12. Let A be an $m \times n$ matrix. Explain why the following are true:

 (a) Any vector $\mathbf{x}$ in $\mathbb{R}^n$ can be uniquely written as a sum $\mathbf{y} + \mathbf{z}$, where $\mathbf{y} \in N(A)$ and $\mathbf{z} \in R(A^T)$.

 (b) Any vector $\mathbf{b} \in \mathbb{R}^m$ can be uniquely written as a sum $\mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in N(A^T)$ and $\mathbf{v} \in R(A)$.

13. Let A be an $m \times n$ matrix. Show that

 (a) if $\mathbf{x} \in N(A^TA)$, then $A\mathbf{x}$ is in both $R(A)$ and $N(A^T)$.

 (b) $N(A^TA) = N(A)$.

 (c) A and A^TA have the same rank.

 (d) if A has linearly independent columns, then A^TA is nonsingular.

14. Let A be an $m \times n$ matrix, B an $n \times r$ matrix, and $C = AB$. Show that

 (a) $N(B)$ is a subspace of $N(C)$.

 (b) $N(C)^{\perp}$ is a subspace of $N(B)^{\perp}$ and, consequently, $R(C^T)$ is a subspace of $R(B^T)$.

15. Let U and V be subspaces of a vector space W. Show that if $W = U \oplus V$, then $U \cap V = \{\mathbf{0}\}$.

16. Let A be an $m \times n$ matrix of rank r and let $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ be a basis for $R(A^T)$. Show that $\{A\mathbf{x}_1, \ldots, A\mathbf{x}_r\}$ is a basis for $R(A)$.

17. Let $\mathbf{x}$ and $\mathbf{y}$ be linearly independent vectors in $\mathbb{R}^n$ and let $S = \text{Span}(\mathbf{x}, \mathbf{y})$. We can use $\mathbf{x}$ and $\mathbf{y}$ to define a matrix A by setting

$$A = \mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{x}^T$$

 (a) Show that A is symmetric.

 (b) Show that $N(A) = S^{\perp}$.

 (c) Show that the rank of A must be 2.