18. Let A be a nonsingular $n \times n$ matrix. Use mathematical induction to prove that A^m is nonsingular and
$$(A^m)^{-1} = (A^{-1})^m$$
for $m = 1, 2, 3, \ldots$.

19. Let A be an $n \times n$ matrix. Show that if $A^2 = O$, then $I - A$ is nonsingular and $(I - A)^{-1} = I + A$.

20. Let A be an $n \times n$ matrix. Show that if $A^{k+1} = O$, then $I - A$ is nonsingular and
$$(I - A)^{-1} = I + A + A^2 + \cdots + A^k$$

21. Given
$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
show that R is nonsingular and $R^{-1} = R^T$.

22. An $n \times n$ matrix A is said to be an *involution* if $A^2 = I$. Show that if G is any matrix of the form
$$G = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$
then G is an involution.

23. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$ (i.e., $\mathbf{u}^T\mathbf{u} = 1$) and let $H = I - 2\mathbf{u}\mathbf{u}^T$. Show that H is an involution.

24. A matrix A is said to be *idempotent* if $A^2 = A$. Show that each of the following matrices are idempotent:

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

25. Let A be an idempotent matrix.
(a) Show that $I - A$ is also idempotent.
(b) Show that $I + A$ is nonsingular and $(I + A)^{-1} = I - \frac{1}{2}A$

26. Let D be an $n \times n$ diagonal matrix whose diagonal entries are either 0 or 1.
(a) Show that D is idempotent.
(b) Show that if X is a nonsingular matrix and $A = XDX^{-1}$, then A is idempotent.

27. Let A be an involution matrix, and let
$$B = \frac{1}{2}(I + A) \quad \text{and} \quad C = \frac{1}{2}(I - A)$$
Show that B and C are both idempotent and $BC = O$.

28. Let A be an $m \times n$ matrix. Show that A^TA and AA^T are both symmetric.

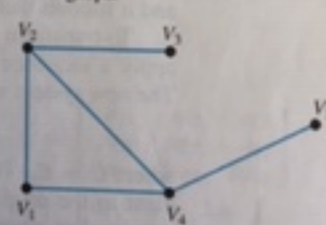
29. Let A and B be symmetric $n \times n$ matrices. Prove that $AB = BA$ if and only if AB is also symmetric.

30. Let A be an $n \times n$ matrix and let
$$B = A + A^T \quad \text{and} \quad C = A - A^T$$
(a) Show that B is symmetric and C is skew symmetric.
(b) Show that every $n \times n$ matrix can be represented as a sum of a symmetric matrix and a skew-symmetric matrix.

31. In Application 1, how many married women and how many single women will there be after 3 years?

32. Consider the matrix
$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$
(a) Draw a graph that has A as its adjacency matrix. Be sure to label the vertices of the graph.
(b) By inspecting the graph, determine the number of walks of length 2 from V_2 to V_3 and from V_2 to V_5.
(c) Compute the second row of A^3, and use it to determine the number of walks of length 3 from V_2 to V_3 and from V_2 to V_5.

33. Consider the graph



(a) Determine the adjacency matrix A of the graph.
(b) Compute A^2. What do the entries in the first row of A^2 tell you about walks of length 2 that start from V_1?
(c) Compute A^3. How many walks of length 3 are there from V_2 to V_4? How many walks of length less than or equal to 3 are there from V_2 to V_4?