13. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V, and let L_1 and L_2 be two linear transformations mapping V into a vector space W. Show that if

$$L_1(\mathbf{v}_i) = L_2(\mathbf{v}_i)$$

for each $i = 1, \ldots, n$, then $L_1 = L_2$ [i.e., show that $L_1(\mathbf{v}) = L_2(\mathbf{v})$ for all $\mathbf{v} \in V$].

14. Let L be a linear operator on $\mathbb{R}^1$ and let $a = L(1)$. Show that $L(x) = ax$ for all $x \in \mathbb{R}^1$.

15. Let L be a linear operator on a vector space V. Define L^n, $n \geq 1$, recursively by

$$L^1 = L$$
$$L^{k+1}(\mathbf{v}) = L(L^k(\mathbf{v})) \qquad \text{for all } \mathbf{v} \in V$$

Show that L^n is a linear operator on V for each $n \geq 1$.

16. Let $L_1: U \to V$ and $L_2: V \to W$ be linear transformations, and let $L = L_2 \circ L_1$ be the mapping defined by

$$L(\mathbf{u}) = L_2(L_1(\mathbf{u}))$$

for each $\mathbf{u} \in U$. Show that L is a linear transformation mapping U into W.

17. Determine the kernel and range of each of the following linear operators on $\mathbb{R}^3$:

(a) $L(\mathbf{x}) = (x_3, x_2, x_1)^T$

(b) $L(\mathbf{x}) = (x_1, x_2, 0)^T$

(c) $L(\mathbf{x}) = (x_1, x_1, x_1)^T$

18. Let S be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. For each linear operator L in Exercise 17, find $L(S)$.

19. Find the kernel and range of each of the following linear operators on P_3:

(a) $L(p(x)) = xp'(x)$

(b) $L(p(x)) = p(x) - p'(x)$

(c) $L(p(x)) = p(0)x + p(1)$

20. Let $L: V \to W$ be a linear transformation, and let T be a subspace of W. The *inverse image* of T, denoted $L^{-1}(T)$, is defined by

$$L^{-1}(T) = \{\mathbf{v} \in V \mid L(\mathbf{v}) \in T\}$$

Show that $L^{-1}(T)$ is a subspace of V.

21. A linear transformation $L: V \to W$ is said to be *one-to-one* if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies that $\mathbf{v}_1 = \mathbf{v}_2$ (i.e., no two distinct vectors $\mathbf{v}_1, \mathbf{v}_2$ in V get mapped into the same vector $\mathbf{w} \in W$). Show that L is one-to-one if and only if $\ker(L) = \{\mathbf{0}_V\}$.

22. A linear transformation $L: V \to W$ is said to map V *onto* W if $L(V) = W$. Show that the linear transformation L defined by

$$L(\mathbf{x}) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)^T$$

maps $\mathbb{R}^3$ onto $\mathbb{R}^3$.

23. Which of the operators defined in Exercise 17 are one-to-one? Which map $\mathbb{R}^3$ onto $\mathbb{R}^3$?

24. Let A be a 2×2 matrix, and let L_A be the linear operator defined by

$$L_A(\mathbf{x}) = A\mathbf{x}$$

Show that

(a) L_A maps $\mathbb{R}^2$ onto the column space of A.

(b) if A is nonsingular, then L_A maps $\mathbb{R}^2$ onto $\mathbb{R}^2$.

25. Let D be the differentiation operator on P_3, and let

$$S = \{p \in P_3 \mid p(0) = 0\}$$

Show that

(a) D maps P_3 onto the subspace P_2, but $D: P_3 \to P_2$ is not one-to-one.

(b) $D: S \to P_3$ is one-to-one but not onto.