(e) $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & -2 \\ 2 & 1 & 3 \end{vmatrix}$ (f) $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{vmatrix}$

(g) $\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$

(h) $\begin{vmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{vmatrix}$

4. Evaluate the following determinants by inspection:

(a) $\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 3 & -2 \end{vmatrix}$

(c) $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}$ (d) $\begin{vmatrix} 4 & 0 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 4 \\ 1 & 0 & 2 & 3 \end{vmatrix}$

5. Evaluate the following determinant. Write your answer as a polynomial in x.

$$\begin{vmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{vmatrix}$$

6. Find all values of λ for which the following determinant will equal 0:

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix}$$

7. Let A be a 3×3 matrix with $a_{11} = 0$ and $a_{21} \neq 0$. Show that A is row equivalent to I if and only if

$$\begin{aligned} &-a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} \\ &+ a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0 \end{aligned}$$

8. Write out the details of the proof of Theorem 1.3.

9. Prove that if a row or a column of an $n \times n$ matrix A consists entirely of zeros, then $\det(A) = 0$.

10. Use mathematical induction to prove that if A is an $(n+1) \times (n+1)$ matrix with two identical rows, then $\det(A) = 0$.

11. Let A and B be 2×2 matrices.
    (a) Does $\det(A + B) = \det(A) + \det(B)$?
    (b) Does $\det(AB) = \det(A)\det(B)$?
    (c) Does $\det(AB) = \det(BA)$?
    Justify your answers.

12. Let A and B be 2×2 matrices and let

$$C = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad D = \begin{bmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{bmatrix},$$
$$E = \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}$$

   (a) Show that $\det(A + B) = \det(A) + \det(B) + \det(C) + \det(D)$.
   (b) Show that if $B = EA$ then $\det(A + B) = \det(A) + \det(B)$.

13. Let A be a symmetric tridiagonal matrix (i.e., A is symmetric and $a_{ij} = 0$ whenever $|i - j| > 1$). Let B be the matrix formed from A by deleting the first two rows and columns. Show that

$$\det(A) = a_{11}\det(M_{11}) - a_{12}^2 \det(B)$$

2 Properties of Determinants

In this section, we consider the effects of row operations on the determinant of a matrix. Once these effects have been established, we will prove that a matrix A is singular if and only if its determinant is zero, and we will develop a method for evaluating determinants by using row operations. Also, we will establish an important theorem about the determinant of the product of two matrices. We begin with the following lemma:

Lemma 2.1 Let A be an $n \times n$ matrix. If A_{jk} denotes the cofactor of a_{jk} for $k = 1, \ldots, n$, then

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \end{cases}$$