Math 311-Zhou **Review for Exam I**

1. Put $Ax = b$ into its augmented matrix $[A|b]$. Use three elementary row operations (3ERO): (1) $(i) \leftrightarrow (j)$, (2) $\alpha(i)$, (3) $\alpha(i) + \to (j)$. (Gauss Elimination and Gauss-Jordan Reduction)

2. Reduce $[A|b]$ to its (reduced) Echelon form and determine if $Ax = b$ has (1) no solution, (2) unique solution or (3) infinite many solutions. In cases of (2)/(3), find all solutions.

3. Find A^{-1}, $A^{-1}_{2\times 2} = \frac{1}{d}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$, $(d = |A|)$ or $[A|I] \to \cdots \to [I|A^{-1}]$ using 3ERO.

4. Evaluate $|A|$. $n = 2, 3$, use the diagram. $n = 4$, use cofactor expansion along a row/column $|A| = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} |M_{ik}| = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} |M_{kj}|$, use properties and 3ERO $(-1, \alpha, 1)$.

5. Vector spaces and subspaces (closed in $\alpha v_1 + \beta v_2$). Find $\mathcal{N}(A) \Leftrightarrow$ solve $Ax = 0$ for all x.

6. Linear Combination: $c_1 A_1 + \cdots c_n A_n = b \Rightarrow Ax = b$ where $x = (c_1, ..., c_n)^T, A = [A_1....A_n]$.

7. Check $S = \text{span}\{v_1, ..., v_m\}$. Any $v \in S, v = c_1 v_1 + \cdots c_m v_m$, e.g., $S \subset V = \mathbb{R}^n$ and $V = P_n$.

8. Linear independent/dependent: ask if $c_1 v_1 + \cdots c_n v_n = 0$ has a nonzero solution $(c_1, ..., c_n)$?

9. Basis $\{v_1, ..., v_n\}$ to V if (a) $v_1, ..., v_n$ are LI and (b) $V = \text{span}\{v_1, ..., v_n\}$. $(n = \dim(V))$. Given $\{u_1, ..., u_m\}$ in V with $\dim(V) = n$, LD if $m > n$, cannot span if $m < n$ and LI $\Leftrightarrow$ span if $m = n$.

10. Reduce a spanning set to a basis by (11) and extend a LI set to a basis by $[A\, I]$ and (11).

11. $v_1 = (a_{11}, a_{21}, ..., a_{n1})^T, ...v_m = (a_{1m}, a_{2m}, ..., a_{nm})^T$, $A = [v_1\, v_2 \ldots v_m] \to \cdots \to U$(Echelon form by 3ERO): (a) Columns of U with a leading 1 are LI, (b) A column of U without a leading 1 can be written as LC of columns with leading 1's to the left. (c) The same to A.

12. To be able to use Equivalent Statements on $A_{n\times n}$:
(a) A is nonsingular; (b) $Ax = 0$ has only zero solution $x = 0$ $(\mathcal{N}(A) = \{0\})$;
(c) A is row equivalent to I; (d) $Ax = b$ has a unique solution for each $b \in \mathbb{R}^n$; (e) $|A| \neq 0$; (f) Columns/rows of A are LI, span $\mathbb{R}^n$, form a basis for $\mathbb{R}^n$.

13. To be able to solve problems in $P_n = \text{span}\{1, x, ..., x^n\}$, collect and equit coefficients of like powers, then turn to solve a linear system $Ax = b$ or $Ax = 0$ or use equivalent statements.

14. Coordinates w.r.t. a basis $E = \{v_1, ..., v_n\}$. $[v]_E = (c_1, ..., c_n)^T \in \mathbb{R}^n \Leftrightarrow v = c_1 v_1 + ... + c_n v_n$.

15. Transition matrix T from $F = \{u_1, ..., u_n\}$ to $E = \{v_1, .., v_n\}$. $T = \Big[[u_1]_E [u_2]_E ... [u_n]_E\Big]$.
$[v]_E = T[v]_F$, $\forall v \in V$.

16. In $\mathbb{R}^n$, if $U = [u_1 ... u_n], V = [v_1 ... v_n]$ are two bases, then $U[w]_U = V[w]_V$.
So from $U \to V$, $[w]_V = V^{-1} U [w]_U$ and from $V \to U, [w]_U = U^{-1} V [w]_V$.

17. For $A_{m\times n}$, find bases for $\mathcal{N}(A)$, $CS(A)$, $RS(A)$. $n = r + k$ where $k = \dim(\mathcal{N}(A))$ and r=rank(A)= # of leading 1's= $\dim(CS(A)) = \dim(RS(A))$.