Cross Product,

Subject math311 §7/8

Recall $\mathbb{R}^3 = \text{span}\{e_1, e_2, e_3\}$ where

$e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$, $e_3 = (0,0,1)^T$.

If denote $i = e_1$, $j = e_2$, $k = e_3$, then

any $x = (x_1, x_2, x_3)^T$ in $\mathbb{R}^3$, $x = x_1 i + x_2 j + x_3 k$,

$y = (y_1, y_2, y_3)^T$ in $\mathbb{R}^3$, $y = y_1 i + y_2 j + y_3 k$.

dot (scalar) product $x \cdot y = y^T x = x_1 y_1 + x_2 y_2 + x_3 y_3$ (scalar)

cross (vector) product,

$$x \times y = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2) i + (x_3 y_1 - x_1 y_3) j + (x_1 y_2 - x_2 y_1) k$$

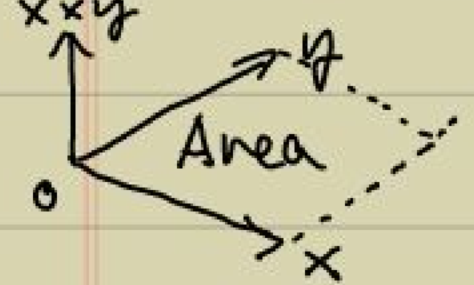
(vector)

<u>Ex.</u> $x = (1, 2, -3)$, $y = (-5, 4, 2)$

$$x \times y = \left| \begin{array}{ccc|cc} i & j & k & i & j \\ 1 & 2 & -3 & 1 & 2 \\ -5 & 4 & 2 & -5 & 4 \end{array} \right| = \begin{array}{l} 4i + 15j + 4k \\ +10k + 12i - 2j \end{array}$$

$= 16 i + 13 j + 14 k = (16, 13, 14)$

Geometric meaning of $x \times y$.



$x \times y$ is $\perp$ x, y, by the right-hand-rule.

$|x \times y| =$ the arear of the parallelogram

In the last example, $x = (1, 2, -3)$, $y = (-5, 4, 2)$, $x \times y = (16, 13, 14)$

$\Rightarrow (x \times y) \cdot x = 16 + 26 - 42 = 0$, $(x \times y) \cdot y = -80 + 52 + 28 = 0$.

Subject

Properties:

1) $i \times j = k$, $j \times k = i$, $k \times i = j$;

2) $X \times Y = -Y \times X$;

3) $X \times (Y + Z) = X \times Y + X \times Z$;

4) $(X + Y) \times Z = X \times Z + Y \times Z$;

5) $\alpha (X \times Y) = (\alpha X) \times Y = X \times (\alpha Y)$.

Ex: Find the area A of the triangle formed by three points $(0,0,0)$, $(1,2,-3)$, $(2,0,1)$.

$$A = \frac{1}{2} \| (1,2,-3) \times (2,0,1) \| = \frac{1}{2} \left\| \begin{matrix} i & j & k \\ 1 & 2 & -3 \\ 2 & 0 & 1 \end{matrix} \right\| = \frac{1}{2} \| 2i - 7j - 4k \|$$

$$= \frac{1}{2} \sqrt{4 + 49 + 16} = \frac{1}{2} \sqrt{69}.$$

The equation of a plane passing throug (x_0, y_0, z_0) with a normal vector $n = (a, b, c)$ is given by

$$n \cdot ((x, y, z) - (x_0, y_0, z_0)) = 0 \Leftrightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

$\Leftrightarrow$ general form $ax + by + cz = ax_0 + by_0 + cz_0$.

The parametric equation of a plane containing two given vectors u_1 and u_2 is

$$p(s, t) = s u_1 + t u_2.$$

when the plane is translated by a vector v,
then its parametric equation is

$$p(s,t) = s u_1 + t u_2 + v.$$

To find its general form, we have $(x_0, y_0, z_0) = v$. need a normal vector. So we do $n = u_1 \times u_2 = (a, b, c)$.

Ex: Given $p(s,t) = s u_1 + t u_1 + v$ where $u_1 = (3, -1, 2)$
$u_2 = (2, 5, -2)$, $v = (0, 0, 4)$. Thus we have
$(x_0, y_0, z_0) = v = (0, 0, 4)$ and

$$n = u_1 \times u_2 = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 2 & 5 & -2 \end{vmatrix} = -8i + 10j + 17k = (-8, 10, 17) = (a, b, c).$$

Its general form is

$$-8x + 10y + 17z = 17 \times 4 = 68.$$

Path and Vector Field

1) A continuous mapping $X: I \subseteq \mathbb{R} \to \mathbb{R}^n$ is called a path where I is an interval in $\mathbb{R}$, say $I = [a, b]$. Then $X(a)$ and $X(b)$ are called the endpoints of the path;

2) A mapping F from (a subset of) $\mathbb{R}^n$ into $\mathbb{R}^n$ is called a vector field.

<u>Ex.</u> $X(t) = b + ta$ for $-\infty < t < \infty$, and given vectors a and b is a straight line.

<u>Ex:</u> $X(t) = (r\cos\omega t, r\sin\omega t)$ for given $r, \omega > 0$ and $0 \le t \le 2\pi$, gives a circle of radius r centered at 0, moving counterclockwise with a circular speed ω from $(r, 0)$ for one revolution.

note $X(t) = (x(t), y(t)) \Rightarrow x^2(t) + y^2(t) = r^2$.

<u>Ex:</u> $X(t) = (a\cos t, a\sin t, bt)$ for given $a > 0$, b. is called a circular helix

$X(t) = (x(t), y(t), z(t)) \Rightarrow x^2(t) + y^2(t) = a^2$, $z(t) = bt$ height.

$X(t) = (x_1(t), \cdots, x_n(t))$ --- position of a particle moving along the path.

$V(t) = X'(t) = (x_1'(t), \cdots, x_n'(t))$ --- velocity

$\|V(t)\| = \|X'(t)\| = \left((x_1'(t))^2 + \cdots + (x_n'(t))^2\right)^{1/2}$ --- speed

Ex: $X(t) = b + ta$, $V(t) = X'(t) = a$, constant vector, straight line.

Ex. $X(t) = (r\cos\omega t, r\sin\omega t)$, $V(t) = X'(t) = (-\omega r\sin\omega t, \omega r\cos\omega t)$.

$\|V(t)\| = \omega r$ --- speed.

Def: For a C^1 path $X(t)$, the line tangent to $X(t)$ at t_0 is given by

$$L(s) = X(t_0) + sX'(t_0) = X(t_0) + sV(t_0)$$

or $\quad L(t) = X(t_0) + (t - t_0)V(t_0)$

Def: For C^1 path $X(t)$, $\|X'(t)\|dt$ is called the arc-length element. The length of the path $X(t)$ from a to b is

$$L(X) = \int_a^b \|X'(t)\|dt.$$

$s(t) = \int_a^t \|X'(t)\|dt$ is called the arc length reparameterization. We have

$$\frac{ds(t)}{dt} = \|X'(t)\| = \text{speed}.$$

Ex: $x:[0,2\pi]\to\mathbb{R}^2$ with $x(t)=(a\cos t, a\sin t)$, $a>0$.

Then $\|x'(t)\|=\sqrt{a^2\sin^2 t+a^2\cos^2 t}=a$.

$L(x)=\int_0^{2\pi} a\,dt = 2\pi a$.

Ex: Helix: $x(t)=(a\cos t, a\sin t, bt)$ $0\le t\le 2\pi$.

$x'(t)=(-a\sin t, a\cos t, b)$. $\|x'(t)\|=\sqrt{a^2+b^2}$.

$L(x)=\int_0^{2\pi}\sqrt{a^2+b^2}\,dt = 2\pi\sqrt{a^2+b^2}$

If $a<t_1<b$, then

$\int_a^b\|x'(t)\|dt=\int_a^{t_1}\|x'(t)\|dt+\int_{t_1}^b\|x'(t)\|dt$.

Gradient, Divergence, curl, the Del operator in $\mathbb{R}^3$

$f:\mathbb{R}^3\to\mathbb{R}$ a function

its gradient $\nabla f(x,y,z)=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})$ a vector field.

Define the Del-operator $\nabla=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})$.

Let $F:\mathbb{R}^3\to\mathbb{R}^3$ be a vector field.

$F(x,y,z)=M(x,y,z)\,i+N(x,y,z)\,j+P(x,y,z)\,k$.

The divergence of F at (x,y,z)

$\nabla\cdot F(x,y,z)=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}=$ the rate of particles leaving (x,y,z)

the curl of F at (x,y,z) is $\nabla\times F(x,y,z)=\begin{vmatrix} i & j & k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ M & N & P\end{vmatrix}$.

§10.1 Scalar & Vector Line Integrals

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a continuous function and $x: [a,b] \to \mathbb{R}^3$ be a C^1 path.

$ds = \|x'(t)\| dt$ --- scalar

$d\vec{s} = x'(t) dt$ --- vector

The scalar line integral of f along the path $x(t)$ from a to b is

$$\int_a^b f(x)\, \|x'(t)\|\, dt.$$

Ex. The helix $x(t) = (\cos t, \sin t, t)$, $0 \le t \le 2\pi$.

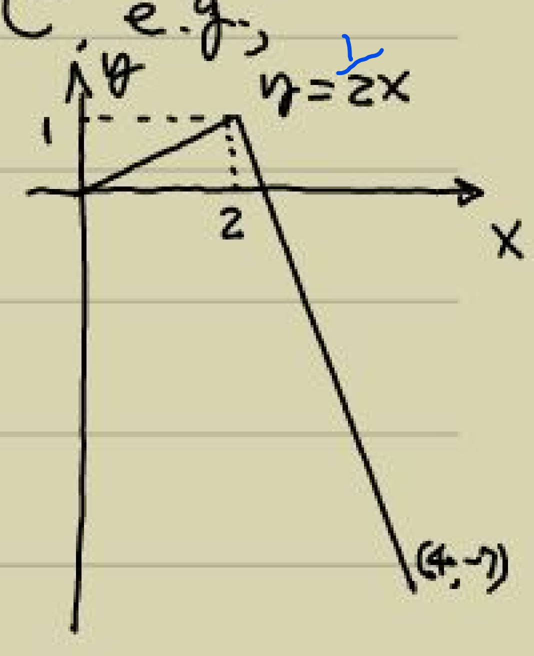
$f(x,y,z) = xy + z$. Then

$$\int_a^b f(x(t))\, \|x'(t)\|\, dt = \int_0^{2\pi} (\cos t \cdot \sin t + t)\sqrt{\sin^2 t + \cos^2 t + 1}\, dt$$

$$= \sqrt{2}\int_0^{2\pi} (\cos t \sin t + t)\, dt = \sqrt{2}\left(\frac{\sin^2 t}{2} + \frac{t^2}{2}\right)\Big|_0^{2\pi} = 2\sqrt{2}\pi^2.$$

When the path $x(t)$ is piecewise C^1, e.g.,

Ex: A path $x(t) = \begin{cases} (2t, t), & 0 \le t \le 1 \\ (t+1, 5-4t), & 1 \le t \le 3 \end{cases}$



$f(x,y) = y - x$

$$f(x(t)) = \begin{cases} t - 2t = -t, & 0 \le t \le 1 \\ 5-4t-(t+1) = 4-5t, & 1 \le t \le 3 \end{cases}$$

Thus $\int_X f(X(t))\,ds = \int_0^3 f(X(t))\,\|X'(t)\|\,dt$

$$= \int_0^1 (-t)\sqrt{4+1}\,dt + \int_1^3 (4-5t)\sqrt{1+16}\,dt$$
$$= -\sqrt{5}\,\frac{t^2}{2}\Big|_0^1 + \sqrt{17}\left(4t - \frac{5}{2}t^2\right)\Big|_1^3 = -\frac{\sqrt{5}}{2} - 12\sqrt{17}.$$

Next let F be a vector field from $\mathbb{R}^3 \to \mathbb{R}^3$

and $X(t)$ be a C^1 path $a \le t \le b$.

the vector-line integral of F along $X: [a,b] \to \mathbb{R}^3$ is

$$\int_X F \cdot d\vec{s} = \int_a^b F(X(t)) \cdot X'(t)\,dt = \text{In physics work done by } F \text{ along } X(t).$$

Ex: $F = x\hat{i} + y\hat{j} + z\hat{k}$, $X(t) = (t, 3t^2, 2t^3)$, $0 \le t \le 1$.

Then $X'(t) = (1, 6t, 6t^2)$, $F(X(t)) = (t, 3t^2, 2t^3)$.

$$\int_X F \cdot d\vec{s} = \int_0^1 (t, 3t^2, 2t^3)\cdot(1, 6t, 6t^2)\,dt = \int_0^1 (t + 18t^3 + 12t^5)\,dt$$
$$= \frac{1}{2} + 18\cdot\frac{1}{4} + 12\cdot\frac{1}{6} = 7.$$

Since the unit tangent vector $T(t) = \frac{X'(t)}{\|X'(t)\|}$, we have

$$\int_X F \cdot d\vec{s} = \int_a^b F(X(t)) \cdot X'(t)\,dt = \int_a^b \left(F(X(t)) \cdot T(t)\right)\|X'(t)\|\,dt.$$

when $X(t) = (x(t), y(t), z(t))$, $a \le t \le b$.

$F(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$

$$\begin{aligned} F(X(t))\cdot X'(t)\,dt &= (M,N,P)\cdot(x'(t),y'(t),z'(t))\,dt \\ &= \left(Mx'(t)+Ny'(t)+Pz'(t)\right)dt \\ &= M\,dx + N\,dy + P\,dz \end{aligned}$$

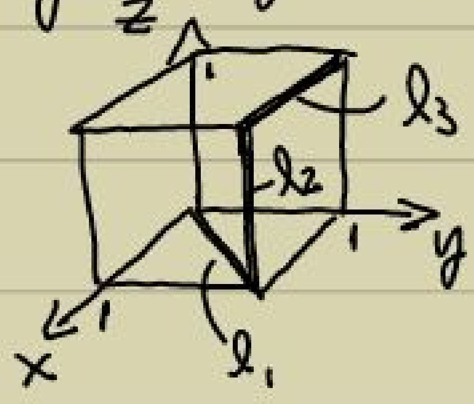
Thus

$$\int_X F\cdot d\vec{s} = \int_X M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$$

<u>Ex</u>: Compute $\int_X (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$.

1). $X(t) = (\underset{x}{t}, \underset{y}{t^2}, \underset{z}{t^3})$ $0 \le t \le 1$.

2)



1) $\int_0^1 (t^2+t^3)\,dt + (t+t^3)2t\,dt + (t+t^2)3t^2\,dt$
$= 3$.

2). l_1: $z=0$, $x=y$ from 0 to 1. $dz=0$

l_2: $x=y=1$, $dx=dy=0$, z from 0 to 1.

l_3: $y=z=1$, $dy=dz=0$, x from 1 to 0.

$$\int_{l_1} + \int_{l_2} + \int_{l_3} = \int_0^1 2x\,dx + \int_0^1 2\,dz + \int_1^0 2\,dx = 1+2-2 = 1.$$

substitution in line integrals.

Def: Let $X: [a,b] \to \mathbb{R}^3$ be a piecewise C' path.

A path $Y: [c,d] \to \mathbb{R}^3$ is said to be a reparameterization of X if there is a one-to-one onto C' function $u: [c,d] \to [a,b]$ such that $u^{-1}: [a,b] \to [c,d]$ is also C' and $Y(t) = X(u(t))$.

Ex. $X(t) = (1+2t, 2-t, 3+3t)$, $0 \le t \le 1$, traces a line segment from $(1,2,3)$ to $(3,1,6)$.

$Y(t) = (1+2t^2, 2-t^2, 3+3t^2)$, $0 \le t \le 1$, traces the same line segment. Set $u = t^2$, then $Y(t) = X(u(t))$.

THM. Let $X: [a,b] \to \mathbb{R}^n$ be a piecewise C' path and $f: \mathbb{R}^n \to \mathbb{R}$ be a continuous function. If $Y: [c,d] \to \mathbb{R}^n$ is any reparameterization of X, then

$$\int_Y f\,ds = \int_c^d f(Y(t))\|Y'(t)\|\,dt$$
$$= \int_X f\,ds = \int_a^b f(X(t))\|X'(t)\|\,dt.$$

THM Let $X:[a,b]\to\mathbb{R}^n$ be a piecewise C^1 path and $Y:[c,d]\to\mathbb{R}^n$ be a reparameterization of X.
Let $F:\mathbb{R}^n\to\mathbb{R}^n$ be a continuous vector field. Then

1) If Y is orientation-preserving, then $\int_Y F\cdot d\vec{s}=\int_X F\cdot d\vec{s}$
2) If Y is orientation-reversing, then $\int_Y F\cdot d\vec{s}=-\int_X F\cdot d\vec{s}$

__Ex.__ $X(t)=(t,t)$, $0\le t\le 1$.
$Y(t)=(2t,2t)$, $0\le t\le \frac{1}{2}$, is orientation preserving.
$Z(t)=(1-t,1-t)$, $0\le t\le 1$, is orientation reversing.

__Ex.__ Find the work done by the force
$F=x\mathbf{i}-y\mathbf{j}+(x+y+z)\mathbf{k}$ on a particle moving along the path $y=3x^2$, $z=0$ from $(0,0,0)$ to $(2,12,0)$.

set $\begin{cases} x=t \\ y=3t^2, \\ z=0. \end{cases}$ $\quad 0\le t\le 2.$

$$W=\int_0^2 F(X(t))\cdot X'(t)\,dt=\int_0^2 (t,-3t^2,t+3t^2)\cdot(1,6t,0)\,dt$$
$$=\int_0^2(t-18t^3)\,dt=\left(\frac{t^2}{2}-18\frac{t^4}{4}\right)\Big|_0^2=-70.$$

The particle actually moves from $(2,12,0)$ to $(0,0,0)$.