3. Find an equation for the plane tangent to the surface

$$x = e^s, \qquad y = t^2 e^{2s}, \qquad z = 2e^{-s} + t$$

at the point $(1, 4, 0)$.

4. Let $\mathbf{X}(s,t) = (s^2 \cos t, s^2 \sin t, s)$, $-3 \le s \le 3$, $0 \le t \le 2\pi$.

(a) Find a normal vector at $(s,t) = (-1, 0)$.

(b) Determine the tangent plane at the point $(1, 0, -1)$.

(c) Find an equation for the image of $\mathbf{X}$ in the form $F(x, y, z) = 0$.

5. Consider the parametrized surface $\mathbf{X}(s,t) = (s, s^2 + t, t^2)$.

(a) Graph this surface for $-2 \le s \le 2$, $-2 \le t \le 2$. (Using a computer may help.)

(b) Is the surface smooth?

(c) Find an equation for the tangent plane at the point $(1, 0, 1)$.

6. Describe the parametrized surface of Exercise 1 by an equation of the form $z = f(x, y)$.

7. Let S be the surface parametrized by $x = s\cos t$, $y = s\sin t$, $z = s^2$, where $s \ge 0$, $0 \le t \le 2\pi$.

(a) At what points is S smooth? Find an equation for the tangent plane at the point $(1, \sqrt{3}, 4)$.

(b) Sketch the graph of S. Can you recognize S as a familiar surface?

(c) Describe S by an equation of the form $z = f(x, y)$.

(d) Using your answer in part (c), discuss whether S has a tangent plane at every point.

8. Verify that the image of the parametrized surface

$$\mathbf{X}(s,t) = (2\sin s\,\cos t, 3\sin s\,\sin t, \cos s),$$
$$0 \le s \le \pi, \quad 0 \le t \le 2\pi,$$

is an ellipsoid.

9. Verify that, for the torus of Example 5, the s-coordinate curve, when $t = t_0$, is a circle of radius $a + b\cos t_0$.

10. The surface in $\mathbf{R}^3$ parametrized by

$$\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, \theta), \quad r \ge 0, \quad -\infty < \theta < \infty,$$

is called a **helicoid**.

(a) Describe the r-coordinate curve when $\theta = \pi/3$. Give a general description of the r-coordinate curves.

(b) Describe the θ-coordinate curve when $r = 1$. Give a general description of the θ-coordinate curves.

(c) Sketch the graph of the helicoid (perhaps using a computer) for $0 \le r \le 1$, $0 \le \theta \le 4\pi$. Can you see why the surface is called a helicoid?

11. Given the sphere of radius 2 centered at $(2, -1, 0)$, find an equation for the plane tangent to it at the point $(1, 0, \sqrt{2})$ in three ways:

(a) by considering the sphere as the graph of the function

$$f(x, y) = \sqrt{4 - (x-2)^2 - (y+1)^2};$$

(b) by considering the sphere as a level surface of the function

$$F(x, y, z) = (x-2)^2 + (y+1)^2 + z^2;$$

(c) by considering the sphere as the surface parametrized by

$$\mathbf{X}(s,t) = (2\sin s\,\cos t + 2, 2\sin s\,\sin t - 1, 2\cos s).$$

In Exercises 12–15, represent the given surface as a piecewise smooth parametrized surface.

12. The lower hemisphere $x^2 + y^2 + z^2 = 9$, including the equatorial circle.

13. The part of the cylinder $x^2 + z^2 = 4$ lying between $y = -1$ and $y = 3$.

14. The closed triangular region in $\mathbf{R}^3$ with vertices $(2, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 5)$.

15. The hyperboloid $z^2 - x^2 - y^2 = 1$. (Hint: Use two maps to parametrize the surface.)

16. This problem concerns the parametrized surface $\mathbf{X}(s,t) = (s^3, t^3, st)$.

(a) Find an equation of the plane tangent to this surface at the point $(1, -1, -1)$.

(b) Is this surface smooth? Why or why not?

(c) Use a computer to graph this surface for $-1 \le s \le 1$, $-1 \le t \le 1$.

(d) Verify that this surface may also be described by the xyz-coordinate equation $z = \sqrt[3]{xy}$. Try using a computer to graph the surface when described in this form. Many software systems will have trouble, or will provide an incomplete graph, which is one reason why parametric descriptions of surfaces are desirable.

17. The surface given parametrically by $\mathbf{X}(s,t) = (st, t, s^2)$ is known as the **Whitney umbrella**.

(a) Verify that this surface may also be described by the xyz-coordinate equation $y^2 z = x^2$.

(b) Is $\mathbf{X}$ smooth?

(c) Use a computer to graph this surface for $-2 \le s \le 2$, $-2 \le t \le 2$.

(d) Some points (x, y, z) of the surface do not correspond to a single parameter point (s, t). Which ones? Explain how this relates to the graph.

(e) Give an equation of the plane tangent to this surface at the point $(2, 1, 4)$.