Subject §11.1

$$C: \begin{cases} x = x(t), & a \le t \le b \text{ describe a curve } C \text{ in } \mathbb{R}^3. \\ y = y(t), \\ z = z(t). \end{cases} \quad X(t) = x(t)\,i + y(t)\,j + z(t)\,k = (x(t), y(t), z(t)).$$

$$S: \begin{cases} x = x(s,t), & (s,t) \in D, \text{ describe a surface in } \mathbb{R}^3. \\ y = y(s,t), \\ z = z(s,t), \end{cases} \quad X(s,t) = x(s,t)\,i + y(s,t)\,j + z(s,t)\,k.$$

Ex: $S: \begin{cases} x = a\cos s \sin t \\ y = a \sin s \sin t \\ z = a \cos t \end{cases}$ $(s,t) \in D = [0, 2\pi) \times [0, \pi]$ describe a sphere of radius $a > 0$.

check: $x^2 + y^2 + z^2 = a^2\cos^2 s \sin^2 t + a^2 \sin^2 s \sin^2 t + a^2 \cos^2 t = a^2$.

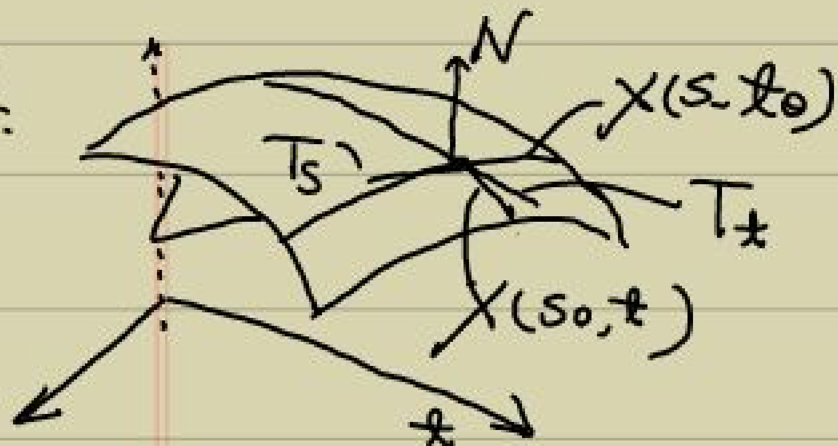
S transforms a rectangular domain D in $\mathbb{R}^2$ into a sphere in $\mathbb{R}^3$.

Ex. $T: \begin{cases} x = a\cos s, & 0 \le s \le 2\pi \\ y = a \sin s, & -\infty < t < \infty \\ z = t, \end{cases}$ describe a circular cylinder surface along z-axis in $\mathbb{R}^3$.

Let $X(s,t)$ be a surface S in $\mathbb{R}^3$, $(s,t) \in D$.

For each fixed t_0, for $(s, t_0) \in D$, $s \to X(s, t_0)$ is a curve on S, called the s-coordinate curve at $t = t_0$. Similarly for each fixed s_0, $t \to X(s_0, t)$ is the t-coordinate curve at $s = s_0$.

S:



Denote $T_s(s_0, t_0) = \frac{\partial X}{\partial s}(s_0, t_0) = \frac{\partial x}{\partial s}(s_0,t_0)i + \frac{\partial y}{\partial s}(s_0,t_0)j + \frac{\partial z}{\partial s}(s_0,t_0)k$,

the tangent of S in s-direction at (s_0, t_0), while

$T_t(s_0,t_0) = \frac{\partial X}{\partial t}(s_0,t_0) = \frac{\partial x}{\partial t}(s_0,t_0)i + \frac{\partial y}{\partial t}(s_0,t_0)j + \frac{\partial z}{\partial t}(s_0,t_0)k$.

the tangent of S in t-direction at (s_0, t_0).

$N = T_s \times T_t$ is the outward normal vector of X at (s_0, t_0) or S at $X(s_0, t_0)$.

<u>Def</u>: The surface $S = X(D) = \{x(s,t)i + y(s,t)j + z(s,t)k : (s,t) \in D\}$ is smooth at $X(s_0, t_0)$, $(s_0,t_0) \in \text{Int}\,D$, the outward normal $N(s_0,t_0) = T_s(s_0,t_0) \times T_t(s_0,t_0) \neq 0$.

$S = X(D)$ is smooth, if S is smooth at each $X(s_0, t_0)$ $(s_0, t_0) \in D$.

<u>Ex</u>: The cone $z^2 = x^2 + y^2$. $\begin{cases} x = s\cos t \\ y = s\sin t \\ z = s \end{cases}$ $0 \le t \le 2\pi$.

$$N(s,t) = T_s(s,t) \times T_t(s,t) = \begin{vmatrix} i & j & k \\ \cos t & \sin t & 1 \\ -s\sin t & s\cos t & 0 \end{vmatrix} = \begin{matrix} -s\cos t\, i - s\sin t + sk \\ = s(-\cos t, -\sin t, 1) \end{matrix}$$

Thus $N(s,t) = 0 \Leftrightarrow s = 0. \Rightarrow$ [redacted] $(x, y, z) = (0, 0, 0)$

* The plane T tangent to S at $X(s_0,t_0)$ is given by
$N(s_0,t_0)\cdot\big((x,y,z)-X(s_0,t_0)\big)=0.$

If $N(s_0,t_0)=(a,b,c)$ and $X(s_0,t_0)=(x_0,y_0,z_0)$, then

$T:\quad a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$

Ex: The cone $S: X(s,t)=(s\cos t, s\sin t, s)$.

Find the plane T tangent to S at $X(1,\frac{\pi}{2})$ =(0,1,1)

$T_s(1,\frac{\pi}{2})=(\cos t\, i+\sin t\, j+k)\Big|_{(1,\frac{\pi}{2})}=j+k$

$$N=\begin{vmatrix} i & j & k\\ 0 & 1 & 1\\ -1 & 0 & 0\end{vmatrix}=-j+k=(0,-1,1)$$

$T_t(1,\frac{\pi}{2})=(-s\sin t\, i+s\cos t\, j)\Big|_{(1,\frac{\pi}{2})}=-i$

$T:\quad 0(x-0)-(y-1)+(z-1)=0.$ or $z=y.$ #.

Recall for given vectors u and v,

$\|u\times v\|$ = area of the parallelogram formed by u and v,

For a curve $X(t)=(x(t),y(t),z(t))$, $X'(t)=(x'(t),y'(t),z'(t))$, tangent vector to $X(t)$. Arc-length element. $\|X'(t)\|dt$.

Now for a surface $S:\begin{cases} x=x(s,t)\\ y=y(s,t)\\ z=z(s,t)\end{cases}\quad (s,t)\in D.$

$\|T_s\times T_t\|$ = area of the parallelogram formed by T_s and T_t

$\|T_s\times T_t\|\,ds\,dt$ = surface element $=dS$ ... scalar

$N(s,t)\,ds\,dt=T_s\times T_t\,ds\,dt=d\vec{S}$ ... vector

$\iint_D \|T_s \times T_t\| \, ds\,dt =$ surface area of S.

Ex: For a sphere of radius a $S: \begin{cases} x = a\cos s \sin t, \\ y = a \sin s \sin t, \\ z = a\cos t, \end{cases}$

$0 \le s \le 2\pi, \ 0 \le t \le \pi.$

$T_s = (-a\sin s \sin t, a\cos s \sin t, 0)$

$T_t = (a\cos s \cos t, a\sin s \cos t, -a\sin t)$

$$T_s \times T_t = \begin{vmatrix} i & j & k \\ -a\sin s \sin t & a\cos s \sin t & 0 \\ a\cos s\cos t & a\sin s\cos t & -a\sin t \end{vmatrix} = -a^2\sin t(\cos s \sin t, \sin s \sin t, \cos t)$$

$\|T_s \times T_t\| = a^2 \sin t$

$$\text{surface area} = \int_0^{\pi}\int_0^{2\pi} a^2 \sin t \, ds\,dt = a^2 2\pi(-\cos t)\Big|_0^{\pi} = 4a^2\pi$$

Since

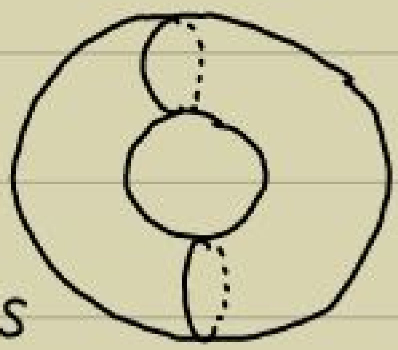
$$N(s,t) = T_s \times T_t = \frac{\partial(y,z)}{\partial(s,t)} i - \frac{\partial(x,z)}{\partial(s,t)} j + \frac{\partial(x,y)}{\partial(s,t)} k$$

where $\dfrac{\partial(u,v)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{vmatrix} = \dfrac{\partial u}{\partial s}\dfrac{\partial v}{\partial t} - \dfrac{\partial u}{\partial t}\dfrac{\partial v}{\partial s}$

$$\Rightarrow \|N(s,t)\| = \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2}$$

$$\Rightarrow \text{surface area} = \iint_D \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2} \, ds\,dt.$$

Ex: Torus. Let $S: \begin{cases} x=(a+b\cos t)\cos s, & 0 \le s, t \le 2\pi, \\ y=(a+b\cos t)\sin s, & a, b>0 \\ z=b\sin t. \end{cases}$



$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} -(a+b\cos t)\sin s & -b\sin t\cos s \\ (a+b\cos t)\cos s & -b\sin t \sin s \end{vmatrix}$$

$$= b(a+b\cos t)\sin t \sin^2 s + b(a+b\cos t)\sin t\cos^2 s$$

$$= b(a+b\cos t)\sin t.$$

$$\frac{\partial(x,z)}{\partial(s,t)} = -b(a+b\cos t)\cos t\sin s.$$

$$\frac{\partial(y,z)}{\partial(s,t)} = b(a+b\cos t)\cos t\cos s,$$

$\|N(s,t)\| = b(a+b\cos t)$

surface area $= \int_0^{2\pi}\int_0^{2\pi} b(a+b\cos t)\,dt\,ds = 4ab\pi^2.$

when $S: z=f(x,y),\ x,y\in D \Rightarrow S: \begin{cases} x=s \\ y=t \\ z=f(s,t) \end{cases} (s,t)\in D$

$T_s = i + f_s k,\quad T_t = j + f_t k$

$$\Rightarrow N(s,t) = T_s \times T_t = \begin{vmatrix} i & j & k \\ 1 & 0 & f_s \\ 0 & 1 & f_t \end{vmatrix} = -f_s i - f_t j + k.$$

$\|N(s,t)\| = \sqrt{f_x^2 + f_y^2 + 1}.$

surface area $= \iint_D \sqrt{f_x^2+f_y^2+1}\,dx\,dy.$

Add more examples here, #11, etc

§11.2

If f is a continuous function defined on $S=X(D)$,

then

$$\iint_D f(x(s,t),y(s,t),z(s,t))\,\|T_s \times T_t\|\,ds\,dt$$

is called the scalar surface integral of f on S.

If F is a continuous vector field defined on $S=X(D)$,

then

$$\iint_D F \cdot N\,ds\,dt$$

is called the vector surface integral of F on S.

In particular

$$\iint_D F\cdot n\,dS = \iint_D F\cdot N\,ds\,dt$$ where $n=N/\|N\|$.

is called the <u>flux</u> of F across $S=X(D)$. out.

<u>Ex</u>: Consider a lateral cylindrical surface

$$S_1: \begin{cases} x = 3\cos s \\ y = 3\sin s \\ z = t. \end{cases} \quad \begin{array}{l} 0 \le s \le 2\pi, \\ 0 \le t \le 15. \end{array}$$

$$S_2: \text{bottom}: \begin{cases} x = s\cos t, \\ y = s\sin t, \\ z = 0. \end{cases} \qquad S_3: \text{Top}: \begin{cases} x = s\cos t \\ y = s\sin t \\ z = 15. \end{cases}$$

$$0 \le s \le 3,\ 0 \le t \le 2\pi.$$

$f(x,y,z)=z$, $\quad I = \iint_D f\,\|N\|\,ds\,dt = I_1+I_2+I_3$, $N=T_s\times T_t$.

(scalar surface integral of f on S)

S_1: $T_s \times T_t = \begin{vmatrix} i & j & k \\ -3\sin s & 3\cos s & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\cos s\, i + 3\sin s\, j$. horizontal

$\|T_s \times T_t\| = 3$.

$I_1 = \int_0^{2\pi}\int_0^{15} t \cdot 3\, dt\, ds = 2\pi \frac{3}{2} 15^2 = 675\pi$.

S_2: $f = z = 0$. $\Rightarrow I_2 = 0$

S_3: $f = z = 15$. $N = \begin{vmatrix} i & j & k \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 0 \end{vmatrix} = s k$, $\|N\| = s$

$I_3 = \int_0^{2\pi}\int_0^{3} 15 \cdot s\, ds\, dt = 30\pi \frac{9}{2} = 135\pi$

$I = I_1 + I_2 + I_3 = 675\pi + 135\pi = 810\pi$.

When S: $z = g(x,y)$, $(x,y) \in D$, we have (scalar surface integral of f on S)

$$\iint_D f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + 1}\, dx\, dy.$$

Ex. S: $z = 4 - x^2 - y^2$, $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 4\}$

$f = 4 - z$.

scalar surface integral of on S

$$\iint_D f \|N\| dA = \iint_D (4 - (4 - x^2 - y^2))\sqrt{4x^2 + 4y^2 + 1}\, dx\, dy$$

$$= \iint_D (x^2 + y^2)\sqrt{4(x^2+y^2)+1}\, dx\, dy.$$

Polar coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

D: $0 \le \theta \le 2\pi$, $0 \le r \le 2$

$$= \int_0^{2\pi}\int_0^{2} r^2 \sqrt{4r^2+1}\, r\, dr\, d\theta.$$ substitute $4r^2 + 1 = t$, $dt = 8r\,dr$

$r\big|_0^2 \Rightarrow t\big|_1^{17}$

$$= 2\pi \int_1^{17} \frac{t-1}{4}\sqrt{t}\,\frac{1}{8}\, dt$$

$$= \frac{\pi}{16}\int_1^{17} (t^{3/2} - t^{1/2})\, dt = \cdots \frac{\pi}{16}\left[\frac{2}{5}(17^{5/2} - 1) - \frac{2}{3}(17^{3/2} - 1)\right]$$

Ex: $F = x\mathbf{i} + y\mathbf{j} + (z-2y)\mathbf{k}$, $\underset{\sim}{X}(s,t) = (s\cos t, s\sin t, t)$

$0 \le s \le 1,\ 0 \le t \le 2\pi$

Evaluate vector surface integral.

$$\iint_D F \cdot d\vec{S} = \iint_D F \cdot N\, ds\,dt, \quad N = T_s \times T_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 1 \end{vmatrix}$$

$$= \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + s\mathbf{k}$$

$$\| $$

$$\int_0^{2\pi}\int_0^1 (s\cos t, s\sin t, (t-2s\sin t)) \cdot (\sin t, -\cos t, s)\, ds\,dt$$

$$= \int_0^{2\pi}\int_0^1 (st - 2s^2\sin t)\, ds\,dt$$

$$= \int_0^{2\pi} \left(\frac{1}{2}t - \frac{2}{3}\sin t\right) dt = \pi^2.$$

when S: $z = g(x,y)$. $\Rightarrow \underset{\sim}{X}(x,y) = (x, y, g(x,y))$

$$N(s,t) = N(x,y) = T_x \times T_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = (-g_x, -g_y, 1).$$

Then

$$\iint_D F \cdot d\vec{S} = \iint_D F(x,y,g(x,y)) \cdot (-g_x, -g_y, 1)\, dx\,dy.$$

Add examples here.

§11.3

Stokes' Theorem.

Let S be a bounded piecewise smooth oriented surface in $\mathbb{R}^3$, S: $\underset{\sim}{X}(s,t) = (x(s,t), y(s,t), z(s,t))$, $(s,t) \in D$, $N(s,t) \ne 0$.

(the boundary of S)

and assume ∂S consists finitely many piecewise C^1 simple closed curves, $\underset{\sim}{X}(s)$, $s \in I$, each of which is oriented

Subject

consistently with S ($N(s,t)$ is to the left)

Let F be a C' vector field on S, then

$$\iint_S \nabla\times F\cdot d\vec{S} = \oint_{\partial S} F\cdot d\vec{s}$$

$$\iint_D \nabla\times F(\underline{X}(s,t))\cdot N(s,t)\,ds\,dt = \oint F(x(s))\cdot x'(s)\,ds.$$

LHS = curl on S = RHS = circulation along ∂S.

Gauss Theorem (Divergence)

Let D be a bounded solid region in $\mathbb{R}^3$ whose boundary ∂D consists of finitely many piecewise smooth, closed orientable surfaces, each of which is oriented by unit normals that point away from D. Let F be a piecewise C' vector field on D. Then

$$\oiint_{\partial D} F\cdot d\vec{S} = \iiint_D \nabla\cdot F\,dV$$

Flux across ∂D — divergence

Total particles cross ∂D out = total particles leave D.

Ex. $S: z = g(x,y) = 9 - x^2 - y^2 \quad (z \geq 0) \Rightarrow x^2 + y^2 \leq 9$

$F = (2z - y)i + (x+z)j + (3x - 2y)k \quad \Rightarrow \quad D: \begin{cases} x = r\cos\theta, & 0 \leq \theta \leq 2\pi \\ y = r\sin\theta, & 0 \leq r \leq 3. \end{cases}$

verify the Stokes theorem.

$\partial S: z = 0 \Rightarrow x^2 + y^2 = 9. \Rightarrow X(s) = (3\cos s, 3\sin s, 0), \ 0 \leq s \leq 2\pi.$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z - y & x+z & 3x - 2y \end{vmatrix} = -3i - j + 2k$$

$$N(x,y) = -g_x i - g_y j + k = 2x i + 2y j + k$$

$$LHS = \iint_S \nabla \times F \cdot d\vec{S} = \iint_D (-3, -1, 2) \cdot (2x, 2y, 1)\, dx\, dy$$

$$= \iint_D (-6x - 2y + 2)\, dx\, dy = \int_0^{2\pi} \int_0^3 (-6r\cos\theta - 2r\sin\theta + 2)\, r\, dr\, d\theta$$

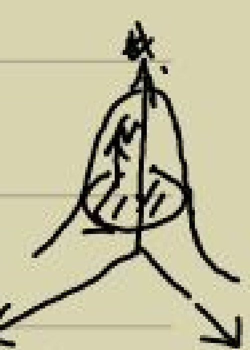
$$= \int_0^{2\pi} \left(-\frac{27}{3}\cos\theta - \frac{2}{3}27\sin\theta + \frac{2}{2}9\right) d\theta = 18\pi.$$

$X(s) = (3\cos s, 3\sin s, 0)$

$$RHS = \oint_{\partial S} F \cdot d\vec{s} = \int_0^{2\pi} F(X(s)) \cdot X'(s)\, ds$$

$$= \int_0^{2\pi} (-3\sin s, 3\cos s, 9\cos s - 6\sin s) \cdot (-3\sin s, 3\cos s, 0)\, ds$$

$$= \int_0^{2\pi} (9\sin^2 s + 9\cos^2 s)\, ds = 18\pi = LHS.$$



****Ex.** Given $S: z = g(x,y) = e^{-(x^2+y^2)}, \ z \geq \frac{1}{e}$

$F = (e^{y+z} - 2y)i + (xe^{y+z} + y)j + e^{x+y}k. \quad D = \{(x,y) : x^2 + y^2 \leq 1\}.$

Evaluate $\iint_S \nabla \times F \cdot d\vec{S} = \oint_{\partial S} F \cdot d\vec{s}$; $\partial S: \begin{cases} x = \cos\theta, \\ y = \sin\theta, & 0 \leq \theta \leq 2\pi. \\ z = 1/e. \end{cases}$

$$\nabla \times F = (e^{x+y} - xe^{y+z})i + (e^{y+z} - e^{x+y})j + 2k$$

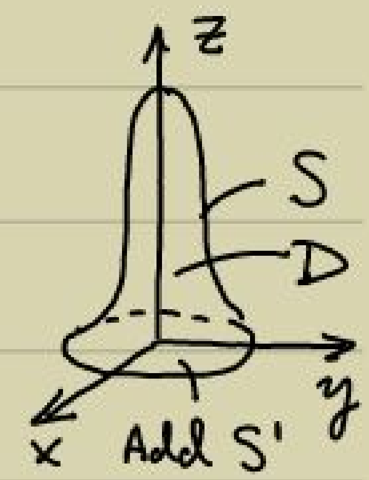
Both LHS and RHS are too hard to do.

* Create an easy surface S_1: $X(r,\theta) = \begin{cases} x = r\cos\theta & 0 \le r \le 1 \\ y = r\sin\theta. & 0 \le \theta \le 2\pi. \\ z = \frac{1}{e} & \end{cases}$

a flat disk with $\underset{\sim\sim\sim}{\partial S_1 = \partial S}$.

Note now $N(r,\theta) = +rk = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

Then LHS $= \iint_{S_1} \nabla \times F \cdot N(r,\theta)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r\,dr\,d\theta = 2|S_1| = 2\pi$.



$\underline{\text{Ex}}$: Let S: $z = (1-x^2-y^2)e^{(1-x^2-3y^2)}$, $z \ge 0$

$F = e^y\cos z\, i + \sqrt{x^3+1}\,\sin z\, j + (x^2+y^2+3)k$,

Ask $\iint_S F \cdot d\vec{S}$, too hard to do it directly.

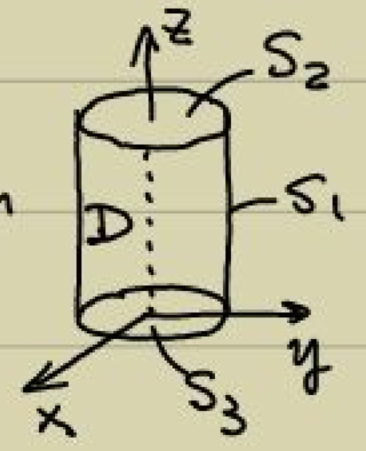
So add S': $x^2+y^2 \le 1$, $z = 0$. $N = -rk$. $\begin{cases} x = r\cos\theta, & 0 \le r \le 1 \\ y = r\sin\theta, & 0 \le \theta \le 2\pi. \end{cases}$

Then by the ~~Stokes~~ Gauss Theorem (divergence)

$$\iint_S F\cdot d\vec{S} + \iint_{S'} F\cdot d\vec{S} = \iiint_D \nabla\cdot F\,dV \qquad \text{but } \nabla\cdot F = 0.$$

Thus

$$\iint_S F\cdot d\vec{S} = -\iint_{S'} F\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 (r^2+3)\,r\,dr\,d\theta = 2\pi\left(\frac{1}{4}+\frac{3}{2}\right) = \frac{7}{2}\pi.$$

Ex: Given  $D: x^2+y^2 \le a^2,\ 0 \le z \le b,$

$S_1: x^2+y^2=a^2,\ 0 \le z \le b.$

$S_2: x^2+y^2 \le a^2,\ z=b.$

$S_3: x^2+y^2 \le a^2,\ z=0.$

$F = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. verify the Gauss theorem.

To find N for S_1, S_2, S_3, we write

$$S_1: \begin{cases} x = a\cos s, & 0 \le s \le 2\pi \\ y = a\sin s, & 0 \le t \le b \\ z = t. \end{cases} ; \quad S_2\,(S_3) \begin{cases} x = t\cos s, & 0 \le t \le a \\ y = t\sin s. & 0 \le s \le 2\pi. \\ z = b\,(0). \end{cases}$$

$\Rightarrow$ For S_1: $N = a\cos s\,\mathbf{i} + a\sin s\,\mathbf{j}$, S_2: $N = t\mathbf{k}$, S_3: $N = -t\mathbf{k}$

$$LHS = \iint_{\partial D} F \cdot d\vec{S} = \Big(\iint_{S_1} + \iint_{S_2} + \iint_{S_3}\Big) F \cdot d\vec{S}$$

$$= \int_0^{2\pi}\int_0^b (a^2\cos^2 s + a^2\sin^2 s)\,ds\,dt + \int_0^{2\pi}\int_0^a bt\,dt\,ds + \int_0^{2\pi}\int_0^a -0\cdot t\,dt\,ds$$

$$= 2\pi a^2 b + \pi a^2 b + 0 = 3\pi a^2 b.$$

$$RHS = \iiint_D \nabla\cdot F\,dv = \iiint_D (1+1+1)\,dv = 3\,|D| = 3a^2\pi b = LHS.$$

* If D is a region and $F = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then

$$\frac{1}{3}\iint_{\partial D} F\cdot d\vec{S} = \frac{1}{3}\iiint_D \nabla\cdot F\,dv = \iiint_D dv = |D| = \text{volume}.$$