# §11.3+

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Gauss (divergence theorem)

$$\oiint_{\partial D} F \cdot d\vec{S} = \iiint_D \nabla \cdot F \, dV$$

$$\begin{array}{ccc} \shortparallel & & \downarrow \\ \oiint_{\partial D} F \cdot n \, dS & & \text{divergence} \\ \downarrow & & /\!/ \\ \text{Flux of } F \text{ across } \partial D & & \end{array}$$

when

$S = \partial D$ is given by a level set

$S: \; G(x, y, z) = C$.

then $\nabla G(x, y, z) = (G_x, G_y, G_z)$

gives a normal vector. The unit normal vector

$$n(x, y, z) = \frac{(G_x, G_y, G_z)}{(G_x^2 + G_y^2 + G_z^2)^{1/2}}$$

We may adjust its sign due to orientation of $\partial D$.

Ex: Let $F = x\,i + y\,j + z\,k$ be a vector field

and $S: \; x^2 + y^2 + z^2 = a^2$

To compute the flux of $F$ across $S$ in two ways:

1) Use the parametric equation system

$$\begin{cases} x = a\cos s \sin t & 0 \le s \le 2\pi \\ y = a \sin s \sin t & 0 \le t \le \pi \\ z = a\cos t \end{cases}$$

$$N(s,t) = T_s \times T_t = \begin{vmatrix} \vec{i} & \vec{j} & k \\ -a\sin s \sin t & a\cos s \sin t & 0 \\ a\cos s\cos t & a\sin s\cos t & -a\sin t \end{vmatrix}$$

$$= (-a^2\cos s \sin^2 t, -a^2 \sin s \sin^2 t, -a^2 \sin t \cos t)$$

$$= -a^2 \sin t\,(\cos s \sin t, \sin s \sin t, \cos t)$$

or

$$= -a\sin t\,(x, y, z).$$

Problem: $s \times t \to N$ ? $t \times s \to N$ ?

it is inward, so add a negative sign

$$\iint_S F \cdot d\vec{S} = -\int_0^{\pi}\int_0^{2\pi} F(x(s,t)) \cdot N(s,t)\,ds\,dt$$

$$= -\int_0^{\pi}\int_0^{2\pi} (a\cos s \sin t, a\sin s \sin t, a\cos t) \cdot \left[-a^2\sin t(\cos s \sin t, \sin s \sin t, \cos t)\right] ds\,dt$$

$$= +a^3\int_0^{\pi}\int_0^{2\pi} \sin t\,(\cos^2 s \sin^2 t + \sin^2 s \sin^2 t + \cos^2 t)\,ds\,dt$$

$$= +a^3 \int_0^{\pi}\int_0^{2\pi} \sin t\,ds\,dt = 2\pi a^3 \int_0^{\pi} \sin t\,dt = 4\pi a^3$$

2). $S: G(x,y,z) = x^2 + y^2 + z^2 = C$

$$\nabla G(x,y,z) = (2x, 2y, 2z)$$

$$n(x,y,z) = \frac{(2x, 2y, 2z)}{\sqrt{(2x)^2+(2y)^2+(2z)^2}} = \frac{(x,y,z)}{a}$$

It is outward, good.

Then
$$\iint_S F \cdot d\vec{S} = \iint_S F \cdot n \, dS = \iint_S \frac{x^2+y^2+z^2}{a} dS$$

$$= a \iint_S dS = a|S| = a \, 4\pi a^2$$

3) If we use $S: z = \pm\sqrt{a^2 - x^2 - y^2}$

$$N(x,y) = (-z_x, -z_y, 1) = \left(\frac{x}{\sqrt{a^2-x^2-y^2}}, \frac{y}{\sqrt{a^2-x^2-y}}, 1\right)$$

$+$

always point upward

From 2) we have $n = \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right)$

we see that

$$N(x,y) = \left(\frac{x}{\sqrt{a^2-x^2-y^2}}, \frac{y}{\sqrt{a^2-x^2-y^2}}, \pm 1\right) \text{ "}\begin{smallmatrix}+\\-\end{smallmatrix}\text{" for } z \begin{smallmatrix}>\\<\end{smallmatrix} 0$$

Ex: Verify Gauss (divergence theorem)

Let $F = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$. $D: a^2 \le x^2+y^2+z^2 \le b^2$.

$$\oiint_{\partial D} F \cdot d\vec{S} = \iiint_D \nabla \cdot F \, dV.$$

RHS: $\nabla \cdot F = F^1_x + F^2_y + F^3_z$ =M_x+N_y+P_z

$F^1_x = (x^2+y^2+z^2)^{-1/2} - \frac{2}{2} x^2 (x^2+y^2+z^2)^{-3/2}$

$\Rightarrow \nabla \cdot F = \dfrac{3}{(x^2+y^2+z^2)^{1/2}} - \dfrac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}} = \dfrac{2}{(x^2+y^2+z^2)^{1/2}}$

$$\begin{cases} x = r\cos s \sin t & 0 \le t \le \pi \\ y = r \sin s \sin t & 0 \le s \le 2\pi \\ z = r\cos t & a \le r \le b \end{cases}$$

$dx\,dy\,dz = r^2 \sin t \, dr\, ds\, dt$

$$\text{RHS} = \int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{2}{r} r^2 \sin t \, dr\, dt\, ds$$

$$= 2\pi \left(-\cos t \Big|_0^{\pi} \cdot r^2 \Big|_a^b\right) = 4\pi(b^2 - a^2)$$

LHS: $\partial D$: $G(x,y,z) = x^2+y^2+z^2 = C$

$$\nabla G(x,y,z) = (2x, 2y, 2z)$$

$n(x,y,z) = \dfrac{\nabla G}{\| \nabla G \|} = \dfrac{(x, y, z)}{\quad}$

$$n(x,y,z) = \frac{\nabla G}{\|\nabla G\|} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$$

$$LHS = \iint_{S_b} F\cdot n\, dS - \iint_{S_a} F\cdot n\, dS$$

$$= \iint_{S_b} \frac{x^2+y^2+z^2}{x^2+y^2+z^2} dS - \iint_{S_a} \frac{x^2+y^2+z^2}{x^2+y^2+z^2} dS$$

$$= \iint_{S_b} dS - \iint_{S_a} dS = 4\pi b^2 - 4\pi a^2 = 4\pi(b^2-a^2)$$

Ex: When $F = Mi + Nj$.

Stokes' theorem reduces to Green's theorem.

$$\iint_S (\nabla\times F)\, d\vec{S} = \oint_{\partial S} F\cdot d\vec{s}$$

$$\Rightarrow \oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

when $F = Mi + Nj$

$$\nabla\times F = \left| \begin{array}{ccc|cc} i & j & k & i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ M & N & 0 & M & N \end{array} \right| = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) k$$

In Stokes'

$$RHS = \oint_{\partial S} (M, N, 0) \cdot (x'(s), y'(s), z'(s))\, ds$$

$$= \oint_{\partial S} (M x'(s) + N y'(s))\, ds = \oint_{\partial S} M\,dx + N\,dy$$

$$LHS = \iint_D \left(0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \cdot (-g_x, -g_y, 1)\, dA$$

$$= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$