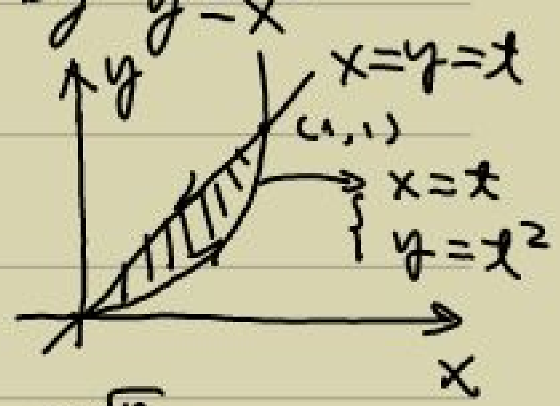
Subject: § 10.2. Green's Theorem

Let D be a closed, bounded region in $\mathbb{R}^2$, whose boundary $C = \partial D$ consists of finitely many simple closed curves that orient the curve C s.t. D is on the left. Let $F(x,y) = M(x,y)\,i + N(x,y)\,j$ be a C^1 vector field. Then

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Ex. Let $F = \underbrace{xy}_{M}\, i + \underbrace{y^2}_{N}\, j$; D bounded by $y = x$, $y = x^2$



$$LHS = \int_0^1 (t\,t^2\,dt + t^4\, 2t\,dt) + \int_1^0 (t\cdot t + t^2)\,dt$$

$$= \frac{1}{4} + 2\,\frac{1}{6} - \frac{2}{3} = -1/12$$

$$RHS = \iint_D (0 - x)\,dx\,dy = \int_0^1 \int_{x=y}^{x=\sqrt{y}} -x\,dx\,dy = \int_0^1 \frac{-x^2}{2}\Big|_y^{\sqrt{y}}\,dy$$

$$= \int_0^1 \left(\frac{-y}{2} + \frac{y^2}{2}\right) dy = -\frac{1}{4} + \frac{1}{6} = -1/12.$$

For $F = -y\,i + x\,j$, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 + 1 = 2$. Then

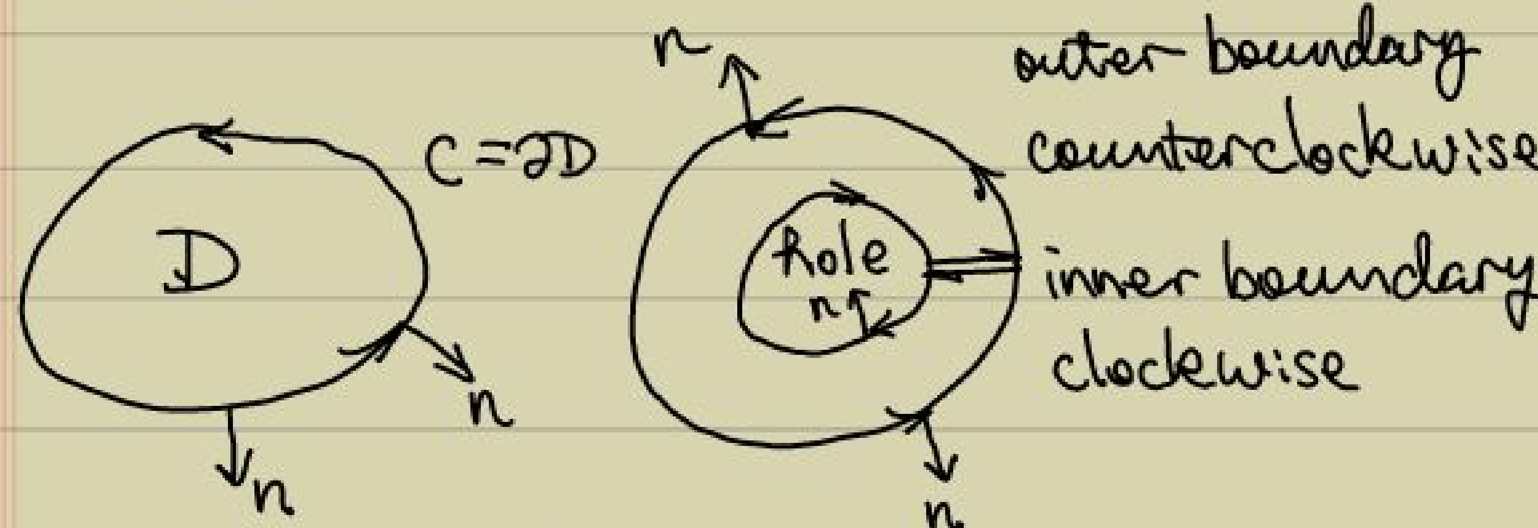
$$\oint_C M\,dx + N\,dy = \oint_C -y\,dx + x\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= 2 \iint_D dx\,dy = 2 \text{ area of } D.$$

Ex: Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$x=a\cos t,\ y=b\sin t,\quad 0\le t\le 2\pi$

$$\text{Area}=\frac{1}{2}\oint_C -y\,dx+x\,dy$$
$$=\frac{1}{2}\int_0^{2\pi}(-b\sin t(-a\sin t)+a\cos t\,(b\cos t))\,dt$$
$$=\frac{1}{2}\int_0^{2\pi} ab\,dt = ab\pi.$$



n = the outward unit normal vector.

Let D be a region bounded by $C=\partial D$ s.t. Green's theorems applies. Let n be the outward unit normal vector at $C=\partial D$, and $F(x,y)=M(x,y)\,i+N(x,y)\,j$ be C^1 vector field on D. Then

$F\cdot n$ = particles crossing C out = Flux

$\nabla\cdot F$ = rate of particles leaving a point

$$\oint_C (F\cdot n)\,ds \quad = \iint_D \nabla\cdot F\,dA \quad \left(\begin{matrix}\text{divergence}\\ \text{theorem}\end{matrix}\right)$$

Total particles crossing C out = total particles left D.

when $C: X(t) = (x(t), y(t))$.

$$T(t) = (x'(t), y'(t)) \perp N(t) = (-y'(t), x'(t)), \quad n(t) = \frac{N(t)}{\|N(t)\|}.$$

§10.3 Path Independence.

A vector field F is said to have a path-independence line integral if

$$\int_{C_1} F \cdot d\vec{s} = \int_{C_2} F \cdot d\vec{s}$$

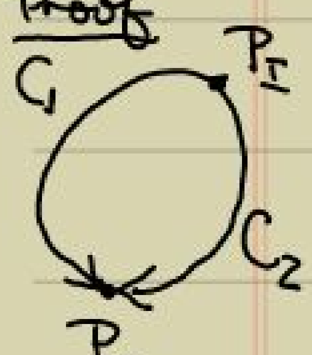
for any simple, piecewise C^1 curves with the same initial and terminal points.

<u>THM</u>: A vector field F has a path-indep line integral if and only if

$$\oint_C F \cdot d\vec{s} = 0$$

for any piecewise C^1 simple, closed curve.

<u>Proof</u>



$$\oint_C F \cdot d\vec{s} = \int_{C_1} F \cdot d\vec{s} - \int_{C_2} F \cdot d\vec{s} = 0.$$

THM. Let F be a continuous vector field on a connected open region D of $\mathbb{R}^n$. Then $F = \nabla f$ if and only if F has a path-indep line integral over curves in D. Moreover, if C is any piecewise C^1, oriented curve in D with initial point A and terminal point B, then

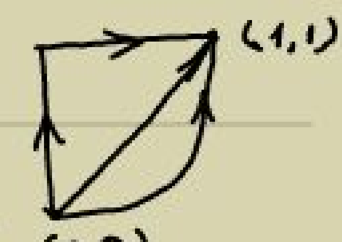
$$\int_C F \cdot d\vec{s} = f(B) - f(A). \qquad (F = \nabla f).$$

Proof: $C: X(t)$, $A = X(a)$, $B = X(b)$

$$\int_C F \cdot d\vec{s} = \int_a^b \nabla f(X(t)) \cdot X'(t)\,dt = \int_a^b \frac{d}{dt} f(X(t))\,dt$$

$$= f(X(t))\Big|_a^b = f(B) - f(A),$$

Ex. $F = M\mathbf{i} + N\mathbf{j} = x\mathbf{i} + y\mathbf{j}$. Note $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$.



$$\int_X F \cdot d\vec{s} = \int M\,dx + N\,dy = \int (M x'(t) + N y'(t))\,dt$$

1) $X: \begin{cases} x = t \\ y = t \end{cases}$, $0 \le t \le 1$, $\int_X F \cdot d\vec{s} = \int_0^1 (t + t)\,dt = 1$;

2) $X: \begin{cases} x = t \\ y = t^2 \to t^r \ (r > 0) \end{cases}$, $0 \le t \le 1$, $\int_X F \cdot d\vec{s} = \int_0^1 (t + t^2 2t)\,dt = \frac{1}{2} + \frac{2}{4} = 1$;

3) $X: \begin{cases} x = 0 \\ y = t \end{cases} + \begin{cases} x = t \\ y = 1 \end{cases}$ $0 \le t \le 1$, $\int_X F \cdot d\vec{s} = \int_0^1 (0 + t\,dt) + \int_0^1 (t\,dt + 1 \cdot 0) = 1$;

4). $f = \frac{1}{2}(x^2 + y^2)$, $F = \nabla f = (f_x, f_y) = (x, y)$,

$$\int_X F \cdot d\vec{s} = f(1,1) - f(0,0) = \frac{1}{2}(1+1) + 0 = 1.$$

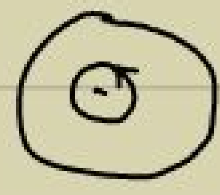
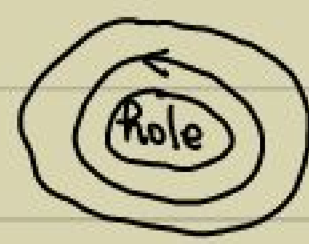
When $F=\nabla f$, F is called a conservative vector field, f is called a scalar potential.

For given F,

1) how to know if F is conservative?

2) Assume F is conservative, how to find f s.t. $F=\nabla f$?

Def. A region D in $\mathbb{R}^2$ is simply connected if any simple closed curve in D can be shrunk to a point.

Yes:  no: 

THM: Let $F=M\,i+N\,j$ be a C^1 vector field in a simply connected region D in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $F=\nabla f$ for some f if and only if $\nabla\times F=0$ in D.

Remark: $\nabla\times F=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \frac{\partial N}{\partial z} i - \frac{\partial M}{\partial z} j + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) k = 0$

$\Rightarrow N=N(x,y),\ M=M(x,y)$ and $\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}$.

Ex: Let $F=x^2y\,i-2xy\,j$

then $\frac{\partial N}{\partial x}=-2y \neq \frac{\partial M}{\partial y}=x^2 \Rightarrow$ not conservative.

Ex: $F=(2xy+\cos 2y)i+(x^2-2x\sin 2y)j = Mi+Nj$

check. $\frac{\partial M}{\partial y} = 2x - 2\sin 2y = \frac{\partial N}{\partial x} = 2x-2\sin 2y$

$\Rightarrow F$ is conservative, how to find f s.t. $F=\nabla f$?

$F = Mi+Nj = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$.

$M = \frac{\partial f}{\partial x} \Rightarrow f = \int M dx + \alpha(y) = x^2y + x\cos 2y + \alpha(y)$

$N = \frac{\partial f}{\partial y} \Rightarrow f = \int N dy + \beta(x) = x^2y + x\cos 2y + \beta(x)$

$f = f \Rightarrow \alpha(y) = \beta(x) = 0 \Rightarrow f = x^2y + x\cos 2y$.

THM. If D is simply connected domain, then

$F = \nabla f$ in D if and only if $\nabla \times F = 0$

Ex. $F = (e^x \sin y - yz)i + (e^x\cos y - xz)j + (z-xy)k$

check $\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x\sin y - yz & e^x\cos y - xz & z-xy \end{vmatrix} = 0.$

$\Rightarrow F = Mi+Nj+Pk = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$. To find f.

$f = \int M dx + \alpha(y,z) = \int (e^x \sin y - yz)dx + \alpha(y,z) = e^x\sin y - xyz + \alpha(y,z)$

$f = \int N dy + \beta(x,z) = \int (e^x\cos y - xz) dy + \beta(x,z) = e^x \sin y - xyz + \beta(x,z)$

$f = \int P dz + \gamma(x,y) = \int (z-xy) dz + \gamma(x,y) = z^2/2 - xyz + \gamma(x,y)$

$f=f=f \Rightarrow \alpha(y,z) = \beta(x,z) = z^2/2,\ \gamma(x,y) = e^x \sin y$

$\Rightarrow f = e^x\sin y - xyz + z^2/2\ (+c)$

Next to compute $\int F \cdot dS$ along a curve from $(0,0,0)$ to $(1, \pi/2, 2)$. we have

$$\int_X F \cdot dS = f(1, \pi/2, 2) - f(0,0,0) = e \cdot 1 - 1 \cdot \frac{\pi}{2} \cdot 2 + \frac{4}{2} = e - \pi + 2$$

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