Several classes of multiple solution problems

Let B and V be Banach spaces and $F : \mathbb{R} \times B \to V$ be an operator.

Multiple Solution Problem: Given $\lambda \in \mathbb{R}$, find all $u \in B$ s.t. $F(\lambda, u) = 0$.

Multiple Fixed Point Problem: Find all $u \in B$ s.t. $F(u) = u$.

Nonlinear Eigen Problem: Find all $(\lambda, u) \in \mathbb{R} \times B$ s.t. $F(\lambda, u) = 0$,

λ is an eigenvalue, u is an eigenfunction related to the eigenvalue λ.

Bifurcation Problem: Find λ across which its multiplicity changes.

Nonlinear Eigenvalue Problem: Find $(\lambda, u) \in \mathbb{R} \times (B \setminus \{0\})$ s.t. $F(\lambda)u = 0$

where $F(\lambda) : B \to B^*$, e.g., $F(\lambda) = \lambda^2 A + \lambda B + C$ (matrix computation).

Nonlinear Eigenfunction Problem: Find $(\lambda, u) \in \mathbb{R} \times B$ s.t. $F(u) = \lambda G(u)$

where F and G are some operators from B into V.

Viable Problems: For given $F, G : B \to \mathbb{R}$ and $\lambda_0 > 0$,

find a largest subspace $S \subset B$ s.t. $F(u) \geq \lambda_0 G(u)\ \forall u \in S$.

Why multiple (unstable) solutions?

Multiple unstable solutions, lowly or highly, singly or multiply excited, to many nonlinear systems have been observed and mathematically proved to exist and have a variety of configurations, instabilities/maneuverabilities.

They used to be considered too hard/elusive to catch and therefore to apply.

Now scientists are able to induce, reach or control them with new advanced (synchrotronic, laser) technologies and search for NEW applications.

So far, people's knowledge on such solutions is still quite limited.

Traditional math analysis/numerical computation focus on stable solutions.

Thus development of efficient and reliable numerical methods to solve such problems becomes very interesting to both research and applications.

Due to Strong Nonlinearity, Multiplicity, unstable solutions are very elusive to traditional numerical methods. For different cases, one must develop Algorithm Implementable Characterization of Solutions in an Order:

(a) Single Equation vs System,

(b) Definite/Indefinite vs Strongly Indefinite

(c) Variational vs Non-Variational $\Rightarrow$2-level optimization vs n-person game,

(d) M-type problems vs W-type problems $\Rightarrow$ min-max vs max-min where

An M-type problem: whose energy function has an M-shaped indefinity (typically a mountain pass structure) along a direction, such as, the Lane-Emden-Flower, Henon, focusing Schrodinger equations/systems, etc. and

A W-type problem: whose energy function has a W-shaped indefinity (typically a double-well structure) along a direction, such as, the Allen-Cahn, defocusing Schrodinger equations/systems, etc.

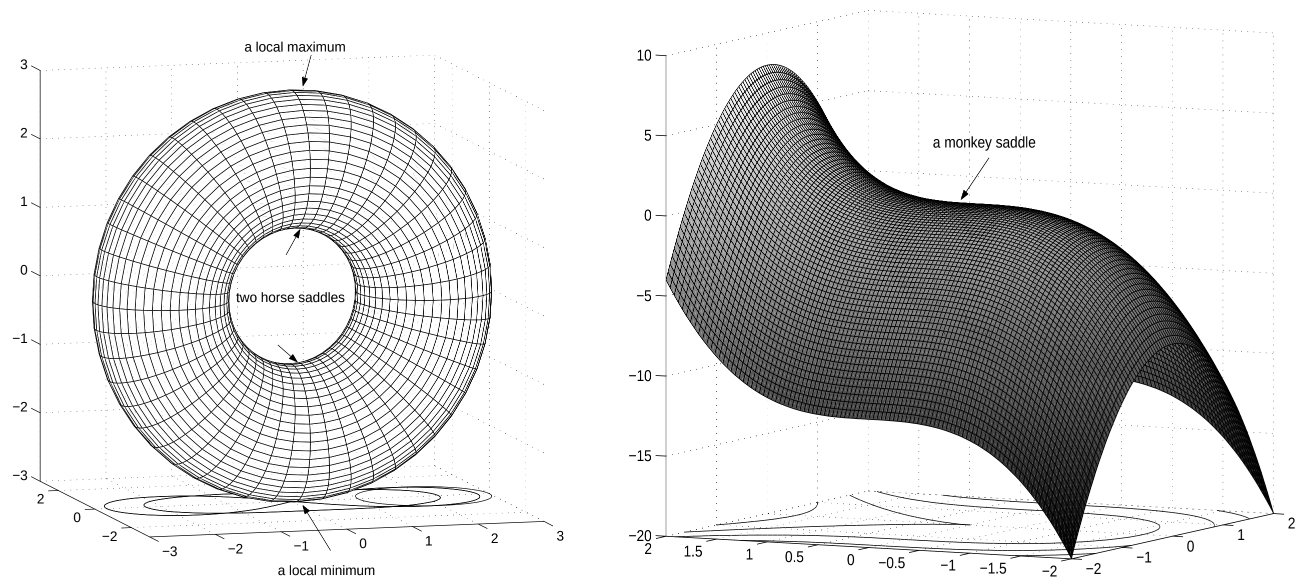

Figure 1: A local maximum, a local minimum and two horse saddles (minimax type) (left) and a monkey saddle (non-minimax type) (right).

Variational Problems: (Euler-Lagrange Equation): $A(u) \equiv J'(u) = 0$
for some $A : B \to B^*$, $J \in C^1(B, \mathbb{R})$ and J' its Frechet derivative

$$\delta J(u; v) = \frac{dJ(u + tv)}{dt}\Big|_{t=0} = \langle J'(u), v\rangle.$$

Solving E-L equation leads to compute multiple critical points.
The most well-studied critical points of J are the local extrema.
The classical critical point theory (Calculus of Variations) and traditional numerical methods focus on solving for such stable solutions.
Critical points u^* that are not local extrema are called saddle points, i.e., for any $\mathcal{N}(u^*) \subset B$, there exist $v, w \in \mathcal{N}(u^*)$ s.t. $J(v) < J(u^*) < J(w)$.
In physical systems, critical points are equilibrium states and saddle points are excited transient equilibrium states, thus unstable solutions.
Various constrained critical point problems:
Let $M \subset B$, $F : D(F) \subset B \to \mathbb{R}$. $u_0 \in M$ is a critical point of F in M, if $D(F)$ contains $\mathcal{N}(u_0)$ of u_0 s.t. (Euler-Lagrange equation)

$$\frac{d}{dt}F(u(t))|_{t=0} = 0, \ \forall u(t) \in M, \ t \in (-\varepsilon, \varepsilon), \ u(0) = u_0, \ u'(0) \text{ exists.}$$

If M has a tangent space TM_{u_0} at u_0, then $F'(u_0)h = 0 \quad \forall h \in \mathrm{TM}_{u_0}$.
If $M = \{u \in B | G(u) = 0\}$ where $G \in C^1(B, \mathbb{R})$, then $\exists \lambda \in \mathbb{R}$ s.t.

$$F'(u_0) = \lambda G'(u_0) \quad \text{or} \quad \mathcal{L}'(u_0) \equiv F'(u_0) - \lambda G'(u_0) = 0.$$

If $u_0 \in \mathrm{int}(M)$ then u_0 is called a (free) critical point of F, i.e., $F'(u_0) = 0$.
Generalized derivative (in the sense of Clarke): $0 \in \partial F(u_0)$.
(Geometric, Topological, Shape derivatives?)

Figure 2: Two typical mountain passes.