SECTION 4 EXERCISES

1. Let $\mathbf{x} = (-1, -1, 1, 1)^T$ and $\mathbf{y} = (1, 1, 5, -3)^T$. Show that $\mathbf{x} \perp \mathbf{y}$. Calculate $\|\mathbf{x}\|_2$, $\|\mathbf{y}\|_2$, $\|\mathbf{x}+\mathbf{y}\|_2$ and verify that the Pythagorean law holds.

2. Let $\mathbf{x} = (1, 1, 1, 1)^T$ and $\mathbf{y} = (8, 2, 2, 0)^T$.
   (a) Determine the angle θ between $\mathbf{x}$ and $\mathbf{y}$.
   (b) Find the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$.
   (c) Verify that $\mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{p}$.
   (d) Compute $\|\mathbf{x}-\mathbf{p}\|_2$, $\|\mathbf{p}\|_2$, $\|\mathbf{x}\|_2$ and verify that the Pythagorean law is satisfied.

3. Use equation (1) with weight vector $\mathbf{w} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)^T$ to define an inner product for $\mathbb{R}^3$, and let $\mathbf{x} = (1, 1, 1)^T$ and $\mathbf{y} = (-5, 1, 3)^T$.
   (a) Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal with respect to this inner product.
   (b) Compute the values of $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ with respect to this inner product.

4. Given

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 1 & 1 \\ -3 & 3 & 2 \\ 1 & -2 & -2 \end{bmatrix}$$

determine the value of each of the following:
   (a) $\langle A, B\rangle$ (b) $\|A\|_F$
   (c) $\|B\|_F$ (d) $\|A+B\|_F$

5. Show that equation (2) defines an inner product on $\mathbb{R}^{m\times n}$.

6. Show that the inner product defined by equation (3) satisfies the last two conditions of the definition of an inner product.

7. In $C[0, 1]$, with inner product defined by (3), compute
   (a) $\langle e^x, e^{-x}\rangle$ (b) $\langle x, \sin \pi x\rangle$ (c) $\langle x^2, x^3\rangle$