Subject §5.1 Orthogonality

Consider $\mathbb{R}^n$. Let $x=(x_1,\cdots,x_n)^T$, $y=(y_1,\cdots,y_n)\in\mathbb{R}^n$.

* The scalar product of x and y is defined by

$$x^Ty=[x_1,\cdots,x_n]\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix}=x_1y_1+\cdots+x_ny_n=\sum_{i=1}^n x_iy_i. \text{ — scalar.}$$

* The length of $x\in\mathbb{R}^n$ is

$$\|x\|=(x^Tx)^{1/2}=(x_1^2+\cdots+x_n^2)^{1/2}.$$

* The distance from x to y is

$$\|x-y\|=((x_1-y_1)^2+\cdots+(x_n-y_n)^2)^{1/2}.$$

Ex: $x=(1,-2,3)^T$, $y=(4,2,3)$.

$x^Ty=(1,-2,3)\begin{bmatrix}4\\2\\3\end{bmatrix}=4-4+9=9.$

$\|x\|=(1^2+(-2)^2+3^2)^{1/2}=\sqrt{14}.$

$\|x-y\|=((1-4)^2+(-2-2)^2+(3-3)^2)^{1/2}=5$

THM. If x and y are two vectors in $\mathbb{R}^n$, and θ is the angle between x and y, then (by the law of cosine)

$$x^Ty=\|x\|\|y\|\cos\theta.$$

Proof: By the law of cosine, we have

$$\|y-x\|^2=\|x\|^2+\|y\|^2-2\|x\|\|y\|\cos\theta$$

$$\| \qquad\qquad$$

$$\sum_{i=1}^n(x_i-y_i)^2=\sum_{i=1}^n(x_i^2+y_i^2-2x_iy_i)=\|x\|^2+\|y\|^2-2x^Ty.$$

$$\Rightarrow \qquad x^Ty=\|x\|\|y\|\cos\theta.$$

For a nonzero vector $x \in \mathbb{R}^n$, the unit vector

$$u = \frac{x}{\|x\|}$$

gives the direction of x.

Cauchy – Schwarz Inequality:

$$|x^T y| \leq \|x\| \|y\|.$$

"=" holds if and only if one is a multiple of the other one.

Proof: If $x = 0$, "=" holds. If $x \neq 0$ and $y \neq 0$, then

$$|x^T y| = \|x\| \|y\| |\cos\theta| \leq \|x\| \|y\|.$$

"=" holds if $\cos\theta = 1$, i.e., $\theta = \begin{cases} 0 \\ \pi \end{cases} \Leftrightarrow x = cy.$ ※

The angle θ between two vectors x and y is

$$\cos\theta = \frac{x^T y}{\|x\| \|y\|} \quad \text{or} \quad \theta = \arccos\left(\frac{x^T y}{\|x\| \|y\|}\right).$$

Note
$$-1 \leq \frac{x^T y}{\|x\| \|y\|} \leq 1.$$

Def: Two vectors x and y in $\mathbb{R}^n$ are orthogonal if

$$x^T y = 0. \qquad (x \perp y).$$

Remark: 1) $x^T y = 0 \begin{cases} \text{either } x \text{ or } y \text{ is zero} \\ \text{or } \cos\theta = 0 \Leftrightarrow \theta = \pm\frac{\pi}{2} \Leftrightarrow x \perp y. \end{cases}$

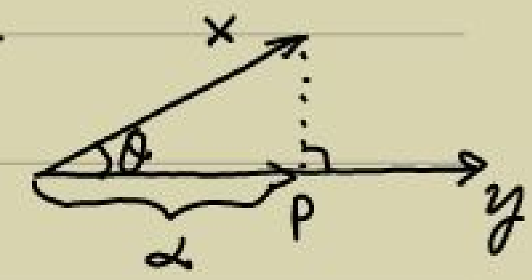
2) 0 is orthogonal to any vector.

3) $x^T x = 0 \Leftrightarrow x = 0.$

Ex: $x=(2,3)$, $y=(-6,4)$. $x^T y = -12+12=0 \Rightarrow x \perp y$.

$x=(1,-2,3)$, $y=(1,-1,-1)$. $x^T y = 1+2-3=0 \Rightarrow x \perp y$.

Scalar and vector Projections.



Given x and y

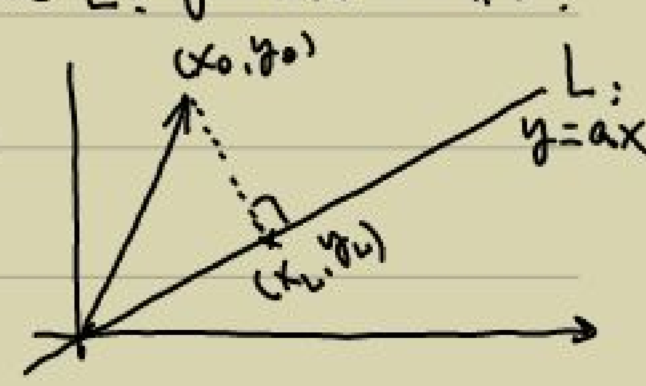
p = vector projection of x onto y.

α = length of p = scalar projection of x onto y.

$$\alpha = \|x\| \cos\theta = \frac{\|x\|\,\|y\| \cos\theta}{\|y\|} = x^T \frac{y}{\|y\|}.$$

$$p = \alpha \frac{y}{\|y\|} = x^T \frac{y}{\|y\|} \frac{y}{\|y\|} = \left(\frac{x^T y}{\|y\|^2} \right) y.$$

Ex: Given a point $(x_0, y_0) \in \mathbb{R}^2$ and a line $L: y=ax$ in $\mathbb{R}^2$.



Find the point (x_L, y_L) on L which gives the shortest distance from (x_0, y_0) to L.

The direction of L is $u = \frac{(1,a)}{\sqrt{1+a^2}} \left(= \frac{(x,ax)}{\sqrt{x^2+a^2x^2}} \right)$.

(x_L, y_L) = vector projection of $\overrightarrow{(x_0,y_0)}$ onto L

$$= (x_0, y_0)^T \frac{(1,a)}{\sqrt{1+a^2}} \frac{(1,a)}{\sqrt{1+a^2}} = \frac{(x_0 + a y_0)(1,a)}{1+a^2}.$$

Distance from (x_0, y_0) to L is $\|(x_0,y_0) - (x_L, y_L)\|$.

Ex: $(x_0, y_0) = (1, 4)$, $L: y = \frac{1}{3}x$, $a = 1/3$.

$(x_L, y_L) = (x_0 + a y_0)(1, a)/(1+a^2) = (1 + 4/3)(1, 1/3)/(1+\frac{1}{9}) \cdot \frac{9}{9}$

$= (9+12, 3+4)/10 = (2.1, 0.7)$

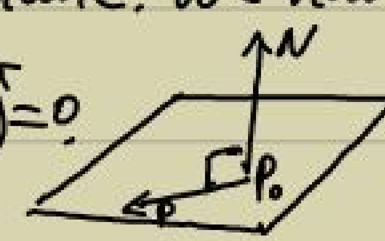
Distance $= \|(x_0, y_0) - (x_L, y_L)\| = \|(-1.1, 3.3)\| = \sqrt{(1.1)^2 + (3.3)^2}$

Hwk #6. Distance from (x_0, y_0) to $y - b = ax$ = distance from $(x_0, y_0 - b)$ to $y = ax$.

Ex: Find the equation of the plane passing through the point $P_0 = (x_0, y_0, z_0)$ and normal to the vector $N = (a, b, c)^T$.

Let $P = (x, y, z)$ be any point on the plane. We have

$(P - P_0) \perp N$ or $(a, b, c) \cdot ((x - x_0), (y - y_0), (z - z_0))^T = 0$



i.e.,

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

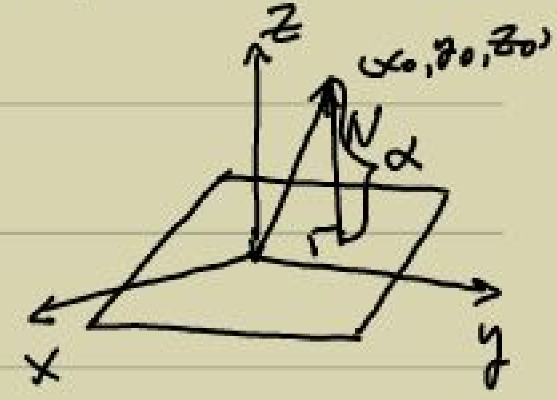
$$ax + by + cz = ax_0 + by_0 + cz_0$$

Ex. $P_0 = (2, -1, 3)$, $N = (2, 3, 4)^T$

P: $2(x-2) + 3(y+1) + 4(z-3) = 0$

or $2x + 3y + 4z = 13$.

Ex: In $\mathbb{R}^3$, find the distance from the point (x_0, y_0, z_0) to a plane $ax+by+cz=0$



The distance $= \alpha =$ scalar projection of (x_0, y_0, z_0) onto $N=(a,b,c)$.

$$d = \alpha = \frac{(x_0, y_0, z_0)\begin{bmatrix} a \\ b \\ c \end{bmatrix}}{\sqrt{a^2+b^2+c^2}} = (ax_0+by_0+cz_0) \Big/ \sqrt{a^2+b^2+c^2}.$$

when $(x_0, y_0, z_0) = (2,0,0)$, $(a,b,c) = (1,2,2)$

$$d = \frac{2}{\sqrt{1+4+4}} = 2/3.$$