{Lip}\,(\mathbb{R}, X)$ *has the uniform approximation by affine property.*

Proof. Assume first that X is super-reflexive. Without loss of generality we may assume it is actually uniformly convex. In Proposition 2.6 we give a soft proof that

$\mathrm{Lip}\,(I\!R, X)$ has the UAAP, but here we give a proof in the spirit of the proof of Theorem 2.3 which can provide a specific estimate of the size of the ball upon which the affine approximation is valid in terms of the modulus of convexity of X and the degree of approximation (however, we do not actually make the estimate).

Let $f : I \to X$ be a function with $\mathrm{Lip}\,(f) \le 1$ defined on an interval I which, without loss of generality, we assume is of length 1. Given ϵ, we have a largest positive d with $d \le 1$

and $$\mathrm{Lip}\,(f; d/4) \le (1 + \epsilon/2)\mathrm{Lip}\,(f; d).$$

As in the proof of Theorem 2.3, d is bounded away from zero by a function of ϵ. By translations in the domain and range, we may assume that 0, $d \in I$,

$$\mathrm{Lip}\,(f; d/4) < (1 + \epsilon)\frac{\|f(d) - f(0)\|}{d},$$

$f(0) = 0$, and $f(d) = dx$ for some $x \in X$ with $0 < \|x\| \le \mathrm{Lip}\,(f) \le 1$.

If $d/4 \le r < 3d/4$,

$$\|f(r) - dx\| = \|f(r) - f(d)\| \le (d - r)\mathrm{Lip}\,(f; d/4) \le (d - r)(1 + \epsilon)\|x\|$$

and $$\|f(r)\| = \|f(r) - f(0)\| \le r(1 + \epsilon)\|x\|.$$

It follows that

$$\|f(r)\| + \|dx - f(r)\| \le (1 + \epsilon)d\|x\| = (1 + \epsilon)\|f(r) + dx - f(r)\|. \tag{2.12}$$

We shall denote below by $\delta(\epsilon)$ a positive function, not the same in every instance, depending only on the modulus of uniform convexity of X, and which tends to zero as ϵ tends to zero. A simple consequence of the uniform convexity definition is that if $\|x\| + \|y\| \le (1 + \epsilon)\|x + y\|$ for two vectors in X for which $\frac{1}{10} \le \frac{\|x\|}{\|y\|} \le 10$ then $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \le \delta(\epsilon)$. When applied to (2.12), this implies that $\|f(r) - ax\| \le \delta(\epsilon)d\|x\|$ for some scalar a depending on r. Considering now the real valued Lipschitz function $\|f(r)\|$, we easily get that, for $d/4 \le r \le 3d/4$, $\big|\|f(r)\| - r\|x\|\big| \le \delta(\epsilon)d\|x\|$. Combining inequalities we get that $\|f(r) - rx\| \le \delta(\epsilon)d\|x\| \le \delta(\epsilon)d$ for $d/4 \le r \le 3d/4$.

If X is not super-reflexive we can assume, by [JS], that X contains, for each positive integer n and for each $\epsilon > 0$, a normalized sequence $\{e_i\}_{i=1}^n$ satisfying

$\|\sum_{i=1}^k a_ie_i - \sum_{i=k+1}^n a_ie_i\| \geq (1-\epsilon)\sum_{i=1}^n a_i$, for all $k = 1,\ldots,n$ and all $a_i \geq 0$, $i = 1,\ldots,n$. Fix n and define $f : [0,n] \to X$ by

$$f(t) = \sum_{i=1}^{[t]} e_i + (t-[t])e_{[t]+1}.$$

Then f is Lipschitz with constant 1 and f^{-1} is Lipschitz with constant $(1-\epsilon)^{-1}$. Assume k and l are positive integers with $k+2l \leq n$ and $A : [k, k+2l] \to X$ is an affine map which approximates f to within $l/4$ on $[k, k+l]$. Then, $\|A(k) - \sum_{i=1}^k e_i\| \leq l/4$, $\|A(k+2l) - \sum_{i=1}^{k+2l} e_i\| \leq l/4$, and $\|A(k+l) - \sum_{i=1}^{k+l} e_i\| \leq l/4$. It then follows that

$$(1-\epsilon)l \leq \frac{1}{2}\|\sum_{i=k+1}^{k+l} e_i - \sum_{i=k+l+1}^{k+2l} e_i\| = \|\sum_{i=1}^{k+l} e_i - \frac{1}{2}(\sum_{i=1}^{k} e_i + \sum_{i=1}^{k+2l} e_i)\| \leq l/2$$

which, for $\epsilon < 1/2$, contradicts the assumption on $\{e_i\}_{i=1}^n$. ∎

Proposition 2.6. *Assume that X is super-reflexive and Y is finite dimensional. Then* Lip (Y, X) *has the uniform approximation by affine property.*

Proof. Suppose that Lip (Y, X) fails the UAAP and let $\mathcal{U}$ be a free ultrafilter on the natural numbers. We claim that then Lip $(Y, X_{\mathcal{U}})$ fails the AAP, where $X_{\mathcal{U}}$ denotes the ultrapower of X with respect to $\mathcal{U}$. Having established the claim, we complete the proof by pointing out that $X_{\mathcal{U}}$ is reflexive since X is super-reflexive, and hence every Lipschitz mapping from a ball in Y to $X_{\mathcal{U}}$ has a point of differentiability; in particular, Lip $(Y, X_{\mathcal{U}})$ has the AAP.

We now prove the claim. Since Lip (Y, X) fails the UAAP, it is easy to see that there exist $\epsilon > 0$ and mappings f_n from $\mathrm{B}_1^Y(0)$ into X with Lip $(f_n) = 1$, $f_n(0) = 0$, so that for all balls $B \subset \mathrm{B}_1^Y(0)$ of radius $r = r(B)$ at least $1/n$ we have for all affine mappings $L : Y \to X$ the estimate

$$||f_n - L||_B \geq \epsilon r, \tag{2.13}$$

where $||g||_C := \sup\limits_{y\in C} ||g(y)||$. Let $f_{\mathcal{U}}$ be the ultraproduct of the mappings f_n, defined for y in $\mathrm{B}_1^Y(0)$ by $f_{\mathcal{U}}(y) = (f_n(y))$. Let $B \subset \mathrm{B}_1^Y(0)$ be a ball and let r be the radius of B. Now if $L : Y \to X_{\mathcal{U}}$ is affine, then since Y is finite dimensional, L is the ultraproduct of some sequence (L_n) of affine mappings from Y to X. But then by (2.13), for all balls $B \subset \mathrm{B}_1^Y(0)$, $||f_{\mathcal{U}} - L||_B = \lim\limits_{n\in U} ||f_n - L_n||_B \geq \epsilon r$. This means that Lip $(Y, X_{\mathcal{U}})$ fails the AAP. ∎

We now characterize those pairs of Banach spaces for which Lip (X, Y) have the uniform approximation by affine property.

Theorem 2.7. *Let X and Y be nonzero Banach spaces. Then* $\mathrm{Lip}\,(X,Y)$ *has the uniform approximation by affine property if and only if one of the spaces is super-reflexive and the other is finite dimensional.*

Proof. If one of the spaces is super-reflexive and the other is finite dimensional, then $\mathrm{Lip}\,(X,Y)$ has the UAAP by Theorem 2.3 or by Proposition 2.6. So assume that the spaces are nonzero and $\mathrm{Lip}\,(X,Y)$ has the UAAP. It is easy to see that if X_0 is a complemented subspace of X and Y_0 is a complemented subspace of Y, then $\mathrm{Lip}\,(X_0,Y_0)$ has the UAAP. Therefore, $\mathrm{Lip}\,(X,\mathbb{R})$ and $\mathrm{Lip}\,(\mathbb{R},Y)$ have the UAAP and hence, by Proposition 2.4 and Proposition 2.5, X and Y are super-reflexive. Now if both are infinite dimensional, then both contain uniformly complemented copies of ℓ_2^n for all n ([FT], [Pis]). Although $\mathrm{Lip}\,(\ell_2^n,\ell_2^n)$ has the UAAP for each fixed n, the estimate for the size of the ball on which a Lipschitz constant one mapping from $\mathrm{B}_1^{\ell_2^n}(0)$ into ℓ_2^n must have a linear approximate within a given error tends to zero as $n\to\infty$. Indeed, consider the mapping which takes $\{a_i\}_{i=1}^n$ to $\{|a_i|\}_{i=1}^n$. Since any $x\in \mathrm{B}_1^{\ell_2^n}(0)$ has a coordinate whose absolute value is $\le 1/\sqrt{n}$, any ball of radius $r>2/\sqrt{n}$ contained in $\mathrm{B}_1^{\ell_2^n}(0)$ contains a segment of the form $x+(-r,r)e_i$ for every basis vector e_i. Clearly, on this segment, the mapping above cannot be approximated by an affine mapping to a degree better than $r/4$. This easily implies that $\mathrm{Lip}\,(X,Y)$ fails the UAAP if both spaces contain uniformly complemented copies of ℓ_2^n for all n. ■

For examples of pairs of infinite dimensional spaces for which $\mathrm{Lip}\,(X,Y)$ has the approximation by affine property we refer to [JLPS2]. In the rest of this section we restrict attention to the situation in which either X or Y is the scalar field.

Proposition 2.8. *The following are equivalent for a Banach space X.*
(i) $\mathrm{Lip}\,(X,\mathbb{R})$ *has the approximation by affine property.*
(ii) X is Asplund.
(iii) Every real valued Lipschitz mapping from a domain in X has a point of Fréchet differentiability.

Proof. Recall that X is Asplund provided every real valued convex continuous function on a convex domain U in X is Fréchet differentiable on a dense G_δ subset of U. Various equivalents to this can be found e.g. in section I.5 of [DGZ].

That condition (iii) implies both (i) and (ii) is clear. In [Pre] it is proved that (ii) implies (iii). The implication (i) $\Rightarrow$ (ii) is a simple consequence of a result of Leach and Whitfield [LW] (or see Theorem 5.3 in [DGZ]). They proved that if X is not Asplund, then there is an equivalent norm (which we might as well take to be the original norm $||\cdot||$) on X and a $\delta>0$ so that every weak*-slice of the dual ball in X^* has diameter larger than δ. This implies that for every x in X and $\epsilon>0$ the diameter of the set $\{x^*\in \mathrm{B}_1^{X^*}(0) : x^*(x)>||x||-\epsilon\}$ is larger than

δ. Their further computations used to prove that $||\cdot||$ is a so-called rough norm on X also yield that $\mathrm{Lip}\,(X, I\!R)$ fails the AAP. Explicitly, it is enough to check that if x is in X and $r > 0$, then there is a vector u in X with $||u|| = r$ so that $||x+u|| + ||x-u|| - 2||x|| > \frac{\delta}{2}r$. To do this, choose norm one linear functionals x_1^*, x_2^* so that $x_i^*(x) > ||x|| - \frac{\delta r}{4}$ for $i = 1, 2$ but $||x_1^* - x_2^*|| > \delta$. So we can take u in X with $||u|| = r$ and $(x_1^* - x_2^*)(u) > \delta r$. Putting things together, we see that

$$||x+u|| + ||x-u|| \geq x_1^*(x+u) + x_2^*(x-u) > 2||x|| - \frac{\delta r}{2} + \delta r,$$

as desired. ■

It is also possible to characterize the spaces X for which $\mathrm{Lip}\,(I\!R, X)$ has the approximation by affine property. Let $\{\mathcal{F}_n\}_{n=0}^{\infty}$ be a sequence of σ-fields on $[0,1)$ with $\mathcal{F}_0 = \{\emptyset, [0,1)\}$, $\mathcal{F}_n$ is generated by exactly 2^n intervals of the form $[a,b)$, and the atoms of $\mathcal{F}_{n+1}$ are obtained from those of $\mathcal{F}_n$ by splitting each of the intervals generating $\mathcal{F}_n$ into two subintervals.

An X valued martingale $\{M_n\}$ with respect to such a sequence of σ-fields is called a *generalized dyadic martingale*. If each atom of $\mathcal{F}_n$ has measure exactly 2^{-n} we call the martingale *dyadic*. $\{M_n\}$ is said to be *δ-separated* if $\|M_n - M_{n+1}\| \geq \delta$ a.e. for all n. It is *bounded* if $\sup \|M_n\|_\infty < \infty$. See [KR] or the book [Bou] for connections between the geometry of X and the existence of δ-separated, bounded, dyadic or generalized dyadic martingales with values in X. It seems not to be known whether the existence of an X valued, bounded, δ-separated generalized dyadic martingale implies the existence of a dyadic martingale with the same properties (possibly with a different $\delta > 0$).

Proposition 2.9. *For a Banach space X, $\mathrm{Lip}\,(I\!R, X)$ has the approximation by affine property if and only if for every $\delta > 0$ there exists no X valued, bounded, δ-separated generalized dyadic martingale.*

Proof. If $\{M_n\}$ is an X valued, δ-separated generalized dyadic martingale satisfying $\|M_n\|_\infty \leq 1$, say, define, for each n, $f_n : [0,1) \to X$ by $f_n(t) = \int_0^t M_n(s)ds$. Then $\mathrm{Lip}\,(f_n) \leq 1$ for all n and the sequence $\{f_n(t)\}$ is eventually constant for each t which is an end point of one of the intervals generating one of the $\mathcal{F}_n$'s. The separation and boundedness conditions imply that the set of such t's is dense in $[0,1]$. Thus $f_n(t)$ converges for all t in $[0,1)$ to a function f satisfying $\mathrm{Lip}\,(f) \leq 1$.

If $a < b < c$ with $[a,b)$, $[b,c)$ atoms of $\mathcal{F}_{n+1}$ and $[a,c)$ an atom of $\mathcal{F}_n$, then $f(x) = f_{n+1}(x)$ for $x = a, b, c$. Put $A = M_{n+1}^{[a,b)}$, $C = M_{n+1}^{[b,c)}$ and $B = M_n^{[a,c)}$, where M_k^S denotes the constant value of M_k on the atom S of $\mathcal{F}_k$. Note first that $\frac{c-a}{b-a} \leq \frac{2}{\delta}$. Indeed, $B = \frac{b-a}{c-a}A + \frac{c-b}{c-a}C$ and, if $\frac{b-a}{c-a} < \frac{\delta}{2}$, then $\left|1 - \frac{c-b}{c-a}\right| = \frac{b-a}{c-a} < \frac{\delta}{2}$ and

$$\|B - C\| < \frac{\delta}{2}\|A\| + \frac{\delta}{2}\|C\| < \delta,$$

a contradiction.

Now if L is an affine map on $[a,c)$ with $L(a) = f(a)$ and $\|L(x) - f(x)\| < (\delta^2/4)(c-a)$ on $[a,c)$, then

$$\begin{aligned}\|A-B\| &= \left\|\frac{f(b)-f(a)}{b-a} - \frac{f(c)-f(a)}{c-a}\right\| \\ &\leq \left\|\frac{L(b)-L(a)}{b-a} - \frac{L(c)-L(a)}{c-a}\right\| + \frac{\delta^2}{4}\frac{c-a}{b-a} + \frac{\delta^2}{4} \\ &= \frac{\delta^2}{4}\left(\frac{c-a}{b-a}+1\right) \leq \frac{\delta^2}{4}\left(1+\frac{2}{\delta}\right) < \delta,\end{aligned}$$

a contradiction. Since every interval contains an interval of comparable length which is an atom of one of the $\mathcal{F}_n$'s, it follows that $\mathrm{Lip}\,(\mathbb{R},X)$ does not have the AAP.

If $\mathrm{Lip}\,(\mathbb{R},X)$ does not have the AAP, let $f : [0,1) \to X$ be a function with $\mathrm{Lip}\,(f) = 1$ which for some $\delta > 0$ has the property that for every subinterval $[a,c)$ of $[0,1)$, there exists a b with $a < b < c$ and

$$\left\|f(b) - \frac{c-b}{c-a}f(a) - \frac{b-a}{c-a}f(c)\right\| > \delta(c-a). \tag{2.14}$$

Then define a generalized dyadic martingale as follows: $M_0 = f(1)-f(0)$ everywhere on $[0,1)$. Assume M_n was already defined and, for each atom interval $[a,c)$ in $\mathcal{F}_n$, $M_n = \frac{f(c)-f(a)}{c-a}$ on $[a,c)$. Then, for each such interval pick a b as in (2.14), add the two intervals $[a,b),[b,c)$ to $\mathcal{F}_{n+1}$, and define $M_{n+1} = \frac{f(b)-f(a)}{b-a}$ on $[a,b)$ and $M_{n+1} = \frac{f(c)-f(b)}{c-b}$ on $[b,c)$. Then

$$M_n^{[a,c)} = \frac{b-a}{c-a}M_{n+1}^{[a,b)} + \frac{c-b}{c-a}M_{n+1}^{[b,c)},$$

and

$$\begin{aligned}\|M_{n+1}^{[a,b)} - M_n^{[a,c)}\| &= \left\|\frac{f(b)-f(a)}{b-a} - \frac{f(c)-f(a)}{c-a}\right\| \\ &= \frac{1}{b-a}\left\|f(b) - (1-\frac{b-a}{c-a})f(a) - \frac{b-a}{c-a}f(c)\right\| \\ &= \frac{1}{b-a}\left\|f(b) - \frac{c-b}{c-a}f(a) - \frac{b-a}{c-a}f(c)\right\| > \frac{\delta(c-a)}{b-a} > \delta.\end{aligned}$$

Similarly, also $\|M_{n+1}^{[b,c)} - M_n^{[a,c)}\| > \delta$. So $\{M_n\}$ is an X valued, δ-separated generalized dyadic martingale with $\|M_n\|_\infty \leq 1$. ■

It is well known that X fails the Radon-Nikodym property if and only if there is a bounded, δ-separated, X valued martingale (see e.g. section V.3 in [DU]). There is an ingenious example in [BR] of a subspace X of L_1 which does not have the Radon-Nikodym property, yet for all $\delta > 0$ there is no bounded, δ-separated, X valued generalized dyadic martingale. Thus $\mathrm{Lip}\,(\mathbb{R}, X)$ can have the approximation by affine property when X fails the Radon-Nikodym property. The characterization of the Radon-Nikodym property in terms of differentiability properties is given by the following (mostly known) proposition:

Proposition 2.10. *The following are equivalent for a Banach space X.*

(i) X has the Radon-Nikodym property.

(ii) Every Lipschitz mapping from $\mathbb{R}$ into X is differentiable almost everywhere.

(iii) Every Lipschitz mapping from $\mathbb{R}$ into X has for every $\epsilon > 0$ a point of ϵ-Fréchet differentiability.

Proof. $(i) \Rightarrow (ii)$ is classical, so one only needs to prove $(iii) \Rightarrow (i)$. The proof is basically the same as the proof of the first half of Proposition 2.9: If X fails the Radon-Nikodym property then there is a bounded, δ-separated, X valued martingale. That is, for some sequence $\{\mathcal{F}_n\}_{n=0}^{\infty}$ of σ-fields on $[0,1)$ such that $\mathcal{F}_0 = \{\emptyset, [0,1)\}$ and each $\mathcal{F}_n$ is generated by intervals, there is an X valued martingale $\{M_n\}$ with respect to this sequence of σ-fields such that $||M_n||_\infty \le 1$ and $||M_n(x) - M_{n+1}(x)|| \ge \delta$ for all n and all $x \in [0,1)$. Define f as in the beginning of the proof of Proposition 2.9. If x_0 is a point such that

$$||f(x_0+u) - f(x_0) - Lu|| \le \epsilon|u| \quad \text{for} \quad |u| \le \eta$$

for some linear L, let $a \le b < c \le d$ with $[b,c)$ an atom of $\mathcal{F}_{n+1}$ and $[a,d)$ an atom of $\mathcal{F}_n$ with $x_0 \in [b,c)$ and $d - a < \eta$. Then $f(x) = f_{n+1}(x)$ for $x = a, b, c, d$, and

$$\begin{aligned}
&||f(c) - f(b) - L(c-b)|| \\
&\qquad = ||f(x_0 + c - x_0) - f(x_0) - L(c - x_0) - (f(x_0 + b - x_0) \\
&\qquad - f(x_0) - L(b - x_0))|| \\
&\qquad \le 2\epsilon(c-b).
\end{aligned}$$

Similarly,

$$||f(d) - f(a) - L(d-a)|| \le 2\epsilon(d-a).$$

Consequently,

$$||M_n(x_0) - M_{n+1}(x_0)|| = \left\| \frac{f(d) - f(a)}{d-a} - \frac{f(c) - f(b)}{c-b} \right\| \le 4\epsilon,$$

which is impossible if $\epsilon < \delta/4$. ■

Remark 2.11. The implication $(ii) \Rightarrow (i)$ in Proposition 2.10 has been known to experts for a long time. It is mentioned without proof in [Ben]. The only published proof of which we are aware appears in [Qia], where a more general theorem is proved.

3. Nonlinear quotient mappings

We begin with two definitions. The definition of co-Lipschitz appears in Chapter 1, section 1.25 of [Gro]. A mapping f is co-uniform if and only if the sequence $(f, f, f, \dots)$ is uniformly open in the sense used in Chapter 10, section 2 of [Why]. Co-uniform mappings and uniform quotients in the context of general uniform spaces are discussed in [Jam], where they are used to develop a theory of uniform transformation groups and uniform covering spaces.

Definition 3.1. *A mapping T from a metric space X to a metric space Y is said to be* co-uniformly continuous *provided that for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ so for every x in X, $T\mathrm{B}_\epsilon(x) \supset \mathrm{B}_\delta(Tx)$. If $\delta(\epsilon)$ can be chosen larger than ϵ/C for some $C > 0$, then T is said to be* co-Lipschitz*, and the smallest such C is denoted by co-Lip(T).*

Definition 3.2. *A mapping T from a metric space X to a metric space Y is said to be a* uniform quotient mapping *provided T is uniformly continuous and co-uniformly continuous. If T is Lipschitz and co-Lipschitz, then T is called a* Lipschitz quotient mapping.

A space Y is said to be a *uniform quotient* (respectively, *Lipschitz quotient*) of a space X provided there is a uniform quotient mapping (respectively, a Lipschitz quotient mapping) from X onto Y.

The linear theory is simplified by the fact that a surjective bounded linear operator between Banach spaces is automatically a quotient mapping. Of course, a surjective Lipschitz mapping from $\mathbb{R}$ to $\mathbb{R}$ which is a homeomorphism need not be even a uniform quotient mapping. Moreover, a surjective Lipschitz mapping does not carry any structure: in [Bat] it is shown that if X and Y are Banach spaces with X infinite dimensional, and the density character of X is at least as large as that of Y, then there is a Lipschitz mapping of X onto Y. In this section we show that, in contrast to this, uniform and Lipschitz quotient mappings do preserve some structure.

A connection between uniformly continuous mappings and Lipschitz mappings is provided by the well-known fact that a uniformly continuous mapping from a convex domain is "Lipschitz for large distances". There is a "co" version of this:

Remark 3.3. If T is a uniformly continuous mapping from a convex set X onto a set Y, then T is "Lipschitz for large distances" in the sense that for each $\epsilon_0 > 0$ there exists $C = C(\epsilon_0) > 0$ so that for all $\epsilon \geq \epsilon_0$ and all x in X, $T\mathrm{B}_\epsilon(x) \subset \mathrm{B}_{C\epsilon}(Tx)$. Similarly, if T is a co-uniformly continuous mapping from a set X onto a convex set Y, then T is "co-Lipschitz for large distances" in the sense for each $\epsilon_0 > 0$ there exists $C = C(\epsilon_0) > 0$ so that for all $\epsilon \geq \epsilon_0$ and all x in X, $T\mathrm{B}_\epsilon(x) \supset \mathrm{B}_{\epsilon/C}(Tx)$. To see this, suppose that ϵ and δ satisfy $T\mathrm{B}_\epsilon(x) \supset \mathrm{B}_\delta(Tx)$ for every x in X. We have for any x in X that

$$\begin{aligned} T\mathrm{B}^X_{2\epsilon}(x) &\supset T\left[\cup_{y\in \mathrm{B}^X_\epsilon(x)} \mathrm{B}^X_\epsilon(y)\right] = \cup_{y\in \mathrm{B}^X_\epsilon(x)} T\mathrm{B}^X_\epsilon(y) \\ &\supset \cup_{y\in \mathrm{B}^X_\epsilon(x)} \mathrm{B}^Y_\delta(Ty) \supset \cup_{z\in \mathrm{B}^Y_\delta(Tx)} \mathrm{B}^Y_\delta(z) = \mathrm{B}^Y_{2\delta}(Tx), \end{aligned}$$

where the last equality follows from the convexity of Y.

Proposition 3.4. *If the Banach space Y is a uniform quotient of the Banach space X and $\mathcal{U}$ is a free ultrafilter on the natural numbers, then $Y_\mathcal{U}$ is a Lipschitz quotient of $X_\mathcal{U}$.*

Proof. Let T be a uniform quotient mapping from X onto Y. By the remark, there is a constant $C > 0$ so that for all x in X and $r \geq 1$, $\mathrm{B}^Y_{Cr}(Tx) \supset T\mathrm{B}^X_r(x) \supset \mathrm{B}^Y_{r/C}(Tx)$. This is the only property of T needed in the proof.

For each n, define $T_n : X \to Y$ by $T_n x = \frac{T(nx)}{n}$. Then for each $r \geq \frac{1}{n}$ and x in X, we have that $\mathrm{B}^Y_{Cr}(T_n x) \supset T_n\mathrm{B}^X_r(x) \supset \mathrm{B}^Y_{r/C}(T_n x)$. Let $T_\mathcal{U} : X_\mathcal{U} \to Y_\mathcal{U}$ be the ultraproduct of the mappings T_n, defined for $\tilde{x} = (x_n)$ in $X_\mathcal{U}$ by $T_\mathcal{U}\tilde{x} = (T_n x_n)$. From the preceding comments it follows easily that for each $\tilde{x}$ in $X_\mathcal{U}$ and $r > 0$, $\mathrm{B}^{Y_\mathcal{U}}_{Cr}(T_\mathcal{U}\tilde{x}) \supset T_\mathcal{U}\mathrm{B}^{X_\mathcal{U}}_r(x) \supset \mathrm{B}^{Y_\mathcal{U}}_{r/C}(T_\mathcal{U}\tilde{x})$, so that $T_\mathcal{U}$ is a Lipschitz quotient mapping from $X_\mathcal{U}$ onto $Y_\mathcal{U}$. ■

We do not know whether a Lipschitz quotient of a separable Banach space must be a linear quotient of that space. In the nonseparable setting there is a counterexample: In [AL] it was shown that $c_0(\aleph)$ is Lipschitz equivalent to a certain subspace X of ℓ_∞, where $\aleph$ is the cardinality of the continuum. However, no nonseparable subspace of ℓ_∞ is isomorphic to a quotient of $c_0(\aleph)$ because this space (and hence all of its linear quotients) are weakly compactly generated, while every weakly compact subset of ℓ_∞ is separable.

We now come to the main result we have regarding the linearization of nonlinear quotients. Recall that a Banach space X is said to be *finitely crudely representable* in a Banach space Y provided that there exists λ so that every finite dimensional subspace of X is λ-isomorphic to a subspace of Y. If this is true for every $\lambda > 1$, X is said to be *finitely representable* in Y.

Theorem 3.5. *Assume that X is super-reflexive and Y is a uniform quotient of X. Then Y^* is finitely crudely representable in X^*. Consequently, Y is isomorphic to a linear quotient of some ultrapower of X.*

Proof. By making an arbitrarily small distortion of the norm in X, we can in view of the James-Enflo renorming theorem [DGZ, p. 149] assume that X is uniformly smooth. In view of Proposition 3.4, we also can assume, via replacement of X and Y by appropriate ultrapowers, that Y is a Lipschitz quotient of X. So for some $\delta > 0$, there is a mapping T from X onto Y with $\mathrm{Lip}\,(T) = 1$ and $T\mathrm{B}_r(x) \supset \mathrm{B}_{\delta r}(Tx)$ for every x in X and $r > 0$. Let E be any finite dimensional subspace of Y^*, $E_\perp$ the preannihilator of E in Y, and Q the quotient mapping from Y onto $Y/E_\perp \equiv E^*$. The composition QT is then a Lipschitz quotient mapping with $\mathrm{Lip}\,(QT) \le 1$ and $QT\mathrm{B}_r(x) \supset \mathrm{B}_{\delta r}(QTx)$ for every x in X and $r > 0$. (Formally, we should here replace δ by an arbitrary positive number smaller than δ, but at the end we know that Y is reflexive, so the containment we wrote really is true.) For any finite dimensional space Z, the pair $\mathrm{Lip}\,(X, Z)$ has the UAAP by Theorem 2.3, so given any $\delta > \epsilon > 0$ there is a ball $B = \mathrm{B}_r(x_0)$ in X and an affine mapping G from X into $Y/E_\perp$ so that $\sup_{x\in B} ||QTx - Gx|| \le \epsilon r$. This and the quotient property of QT yield that $G\mathrm{B}_r(x_0) \supset \mathrm{B}_{(\delta-\epsilon)r}(QT(x_0))$. Letting G_1 be the linear mapping $G - G(0)$, we infer that $G\mathrm{B}_1(0)$ contains some ball in $Y/E_\perp$ of radius $\delta - \epsilon$, hence contains the ball around 0 of this radius. Therefore G_1^* is a an isomorphic embedding of E into X^* with isomorphism constant at most $(\delta - \epsilon)^{-1}||G||$. Since $\mathrm{Lip}\,(QT) \le 1$, the inequality $\sup_{x\in B} ||QTx - Gx|| \le \epsilon r$ implies that G maps the unit ball of X into some ball of radius at most $1 + \epsilon$, so that $||G|| \le 1 + \epsilon$.

¿From the above we conclude that every finite dimensional subspace of Y is, for arbitrary $\epsilon > 0$, $(\delta^{-1} + \epsilon)$–isomorphic to a subspace of X^*, so that Y^* is finitely crudely representable in X^*. The "consequently" statement is of course a well-known formal consequence of this (since we now know that Y is reflexive). ■

Say that Y is an *isometric Lipschitz quotient* of X provided that for each $\epsilon > 0$ there is a mapping T from X onto Y so that $\mathrm{Lip}\,(T) = 1$ and for each x in X, $r > 0$, and $0 < \delta < 1$, $T\mathrm{B}_r(x) \supset \mathrm{B}_{\delta r}(Tx)$. The reason we tracked constants in the proof of Theorem 3.5 was to make it clear that the following isometric statement is true:

Corollary 3.6. *If X is super-reflexive and Y is an isometric Lipschitz quotient of X, then Y^* is finitely representable in X^* and Y is isometrically isomorphic to a linear quotient of some ultrapower of X.*

We do not know whether a uniform or even Lipschitz quotient of ℓ_p, $1 < p \neq 2 < \infty$, must be isomorphic to a linear quotient of ℓ_p. However, a separable space which is finitely crudely representable in $L_p \equiv L_p[0,1]$ must isomorphically embed into L_p, [LPe]. So we get the following corollary to Theorem 3.5:

Corollary 3.7. *If Y is a uniform quotient of L_p, $1 < p < \infty$, then Y is isomorphic to a linear quotient of L_p.*

Here is another consequence of Theorem 3.5 which is worth mentioning:

Corollary 3.8. *A uniform quotient of a Hilbert space is isomorphic to a Hilbert space.*

Theorem 3.5 and the arguments in [JLS] can be used to classify the uniform quotients of some spaces other than L_p. As in [JLS], denote by $\mathcal{T}^2$ a certain one of the 2-convex modified Tsirelson-type spaces, defined as the closed span of a certain subsequence of the unit vector basis for the 2-convexification of the modified Tsirelson space first defined in [Joh]. (It is known [CO], [CS] that this space is, up to an equivalent renorming, the space usually denoted by $\mathcal{T}^2$, so our abuse of notation does little harm.) From Theorem 3.5 and the results of [JLS] we deduce the following result, which identifies all the uniform quotient spaces of a space in a situation where the uniform quotients differ from the linear quotients.

Corollary 3.9. *A Banach space Y is a uniform quotient of $\mathcal{T}^2$ if and only if Y is isomorphic to a linear quotient of $\mathcal{T}^2 \oplus \ell_2$.*

Proof. In [JLS] it was proved that $\mathcal{T}^2 \oplus \ell_2$ is uniformly homeomorphic to $\mathcal{T}^2$, so every (isomorph of a) linear quotient of $\mathcal{T}^2 \oplus \ell_2$ is a uniform quotient of $\mathcal{T}^2$.

Conversely, if Y is a uniform quotient of $\mathcal{T}^2$, then by Theorem 3.5 Y is isomorphic to a linear quotient of some ultrapower of $\mathcal{T}^2$. But by [CS, p. 150], an ultrapower of $\mathcal{T}^2$ has the form $\mathcal{T}^2 \oplus H$ for some Hilbert space H. This easily implies that the separable space Y is isomorphic to a linear quotient of $\mathcal{T}^2 \oplus \ell_2$. ■

The next result of this section is the observation that the ϵ-Fréchet derivative of a Lipschitz quotient mapping is a surjective linear mapping (at least when ϵ is smaller than the co-Lipschitz constant of the mapping) and hence the target space is isomorphic to a linear quotient of the domain space. Proposition 3.11 shows that the corresponding statement to Proposition 3.10 for the Gâteaux derivative of a Lipschitz quotient mapping is false.

Proposition 3.10. *Suppose X, Y are Banach spaces and $\delta > \epsilon > 0$. Let $f : X \to Y$ be a map which has an ϵ-Fréchet derivative, T, at some point x_0 and which satisfies for some $r_0 > 0$ and all $0 < r < r_0$ the condition $f[\mathrm{B}_r(x_0)] \supset \mathrm{B}_{\delta r}(f(x_0))$. Then T is surjective.*

Proof. Without loss of generality, we may assume that $x_0 = 0$, $f(0) = 0$, and $\delta = 1$. With this normalization we claim that $T\mathrm{B}_1(0)$ contains the interior of $\mathrm{B}^Y_{1-\epsilon}(0)$. If not, since T is a bounded linear operator, $\overline{T\mathrm{B}_1(0)}$ cannot contain $\mathrm{B}^Y_{1-\epsilon}(0)$, and hence there is a vector y_0 in $\mathrm{B}^Y_1(0)$ so that $d(y_0, T\mathrm{B}_1(0)) > \epsilon > 0$. (Here, $d(\cdot, \cdot)$

denotes the usual infimum "distance" between subsets of Y). Then by linearity, for all $r > 0$,

$$d(ry_0, T\mathrm{B}_r(0)) > r\epsilon. \tag{3.1}$$

Since $f\mathrm{B}_r(0) \supset \mathrm{B}_r(f(0))$ for all $0 < r < r_0$, we can choose for $0 < r < r_0$ points x_r in $\mathrm{B}_r^X(0)$ so that $f(x_r) = ry_0$. By the definition of ϵ-Fréchet derivative, we can then choose $0 < r_1 < r_0$ small enough so that

$$||f(x_r) - Tx_r|| < r\epsilon$$

for all $0 < r < r_1$. In particular, this implies that

$$d(ry_0, T\mathrm{B}_r(0)) \leq ||f(x_r) - Tx_r|| < r\epsilon$$

whenever $0 < r < r_1$, contradicting (3.1) above. ■

Proposition 3.11. *There exists a Lipschitz quotient mapping f from ℓ_p, $1 \leq p < \infty$, onto itself whose Gâteaux derivative at zero is identically zero.*

Define

$$f : \left(\sum_0^\infty \ell_p\right)_p \to \ell_p$$

by

$$f\left(\sum_{n=1}^\infty \sum_{k=0}^\infty a_{nk} e_{nk}\right) = \sum_{n=1}^\infty \left(\left[\sum_{k=0}^\infty g_k(a_{nk}^+)^p\right]^{\frac{1}{p}} - \left[\sum_{k=0}^\infty g_k(a_{nk}^-)^p\right]^{\frac{1}{p}}\right) e_n$$

where $\{e_{nk}\}_{k=o}^\infty$ is the unit vector basis of the $n^{\{textth}$ copy of ℓ_p and

$$g_k(t) = \begin{cases} t, & |t| \geq 2^{-k} \\ 0, & |t| \leq 2^{-k-1} \end{cases}$$

and g_k is linear on the intervals $[2^{-k-1}, 2^{-k}]$ and $[-2^{-k}, -2^{-k-1}]$.

Verifying the Lipschitz condition is easy (check on the positive cone and then use general principles).

The quotient property is a bit more delicate but not difficult. Here is the idea:

Suppose that

$$\|f(x) - y\| = d > 0; \qquad x = \sum_{n=1}^\infty \sum_{k=0}^\infty a_{nk} e_{nk}.$$

We want to find z with $f(z) = y$ and $\|x - z\| \leq Cd$ for some constant C. Write

$$y = f(x) + \sum_{n=1}^{\infty} b_n e_n$$

so that

$$d^p = \sum_{n=1}^{\infty} |b_n|^p.$$

Fix n. We want to perturb x in the coordinates $\{e_{nk}\}_{k=1}^{\infty}$ only by a vector w_n with $\|w_n\| \leq Cb_n$ so that the n^{th} component of $f(x + w_n)$ is the n^{th} component of y. If we can do this for each n, then clearly the vector $z = x + \sum_{n=1}^{\infty} w_n$ satisfies the requirement.

If $b_n = 0$ let $w_n = 0$. Otherwise suppose, for definiteness, that $b_n > 0$. Let $A^p = A_n^p = \sum_{k=0}^{\infty} g_k(a_{nk}^+)^p$. If $A \leq b_n$ then we can let w_n be a multiple, α, of e_{nk_n} where k_n is chosen large enough so that $b_n + a_{nk_n} > 2^{-k_n}$. Then the n^{th} component of $f(x + w_n)$ is

$$\left(\left[\sum_{k \neq k_n} g_k(a_{nk}^+)^p + |\alpha + a_{nk_n}| \right]^{\frac{1}{p}} - \left[\sum_{k \neq k_n} g_k(a_{nk}^-)^p \right]^{\frac{1}{p}} \right) e_n.$$

For $\alpha = 10b_n$ this quantity is larger than or equal the n^{th} component of y and for $\alpha \geq 0$ small enough (explicitly, $\alpha = 0$ if $a_{nk_n} > 0$ and $\alpha = |a_{nk_n}|$ otherwise) it is smaller than the n^{th} component of y. By continuity, one can find the appropriate α.

If $A \geq b_n$, then we let $S = S_n$ be all those k's for which $a_{nk} > 2^{-k-1}$; that is, for which $g_k(a_{nk}) > 0$. In this case we can let w_n be a multiple of $\sum_{k \in S} a_{nk}^+ e_{nk}$. ■

In order to "soup-up" the example in Proposition 3.11, we need the following perturbation lemma for co-Lipschitz mappings:

Lemma 3.12. *Suppose that f and g are continuous mappings from the Banach space X into the Banach space Y and Lip$(g) <$ co-Lip$(f)^{-1}$. Then $f + g$ is co-Lipschitz and co-Lip$(f + g) \leq [1 -$ co-Lip(f)Lip$(g)]^{-1}$.*

Proof. Lemma 3.12 follows from a classical successive approximations argument. By dividing f and g by co-Lip(f), we can assume that $1 = \text{co-Lip}(f) > \text{Lip}(g) \equiv \delta$. Let x be in X and $r > 0$. Given $y \in Y$ with $||y|| < r$, we need to find $z \in X$ with $||z|| < (1 - \delta)^{-1} r$ and $f(x + z) + g(x + z) = f(x) + g(x) + y$. Set $z_0 = 0$ in X. Recursively choose z_k in X so that for each $n = 0, 1, 2, \ldots$, $||z_n|| < \delta^{n-1} r$ and

$$f(x + \sum_{k=0}^{n+1} z_k) + g(x + \sum_{k=0}^{n} z_k) = f(x) + g(x) + y. \tag{3.2}$$

We can make the choice of z_1 because co-Lip$(f) = 1$. If (3.2) holds for n with $||z_{n+1}|| < \delta^n r$, then we have $||g(x + \sum_{k=0}^{n+1} z_k) - g(x + \sum_{k=0}^{n} z_k)|| < \delta^{n+1} r$. Again using the condition co-Lip$(f) = 1$, we can choose z_{n+2} with $||z_{n+2}|| < \delta^{n+1} r$ so that (3.2) holds with n replaced by $n+1$. Setting $z = \sum_{k=1}^{\infty} z_k$, we get the desired conclusion because f is continuous. ■

Corollary 3.13. *Let T be any bounded linear operator on ℓ_p, $1 \leq p < \infty$. Then there is a Lipschitz quotient mapping from ℓ_p onto itself whose Gâteaux derivative at zero is T.*

Proof. Add T to a suitable multiple of the f from Proposition 3.11. ■

If the point of Gâteaux differentiability is an isolated point in its level set, the phenomenon in Proposition 3.11 cannot occur:

Proposition 3.14. *Suppose that f is a Lipschitz quotient mapping from X to Y, f has Gâteaux derivative T at some point p in X, and p is isolated in the level set $[f = f(p)]$. Then T is an isomorphism from X into Y.*

Proof. Without loss of generality, we may assume that $p = 0$, $f(0) = 0$, and $f\mathrm{B}_r(x) \supset \mathrm{B}_r(fx)$ for each $r > 0$ and x in X. For $t > 0$, define $f_t : X \to Y$ by

$$f_t(x) = \frac{f(tx)}{t}.$$

The f_t then converge pointwise as $t \to 0$ to the Gâteaux derivative T, and moreover $f_t\mathrm{B}_r(x) \supset \mathrm{B}_r(f_t x)$ for each $r > 0$, $t > 0$, and x in X.

If T is not an isomorphism, then there exists a unit vector $x \in X$ such that $||Tx|| < \frac{1}{4}$ and hence $||f_t x|| < 1/4$ for all $0 < t < t_0$. This implies that for each $0 < t < t_0$, there exists x_t in X such that $||x_t - x|| \leq 1/4$ and $f_t x_t = 0$. Hence, $f(tx_t) = 0$, and the family of points $\{tx_t\} \subset [f = 0]$ tends to zero as $t \to 0$. ■

If one composes the example in Proposition 3.11 with the projection onto the first coordinate of ℓ_p, then one obtains a Lipschitz quotient mapping– call it g–from ℓ_p onto the real line which has zero Gâteaux derivative at 0. This phenomenon of course cannot happen when the domain space is finite dimensional, since then the Gâteaux derivative is a Fréchet derivative. In particular, the restriction of g to any finite dimensional subspace of ℓ_p is not a Lipschitz quotient mapping. However, a nonlinear quotient mapping onto a separable space does have a separable "pullback":

Proposition 3.15. *Let f be a continuous, co-Lipschitz (respectively, co-uniformly continuous) mapping from the metric space X onto the separable metric space Y and let X_0 be a separable subset of X. Then there is a separable closed subset X_1 of X which contains X_0 so that the restriction of f to X_1 is co-Lipschitz (respectively,* *co-uniformly continuous), with co-Lip($f_{|X_1}$) $\le$co-Lip(f) when f is co-Lipschitz. If X is a Banach space, X_1 can be taken to be a subspace of X.*

Proof. For definiteness, assume that f is co-Lipschitz, normalized so that co-Lip$(f) = 1$, so that for each x in X and $r > 0$, $f[\mathrm{B}_r^X(x)] \supset \operatorname{int}\left(\mathrm{B}_r^Y(f(x))\right)$. Let W_0 be a countable dense subset of X_0 and build countable subsets $W_0 \subset W_1 \subset W_2 \subset \dots$ of X so that for each n, each x in W_n, and each positive rational number r,

$$\overline{f\left[\mathrm{B}_r^X(x) \cap W_{n+1}\right]} \supset \mathrm{B}_r^Y(f(x)). \tag{3.3}$$

When X is a Banach space, we can also make sure that W_n is closed under rational linear combinations for $n \ge 1$.

Note that (3.3) must hold for all positive r. Let X_1 be the closure of $\cup_{n=0}^\infty W_n$. From (3.3) we deduce that for each x in X_1 and each $r > 0$,

$$\overline{f\left[\mathrm{B}_r^{X_1}(x)\right]} \supset \mathrm{B}_r^Y(f(x)). \tag{3.4}$$

We complete the proof by observing that in (3.4) we can remove the closure if we replace the right hand side by its interior. For linear f this is sometimes called the "little open mapping theorem"; in fact, the successive approximation argument requires in addition to (3.4) only continuity of f. (If $||y - f(x_0)|| < r - \tau$, choose x_1 so that $||x_1 - x_0|| < r - \tau$ and $||y - f(x_1)|| < \frac{\tau}{2}$. Then choose x_n recursively to satisfy $||x_n - x_{n-1}|| < \frac{\tau}{2^{n-1}}$ and $||y - f(x_n)|| < \frac{\tau}{2^n}$. Clearly $\{x_n\}_{n=1}^\infty$ converges to some point, x, in $\mathrm{B}_r^{X_1}(x_0)$, and $f(x) = y$ by the continuity of f.) ■

Remark 3.16. Note that the last part of the argument for Proposition 3.15 shows that if f is a continuous mapping from the metric space X onto a metric space Y, $X_0 \subset X$, and the restriction of f to X_0 is co-Lipschitz (respectively, co-uniformly continuous) when considered as a mapping onto $f[X_0]$, then f maps the closure of X_0 onto the closure of $f[X_0]$ and $f_{|\overline{X_0}}$ is co-Lipschitz (respectively, co-uniformly continuous) when considered as a mapping onto $\overline{f[X_0]}$.

Corollary 3.17. *Suppose that f is a continuous, co-Lipschitz (respectively, co-uniformly continuous) mapping from the Banach space X onto the Banach space Y, X_0 is a separable subspace of X, and Y_0 is a separable subspace of Y. Then there exist separable closed subspaces X_1 of X, Y_1 of Y so that $X_0 \subset X_1$, $Y_0 \subset Y_1$, $f[X_1] = Y_1$, and the restriction of f to X_1 is a co-Lipschitz (respectively, co-uniformly continuous) mapping of X_1 onto Y_1.*

Proof. We prove the co-Lipschitz case; the co-uniformly continuous case is similar. Note that if Z is a subset of Y, then the restriction of f to $f^{-1}[Z]$ is co-Lipschitz when considered as a mapping onto Z, with co-Lip$(f_{|f^{-1}[Z]}) \leq$ co-Lip(f). Using this and Proposition 3.15, we build separable closed subsets $X_0 \subset W_0 \subset W_1 \subset W_2 \subset \cdots$ so that for each n, the restriction of f to W_n is a co-Lipschitz mapping onto $f[W_n]$ with co-Lip$(f_{|W_n}) \leq$ co-Lip(f), $f[W_0] \supset Y_0$, $W_{n+1} \supset \operatorname{span} W_n$, and $f[W_{n+1}] \supset \operatorname{span} Y_n$. From this it follows that the restriction of f to the separable (possibly nonclosed) subspace $\cup_{n=0}^{\infty} W_n$ is a co-Lipschitz mapping onto its image, which is a (possibly nonclosed) subspace. The desired conclusion now follows from Remark 3.16. ■

Our final result of this section, while modest, seems to be new even for bi-Lipschitz equivalences. Note that it yields that c_0 is not bi-Lipschitz equivalent to $C[0,1]$, a result first proved in [JLS].

Theorem 3.18. *If X is Asplund and Y is a Lipschitz quotient of X, then Y is Asplund.*

Proof. Recall (see e.g. Theorem 5.7 in [DGZ]) that a Banach space is Asplund if and only if every separable subspace has separable dual. Therefore, in view of Proposition 3.17, we may assume that X is separable, and need to prove that Y^* is separable. Let f be a Lipschitz quotient mapping from X onto Y with $\operatorname{Lip}(f) = 1$, set $C :=$ co-Lip(f), and assume that Y^* is nonseparable. Then there is an uncountable set $\{y_\gamma^*\}_{\gamma\in\Gamma}$ of norm one functionals in Y^* so that for each $\gamma \neq \gamma'$, $||y_\gamma^* - y_{\gamma'}^*|| > 1/2$. Set $f_\gamma = y_\gamma^* f$; then $\operatorname{Lip}(f_\gamma) \leq \operatorname{Lip}(f) = 1$. By [Pre], each f_γ has a Fréchet derivative at some point x_γ in the unit ball of X.

Let $\epsilon = (10C)^{-1}$ and for each γ choose $\delta_\gamma > 0$ so that whenever $||z|| \leq \delta_\gamma$,

$$|f_\gamma(x_\gamma + z) - f_\gamma(x_\gamma) - f'_\gamma(x_\gamma; z)| \leq \epsilon||z||. \tag{3.5}$$

By passing to a suitable uncountable subset of Γ, we may assume that $\delta := \inf_{\gamma\in\Gamma} \delta_\gamma > 0$, and also (since X and X^* are separable) that for all γ, γ'

$$||x_\gamma - x'_\gamma|| < \epsilon\delta, \tag{3.6}$$

$$||f'_\gamma(x_\gamma) - f'_{\gamma'}(x_{\gamma'})|| < \epsilon, \tag{3.7}$$

$$|f_\gamma(x_\gamma) - f_{\gamma'}(x_{\gamma'})| < \epsilon\delta. \tag{3.8}$$

If $||z|| \leq \delta$, we have:

$$\begin{aligned}
|f_\gamma(x_\gamma + z) - f_{\gamma'}(x_\gamma + z)| &= |\,\big(f_\gamma(x_\gamma + z) - f_\gamma(x_\gamma) - f'_\gamma(x_\gamma; z)\big) + (f_\gamma(x_\gamma) - f_{\gamma'}(x_{\gamma'})) \\
&\quad + \big(f'_\gamma(x_\gamma; z) - f'_{\gamma'}(x_{\gamma'}; z)\big) + \big(-f_{\gamma'}(x_{\gamma'} + z) + f_{\gamma'}(x_{\gamma'}) + f'_{\gamma'}(x_{\gamma'}; z)\big) \\
&\quad + (f_{\gamma'}(x_{\gamma'} + z) - f_{\gamma'}(x_\gamma + z))\,| \\
&\leq \epsilon||z|| + \epsilon\delta + ||f'_\gamma(x_\gamma) - f'_{\gamma'}(x_{\gamma'})||||z|| + \epsilon||z| + ||x_\gamma - x_{\gamma'}|| \quad \text{by (3.5), (3.8), (3.5)} \\
&< 5\epsilon\delta = \delta/(2C). \quad \text{by (3.7), (3.6)}
\end{aligned}$$

On the other hand,

$$\begin{aligned}\sup_{z\in \mathrm{B}_\delta^X(0)} |f_\gamma(x_\gamma+z)-f_{\gamma'}(x_\gamma+z)| &= \sup_{z\in \mathrm{B}_\delta^X(0)} \left| \left(y_\gamma^*-y_{\gamma'}^*\right) f(x_\gamma+z)\right| \\ &\geq \sup_{y\in \mathrm{B}_{\delta/C}^Y(0)} \left| \left(y_\gamma^*-y_{\gamma'}^*\right) (f(x_\gamma)+y)\right| \\ &\geq ||y_\gamma^*-y_{\gamma'}^*||\,(\delta/C) > \delta/(2C).\end{aligned}$$

■

Remark 3.19. The analogue of Theorem 3.18 for uniform quotient mappings is false. Ribe [Rib2] gave an example of a separable, reflexive space which is bi-uniformly homeomorphic to a space which contains an isomorphic copy of ℓ_1.

4. Examples related to the Gorelik Principle

One natural way of constructing a uniform or Lipschitz quotient mapping is to follow a bi-uniformly continuous or bi-Lipschitz mapping with a linear quotient mapping. By the Gorelik principle [Gor], [JLS], the resulting mapping cannot map a "large" ball in a finite codimensional subspace of the domain into a "small" ball (Theorem 1.1 in [JLS] gives a precise quantitative meaning to this statement). It is natural to guess that any uniform quotient mapping satisfies this version of the Gorelik principle. If this were true, this would provide the machinery to prove, for example, that every uniform quotient of ℓ_p, $1<p<\infty$, is isomorphic to a linear quotient of ℓ_p. Unfortunately, Proposition 4.1 shows that there is not a Gorelik principle for uniform quotient mappings.

Proposition 4.1. *Let X be a Banach space and set $Z = X\oplus_1 X\oplus_1 \mathbb{R}$. Then there exists a Lipschitz, co-uniform mapping T from Z onto X so that $T\left(\mathrm{B}_1^{X\oplus X}(0)\right) = \{0\}$.*

Proof. We shall define $T(x,y,\lambda) = ag(x,\lambda)+f(y,\lambda)$ for appropriate Lipschitz functions f and g from $X\oplus \mathbb{R}$ into X and for an appropriately small a; namely, for $a=\frac{1}{2}$. For each integer k let $\{u_{k,j}\}_{j=1}^\infty$ be a maximal $4\cdot 2^k$-separated set in X, and define for each y in X

$$f(y,2^k) = \sum_{j=1}^\infty \left(2^k - \left|\left\|y-u_{k,j}\right\| - 2^k\right|\right)^+ \frac{y-u_{k,j}}{\left\|y-u_{k,j}\right\|}$$

(where $\frac{0}{0}\equiv 0$.) Write $f_k(y) = f(y,2^k)$; the formula for f just says that f_k translates the ball in X of radius 2^k around each $u_{k,j}$ to a ball around the origin, that f_k vanishes on the complement of $\cup_{j=1}^\infty \mathrm{B}_{2^{k+1}}(u_{k,j})$, and that f_k is affine on the intersection of $\mathrm{B}_{2^{k+1}}(u_{k,j}) \sim \mathrm{B}_{2^k}(u_{k,j})$ with any ray emanating from $u_{k,j}$. Extend

f to a function from $X \oplus I\!R$ into X by making f affine on each interval of the form $[(y, 2^k), (y, 2^{k+1})]$, by setting $f(y, 0) = 0$, and by defining $f(y, \lambda) = f(y, -\lambda)$ when $\lambda < 0$. It is apparent that each f_k has Lipschitz constant one, that f has Lipschitz constant at most two, and that $||f(y, \lambda)|| \le |\lambda|$.

Define g by $g(x, \lambda) = ([|\lambda| \vee (||x|| - 1)] \wedge 1)\, x$. It is easy to see that g has Lipschitz constant one and hence T has Lipschitz constant at most three (as long as $a \le 1$). Evidently T vanishes on the unit ball of the hyperplane $[\lambda = 0]$, so it remains to check that T satisfies the uniform quotient property.

Let (x_0, y_0, λ_0) be in Z and $r > 0$. We need to check that $T\mathrm{B}_r(x_0, y_0, \lambda_0)$ contains a ball around $T(x_0, y_0, \lambda_0)$ whose radius can be estimated from below in terms of r only. We can assume, without loss of generality, that $r < 1$.

Case 1: $|\lambda_0| < \frac{r}{48}$.

This is the only case in which the mapping f comes into play and also the only case in which we need to consider points off the hyperplane $[\lambda = \lambda_0]$ in order to verify the quotient property. We show in this case that $T\mathrm{B}_r(x_0, y_0, \lambda_0)$ contains a ball around $T(x_0, y_0, \lambda_0)$ of radius $\frac{r}{32}$. By symmetry we can assume that $\lambda_0 \ge 0$.

Define k by $2^k \le \frac{r}{6} < 2^{k+1}$ (so $0 < 2^k - \lambda_0 \le 2^k$) and choose j to satisfy $||y_0 - u_{k,j}|| < 2^{k+2}$. Then $f_k \mathrm{B}_{2^k}(u_{k,j}) = \mathrm{B}_{2^k}(0)$ and $\{x_0\} \oplus \mathrm{B}^X_{2^k}(u_{k,j}) \oplus \{2^k\} \subset \mathrm{B}_{6\cdot 2^k}(x_0, y_0, \lambda_0) \subset \mathrm{B}_r(x_0, y_0, \lambda_0)$. Therefore

$$\begin{aligned}\mathrm{B}_r(x_0, y_0, \lambda_0) &\supset ag(x_0, 2^k) + f_k \mathrm{B}_{2^k}(u_{k,j}) \supset ag(x_0, \lambda_0) + \mathrm{B}^X_{2^k - a(2^k - \lambda_0)}(0) \\ &\supset ag(x_0, \lambda_0) + f(y_0, \lambda_0) + \mathrm{B}^X_{2^{k-1} - (1-a)\lambda_0}(0) \supset \mathrm{B}^X_{\frac{r}{32}}\left(T(x_0, y_0, \lambda_0)\right).\end{aligned}$$

Case 2: $|\lambda_0| \ge \frac{r}{48}$.

Again we assume by symmetry that $\lambda_0 \ge 0$. Fix (x_0, y_0, λ_0) in Z and set $s_0 = [\lambda_0 \vee (||x_0|| - 1)] \wedge 1$ so that $g(x_0, \lambda_0) = s_0 x_0$ and $r \le 48(\lambda_0 \wedge 1) \le 48 s_0$. Fix z in X with $||z|| \le \frac{r}{5}$. We want to find a vector x in X with $||x - x_0|| \le r$ and $g(x, \lambda_0) = s_0(x_0 + z)$ $(= g(x_0, \lambda_0) + s_0 z)$. This will give

$$\begin{aligned}T\mathrm{B}_r(x_0, y_0, \lambda_0) &\supset f(y_0, \lambda_0) + ag\left[T\mathrm{B}^X_r(x_0) \oplus \{\lambda_0\}\right] \\ &\supset f(y_0, \lambda_0) + ag(x_0, \lambda_0) + as_0 \mathrm{B}^X_{\frac{r}{5}}(0) \supset \mathrm{B}^X_{\frac{ars_0}{5}}\left(T(x_0, y_0, \lambda_0)\right).\end{aligned}$$

Since $\frac{ars_0}{5} \ge \frac{ar^2}{240}$, this will verify that T has the uniform quotient property.

The vector x will be of the form $x = t(x_0 + z)$ for appropriate t close to one. Write $s(t) = [\lambda_0 \vee (t||x_0 + z|| - 1)] \wedge 1$, so that $g(t(x_0 + z), \lambda_0) = s(t)t(x_0 + z)$. We need to find t_0 so that $t_0 s(t_0) = s_0$ and $||x_0 - t_0(x_0 + z)|| \le r$. If $||x_0|| \ge 3$, the choice $t_0 = 1$ works, since then $||x_0 + z|| - 1 \ge 1$ and thus $s(1) = s_0$. so assume that $||x_0|| \leq 3$. In this case, by continuity of $s(\cdot)$, it is enough to check that $(1-\frac{r}{4})s(1-\frac{r}{4}) \leq s_0 \leq (1+\frac{r}{4})s(1+\frac{r}{4})$; actually, we check the stronger inequalities:

$$\text{(L)} \qquad s(1-\frac{r}{4}) \leq s_0$$

$$\text{(R)} \qquad s_0 \leq s(1+\frac{r}{4}).$$

We first check (R). If $s_0 = \lambda_0 \wedge 1$, then $s(t) \geq s_0$ for all $t \geq 0$ and so (R) is clear. Otherwise $s_0 = ||x_0|| - 1$ and we get

$$\begin{aligned}(1+\frac{r}{4})||x_0+z|| - 1 \geq (1+\frac{r}{4})(1+s_0) - (1+\frac{r}{4})\frac{r}{5} - 1 \\ \geq (1+\frac{r}{4})s_0 + r\left(\frac{1}{4} - \frac{4+r}{20}\right) \geq (1+\frac{r}{4})s_0.\end{aligned}$$

To check (L), suppose first that $s_0 \geq ||x_0|| - 1$. Then

$$(1-\frac{r}{4})||x_0+z|| - 1 \leq (1-\frac{r}{4})(1+s_0) + (1-\frac{r}{4})\frac{r}{5} - 1 \leq (1-\frac{r}{4})s_0,$$

and (L) follows. On the other hand, if $||x_0|| - 1 > s_0$, then $||x_0|| > 1$, and hence

$$(1-\frac{r}{4})||x_0+z|| - 1 \leq ||x_0|| - 1 - \frac{r}{4} + (1-\frac{r}{4})\frac{r}{5} < ||x_0|| - 1,$$

which also yields (L). ■

Of course, if X is isomorphic to its square and also to its hyperplanes (for example, if X is ℓ_p or L_p, $1 \leq p \leq \infty$), then Proposition 4.1 says that there is a uniform quotient mapping from X onto itself which maps the unit ball of a hyperplane to zero.

The next proposition shows that even for Lipschitz quotients there is no Gorelik principle which holds for general spaces.

Proposition 4.2. *For any separable Y there is a Lipschitz quotient T of ℓ_1 onto Y which maps a hyperplane to zero.*

Proof. Define $\varphi : \ell_1^2 \to R$ by

$$\varphi(s,t) = \begin{cases} s-t, & \text{if } s \geq t \geq 0; \\ 0, & \text{if } t \geq s \geq 0; \\ s & \text{in all remaining points} \end{cases}$$

and define $\psi : \ell_1^2 \to R$ by

$$\psi(s,t) = \begin{cases} t-s, & \text{if } 2s \geq t \geq s \geq 0; \\ s, & \text{if } t \geq 2s \geq 0; \\ 0, & \text{in all remaining points.} \end{cases}$$

Then it is easy to see that $\mathrm{Lip}\,(\varphi) \leq 1$, $\mathrm{Lip}\,(\psi) \leq 1$. Moreover,

$$|\varphi(s,t) - \varphi(s',t')| + |\psi(s,t) - \psi(s',t')| \leq |s-s'| + |t-t'|. \tag{4.1}$$

To prove (4.1) note that, if (s,t) and (s',t') are both in $\{(s,t)\ ;\ t \geq s \geq 0\}$ or both in the closure of the complement of this set, at most one of $\varphi(s,t) - \varphi(s',t')$ and $\psi(s,t) - \psi(s',t')$ is non zero.

Choose $\{(x^n, y^n)\}_{n=0}^{\infty}$ dense in the product of ℓ_1 with the unit sphere of Y such that, for $n = 1, 2, \ldots,$ x^n is in ℓ_1^n - the span of the first n basis vectors in ℓ_1. Define $T_n : \ell_1^n \to Y$ by

$$T_1(x) = xy^0,$$

and inductively,

$$T_{n+1}(x_1, \ldots, x_{n+1}) = T_n(x_1^n + \varphi(x_1 - x_1^n, x_{n+1}), x_2, \ldots, x_n) + y^n \psi(x_1 - x_1^n, x_{n+1})$$

if $x_1^n \geq 0$, and

$$T_{n+1}(x_1, \ldots, x_{n+1}) = T_n(x_1^n - \varphi(-x_1 + x_1^n, x_{n+1}), x_2, \ldots, x_n) + y^n \psi(-x_1 + x_1^n, x_{n+1})$$

if $x_1^n < 0$. Then $T_{n+1} = T_n$ on ℓ_1^n and it follows from (4.1) that $\mathrm{Lip}\,(T_{n+1}) \leq 1$.

Define $T : \bigcup_n \ell_1^n \to Y$ by $T(x) = T_n(x)$ for $x \in \ell_1^n$ and extend it by continuity to ℓ_1. T is then Lipschitz with constant 1 and it is easy to see that T maps the hyperplane consisting of all vectors whose first coordinate is zero to zero. To prove that T is a co-Lipschitz mapping, let $x \in \ell_1$, $v \in Y$ with $T(x) = v$, let $r > 0$ and let $v + y \in \mathrm{B}_r(v)$ with $y \neq 0$. Fix an $\epsilon > 0$ with $\epsilon < r$ and let n be such that $\|x - x^n\| + \|y\| \|\frac{y}{\|y\|} - y^n\| < \epsilon$. Put $z = (x_1^n + \mathrm{sgn}(x_1^n)\|y\|, x_2^n, \ldots, x_n^n, 2\|y\|, 0, 0, \ldots)$ (here sgn(0)=1), then

$$\|z - x\| < 3\|y\| + \epsilon < 4r$$

and

$$T(z) = T_{n+1}(z) = T(x^n) + y^n \|y\|.$$

Thus

$$\|T(z) - v - y\| \leq \|x - x^n\| + \|y - y^n \|y\|\| < \epsilon$$

and we get that $T\mathrm{B}_{4r}(x)$ contains a dense subset of $\mathrm{B}_r(v)$. As in the proof of Proposition 3.15, a simple successive approximation argument shows now that for all $r > 0$, $T\mathrm{B}_{4r}(x)$ contains $\mathrm{B}_r(v)$. ∎

Proposition 4.1 implies that for each n there is a uniform quotient mapping from $\mathbb{R}^{2n+1}$ onto $\mathbb{R}^n$ which maps the unit ball of $\mathbb{R}^{2n}$ to zero. However, in the finite dimensional case, more can be said:

Proposition 4.3. *There is a Lipschitz map T from $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$ onto $\mathbb{R}^2$ such that T is a co-uniform quotient map and $T\mathrm{B}_1^{\mathbb{R}^2\oplus\{0\}}(0) = 0$.*

Proof. . For $\theta \in \mathbb{R}$ let

$$U_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For $0 \leq a \leq 1$ let $r_a : \mathbb{R}^+ \to [0,1]$ be defined by

$$r_a(t) = \begin{cases} a, & \text{if } 0 \leq t \leq 1; \\ 1-(1-a)(2-t), & \text{if } 1 < t < 2; \\ 1, & \text{if } 2 < t. \end{cases}$$

We also let $r_a(t) = 1$ if $a > 1$ and $r_a(t) = r_{|a|}(t)$, and define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x,a) = r_a^2(\|x\|)U_{\theta(r_a(\|x\|))}x,$$

where $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$\theta(t) = 2\pi/t.$$

We check first that T is Lipschitz. It is clearly enough to show that its restriction to the set $\{(x,a) : \|x\| \leq 2\}$ is Lipschitz, which follows immediately by noting that it is the composition of

$$(x,a) \in \{(x,a) : \|x\| \leq 2\} \to (x, r_a(\|x\|)),$$

which is clearly Lipschitz, followed by

$$(x,t) \in \{(x,t) : \|x\| \leq 2, 0 \leq t \leq 1\} \to t^2 U_{\theta(t)}x,$$

which has bounded partial derivatives.

It remains to show that T is co-uniform. We note first that for each $a > 0$ the function

$$f_a(x) = T(x,a)$$

is a Lipschitz homeomorphism of the plane, with inverse given by

$$f_a^{-1}(y) = (s_a(\|y\|)/\|y\|)U_{-\theta(r_a(s_a(\|y\|)))}y,$$

where s_a is inverse of $t \to tr_a^2(t)$. The function $tr_a^2(t)$ has a positive derivative bounded below by $a^2 \wedge 1$; moreover, if $0 < \tau < 1$, this derivative is bounded below by τ^2 on the interval $t > 1+\tau$. Thus, if either $a \geq \tau > 0$ or $\|x\| \geq 1+\tau$, the mapping f_a^{-1} has its derivative at $f_a(x)$ bounded in norm by a constant κ_τ depending only on τ. Consequently,

$$f_a B_\tau(x) \supset f_a B_{\tau/\kappa_\tau}(f_a(x)),$$

if either $a \geq \tau$ or $\|x\| \geq 1+\tau$.

To check co-uniformity of T, take any $(x, a) \in \mathbb{R}^3$ with $a > 0$ and let $0 < r < 1$.

Case 1: $r \leq 10a$ or $\|x\| > 1 + r/10$. In this case the above inclusions show that

$$TB_r(x, a) \supset f_a B_r(x) \supset B_{r\kappa_{r/10}}(f_a(x)) = B_{r\kappa_{r/10}}(T(x, a)).$$

Case 2: $r > 10a$ and $\|x\| \leq 1$. Note first that by the definition of f_a,

$$\{f_\gamma(\beta^2 x/\gamma^2) : 1/(k+1) < \gamma \leq 1/k\} = \{y : \|y\| = \beta^2 \|x\|\},$$

whenever k is an integer and $\beta \leq 1/(k+1)$. Let now k_0 be the integer so that $1/(k_0+1) < r \leq 1/k_0$. Let $k \geq 3k_0$, and let $1/(k+2) < \beta \leq 1/(k+1)$ and $1/(k+1) < \gamma \leq 1/k$. We have

$$\begin{aligned}\left\|(x, a) - (\beta^2 x/\gamma^2, \gamma)\right\|^2 &\leq (1 - \beta^2/\gamma^2)^2 + (a-\gamma)^2 \leq (1 - (k/(k+2))^2)^2 + 1/(3k_0)^2 \\ &\leq 4/(3k_0+2)^2 + 1/(3k_0)^2 \leq r^2.\end{aligned}$$

Hence $TB_r(x, a)$ contains all the points $f_\gamma(\beta^2 x/\gamma^2)$ with β and γ as above. Consequently,

$$TB_r(x, a) \supset B_{\|x\|/(3k_0+1)^2}(0) \supset B_{\|x\| r^2/16}(0).$$

Since $\|T(x, a)\| = a^2 \|x\| \leq \|x\| r^2/100$, we conclude that

$$TB_r(x, a) \supset B_{\|x\| r^2/50}(T(x, a)).$$

If $\|x\| \geq r/2$, this means that

$$TB_r(x, a) \supset B_{r^3/200}(T(x, a)),$$

while if $\|x\| < r/2$, then

$$TB_r(x, a) \supset TB_{r/2}(rx/(2\|x\|), a) \supset B_{(r/2)(r/2)^2/50}(T(x, a)) \supset B_{r^3/400}(T(x, a)).$$

Case 3: $r > 10a$ and $1 < \|x\| \leq 1 + r/10$. Put $u = x/\|x\|$. Then $B_r(x, a) \supset B_{9r/10}(u, a)$ and by the proof of case 2,

$$TB_{9r/10}(u, a) \supset B_{(9r/10)^2/16}(0) \supset B_{r^2/20}(0).$$

Also $r_a(\|x\|) = 1 - (1-a)(2 - \|x\|) \le a + \|x\| - 1 \le r/5$. Thus $\|f_a(x)\| \le 11r^2/250$, and therefore

$$TB_r(x,a) \supset B_{r^2/400}(T(x,a)).$$

■

Proposition 4.2 leaves open the possibility that there is a Gorelik principle for Lipschitz quotient mappings between finite dimensional spaces (or even superreflexive spaces, say). We do not know, for example, if there is a Lipschitz quotient mapping from $\mathbb{R}^n$, $n > 2$, onto $\mathbb{R}^2$ which vanishes on a hyperplane. Notice that if $f : X \to Y$ is a Lipschitz quotient mapping which sends the unit ball of some subspace Z of X to zero, and $\mathcal{U}$ is a free ultrafilter on the positive integers, then there is a Lipschitz quotient mapping $f_{\mathcal{U}}$ from the ultrapower $X_{\mathcal{U}}$ of X onto the ultrapower $Y_{\mathcal{U}}$ of Y which sends the entire *subspace* $Z_{\mathcal{U}}$ of $X_{\mathcal{U}}$ to zero. Indeed, define $f_n : X \to Y$ by $f_n(x) = nf\left(\frac{x}{n}\right)$ and let $f_{\mathcal{U}}$ be the ultraproduct of the f_n's, defined by $f_{\mathcal{U}}(x_1, x_2, \dots) = (f_1(x_1), f_2(x_2), \dots)$. Now if X is finite dimensional, then so is Y, and $X = X_{\mathcal{U}}$, $Y = Y_{\mathcal{U}}$, and $Z = Z_{\mathcal{U}}$, so one obtains a Lipschitz quotient mapping from X onto Y which maps the subspace Z to zero. (Incidentally, if f is a uniform quotient mapping from a space X onto a space Y which maps a subspace Z of X to zero, then the construction of Proposition 3.4 produces a *Lipschitz* quotient mapping from $X_{\mathcal{U}}$ onto $Y_{\mathcal{U}}$ which maps $Z_{\mathcal{U}}$ to zero.)

In the case of mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$, there is a close connection between Lipschitz quotient mappings and quasiregular mappings ([Ric] is the standard source for quasiregular mappings). Recall that a map f from a domain G in $\mathbb{R}^n$ to $\mathbb{R}^n$ is called quasiregular provided

(i) f is ACL^n, i.e., f is continuous, its restriction to every line in the direction of each of the coordinate axes is absolutely continuous and its partial derivatives belong locally to the space $L_n(\mathbb{R}^n)$.

(ii) The n by n matrix $D(x)$ of partial derivatives of f satisfies $\|D(x)\|^n \le KJ(x)$ for almost every x, where K is a constant, $J(x)$ is the determinant of $D(x)$, and $\|D(x)\|$ is its norm as an operator from ℓ_2^n to itself.

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz quotient mapping then (i) holds trivially. As for (2), we have the somewhat weaker statement

(ii') For every x at which f is differentiable

$$\|D(x)\|^n \le K|J(x)|$$

where $K = (M/m)^{n-1}$ with M (respectively, m) the Lipschitz (respectively, co-Lipschitz) constant of f.

It seems likely that some results from the quasiregular theory can be carried over to the case of Lipschitz quotient mappings. A deep result of Reshetnyak (which is nicely presented in [Ric]) states that the level sets of a nonconstant quasiregular mapping from a domain G in $\mathbb{R}^n$ into $\mathbb{R}^n$ are discrete sets. In Proposition 4.4 we prove that the level sets of a co-Lipschitz continuous mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ are discrete. Note that this result implies in particular that a Lipschitz quotient mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ cannot vanish on an interval, which means that some version of the Gorelik principle is true for such mappings.

Proposition 4.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous and co-Lipschitz mapping. Then for every $y \in \mathbb{R}^2$ the set $f^{-1}(y)$ is discrete.*

Proof. We use the following simple lemma concerning the lifting of Lipschitz curves:

Lemma 4.5. *Suppose that $f : \mathbb{R}^n \to X$ is continuous and co-Lipschitz with constant one, $f(x) = y$, and $\xi : [0, \infty) \to X$ is a curve with Lipschitz constant one, and $\xi(0) = y$. Then there is a curve $\phi : [0, \infty) \to \mathbb{R}^n$ with Lipschitz constant one such that $\phi(0) = x$ and $f(\phi(t)) = \xi(t)$ for $t \geq 0$.*

Proof. For $m = 1, 2, \dots$ define $\phi_m(0) = x$, and, by induction, assuming that $f(\phi_m(\frac{k}{m})) = \xi(\frac{k}{m})$, choose $\phi_m(\frac{k+1}{m})$ such that $\|\phi_m(\frac{k+1}{m}) - \phi_m(\frac{k}{m})\| \leq \frac{1}{m}$ and $f(\phi_m(\frac{k+1}{m})) = \xi(\frac{k+1}{m})$. Extend $\phi_m(t)$ to a Lipschitz curve $\phi_m : [0, \infty) \to \mathbb{R}^m$ having Lipschitz constant one. The limit ϕ of any convergent subsequence of ϕ_m has the desired properties. ■

Without loss of generality, assume $\mathrm{B}_r(f(x)) \subset f(\mathrm{B}_r(x))$ for every x in $\mathbb{R}^2$ and $r > 0$, $y = 0$, and $f(0) = 0$. Let $u_k = e^{k\pi i/3}$ and $S = \{tu_k : t \geq 0, k = 0, 2, 4\}$. Let also $0 < \delta < 1$ be such that $\|x\|, \|y\| \leq 2$ and $\|x - y\| < \delta$ imply that $\|f(x) - f(y)\| < 1/2$.

For each $x \in \mathrm{B}_1(0) \cap f^{-1}(0)$ and $k = 1, 3, 5$, use Lemma 4.5 to choose $\phi_{k,x} : [0, \infty) \to \mathbb{R}^2$ having Lipschitz constant one such that $\phi_{k,x}(0) = x$ and $f(\phi_{k,x}(t)) = tu_k$ for $t \geq 0$. Let $D_{k,x}$ be the component of $\mathbb{R}^2 \setminus f^{-1}(S)$ containing $\phi_{k,x}(0, \infty)$. Noting that $\mathrm{B}_\delta(\phi_{k,x}(1)) \subset D_{k,x} \cap \mathrm{B}_2(0)$, a comparison of areas shows that the set of all such $D_{k,x}$ has at most $N \leq 4\delta^{-2}$ elements. Suppose now that $\mathrm{B}_1(0) \cap f^{-1}(0)$ has infinitely many elements, hence it contains elements $x \neq y$ such that $\{D_{1,x}, D_{3,x}, D_{5,x}\} = \{D_{1,y}, D_{3,y}, D_{5,y}\}$. Then $D_{k,x} = D_{k,y}$ for $k = 1, 3, 5$, since the (connected) image of $D_k := D_{k,x}$ contains u_k and so can contain no other u_j, and we infer that there are simple curves $\psi_k : [0, 1] \to \mathbb{R}^2$ such that $\psi_k(0) = x$, $\psi_k(1) = y$ and $\psi_k(t) \in D_k$ for $0 < t < 1$. For each pair $k, l = 1, 3, 5$ of different indices, let $G_{k,l}$ be the interior of the Jordan curve $(\psi_k - \psi_l)$ (difference in the sense of oriented curves). If $j \neq k, l$, we note that $G_{k,l} \cap D_j = \emptyset$ since otherwise D_j would be bounded. In particular, $G_{1,3} \cap \partial G_{3,5} = \emptyset$, so either $G_{1,3} \subset G_{3,5}$ or $G_{1,3} \cap G_{3,5} = \emptyset$. In the former case we would get a contradiction from $\psi_1(0, 1) \subset G_{3,5}$, since $\psi_1(0, 1) \subset D_1$ and in the latter case we would infer from $\partial(G_{1,5}) = \partial(G_{1,3} \cup G_{3,5})$ that $G_{1,5} \supset G_{1,3}$ and get a contradiction from $\psi_3(0, 1) \subset G_{1,5}$. ■

References

[AL] I. Aharoni and J. Lindenstrauss, *Uniform equivalence between Banach spaces*, Bull. Amer. Math. Soc. **84** (1978), 281–283.

[AMM] I. Aharoni, B. Maurey, and B. S. Mityagin, *Uniform embeddings of metric spaces and of Banach spaces into Hilbert spaces*, Israel J. Math. **52, no. 3** (1985), 251–265.

[Bat] S. Bates, *On smooth nonlinear surjections of Banach spaces*, Israel J. Math. **100** (1997), 209–220.

[Ben] Y. Benyamini, *The uniform classification of Banach spaces*, Longhorn Notes, Texas Functional Analysis Seminar, 1984-85 Univ. of Texas, 15–39 (An electronic version can be obtained: ftp.math.okstate.edu/pub/banach
or http://www.math.okstate.edu/~alspach/banach/index.html).

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, Colloquium Publications AMS, 1999 (to appear).

[BR] J. Bourgain and H. P. Rosenthal, *Martingales valued in certain subspaces of L^1*, Israel J. Math. **37** (1980), 54–75.

[Bou] R. D. Bourgin, *Geometric aspects of convex sets with the Radon- Nikodým property*, Lecture Notes in Math. 993, Springer, 1983.

[CO] P. G. Casazza and E. Odell, *Tsirelson's space and minimal subspaces*, Texas functional analysis seminar 1982–1983 (Austin, Tex.) 61–72, Longhorn Notes, Univ. Texas Press, Austin, TX, 1983.

[CS] P. G. Casazza and T. J. Shura, *Tsirelson's space*, Lecture Notes in Math. 1363, Springer, 1989.

[DGZ] R. Deville, G. Godefroy, and V. E. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 64, Longman Scientific, New York, 1993.

[DU] J. Diestel and J. J. Uhl, Jr., *Vector Mesures*, Math. Surveys 15, Amer. Math. Soc. (1977).

[FT] T. Figiel and N. Tomczak-Jaegermann, *Projections onto Hilbertian subspaces of Banach spaces*, Israel J. Math. **33** (1979), 155–179.

[Gor] E. Gorelik, *The Uniform nonequivalence of L_p and ℓ_p*, Israel J. Math. **87** (1994), 1–8.

[Gro] M. Gromov,, *Metric structures for Riemannian manifolds*, Progress in Math., vol. 152, Birkhauser, Boston, 1998.

[HM] S. Heinrich and P. Mankiewicz, *Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces*, Studia Math. **73** (1982), 225–251.

[Jam] I. M. James, *Introduction to uniform spaces*, London Math. Soc. Lecture Note Series **144** (1990).

[JS] R. C. James and J. J. Schäffer, *Super-reflexivity and the girth of spheres*, Israel J. Math. **11** (1972), 398–404.

[Joh] W. B. Johnson, *A reflexive Banach space which is not sufficiently Euclidean*, Studia Math. **55** (1976), 201–205.

[JLPS1] W. B. Johnson, J. Lindenstrauss, D. Preiss, and G. Schechtman, *Uniform quotients of the plane*, preprint.

[JLPS2] W. B. Johnson, J. Lindenstrauss, D. Preiss, and G. Schechtman, *Almost Fréchet differentiability of Lipschitz mappings between infinite dimensional Banach spaces*, in preparation.

[JLS] W. B. Johnson, J. Lindenstrauss, and G. Schechtman, *Banach spaces determined by their uniform structures*, Geom. Funct. Anal. **6** (1996), 430–470.

[KR] K. Kunen and H. P. Rosenthal, *Martingale proofs of some geometrical results in Banach space theory*, Pacific J. Math. **100** (1982), 153–175.

[LW] E. B. Leach and J. H. M. Whitfield, *Differentiable functions and rough norms on Banach spaces*, Proc. Amer. Math. Soc. **33** (1972), 120–126.

[LPe] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$ spaces and their applications*, Studia Math. **29** (1968), 275–326.

[LP] J. Lindenstrauss and D. Preiss, *Almost Fréchet differentiability of finitely many Lipschitz functions*, Mathematika **43** (1996), 393–412.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I, Sequence spaces*, Springer-Verlag, 1977.

[Pis] G. Pisier, *Holomorphic semi-groups and the geometry of Banach spaces*, Ann. of Math. **115** (1982), 375–392.

[Pre] D. Preiss, *Differentiability of Lipschitz functions on Banach spaces*, J. Functional Anal. **91** (1990), 312–345.

[Qia] S. Qian, *Nowhere differentiable Lipschitz maps and the Radon-Nikodym property*, J. Math. Anal. and Appl. **185** (1994), 613–616.

[Rib1] M. Ribe, *On uniformly homeomorphic normed spaces*, Ark. Math. **14** (1976), 237–244.

[Rib2] M. Ribe, *Existence of separable uniformly homeomorphic non isomorphic Banach spaces*, Israel J. Math. **48** (1984), 139–147.

[Ric] S. Rickman, *Quasiregular mappings*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 26, Springer-Verlag, Berlin, 1993.

[Why] G. T. Whyburn, *Topological Analysis*, Princeton University Press, Princeton, N. J., 1958.

S. Bates, Department of Mathematics, Columbia University, New York, NY 10027 U.S.A.,
email: smb@math.columbia.edu

W. B. Johnson, Department of Mathematics, Texas A&M University, College Station, TX 77843–3368 U.S.A.
email: johnson@math.tamu.edu

J. Lindenstrauss, Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel
email: joram@math.huji.ac.il

D. Preiss, Department of Mathematics, University College London, London, Great Britain,
email: dp@math.ucl.ac.uk

G. Schechtman, Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot, Israel,
email: gideon@wisdom.weizmann.ac.il