Computational Theory and Methods for Solving Multiple Solution Problems

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1 Introduction

Let B and V be Banach spaces and $F : \mathbb{R} \times B \to V$ be an operator.

Multiple Solution Problem: Given $\lambda \in \mathbb{R}$, find all $u \in B$ s.t. $F(\lambda, u) = 0$.

Multiple Fixed Point Problem: Find all $u \in B$ s.t. $F(u) = u$.

Nonlinear Eigen Problem: Find all eigensolutions $(\lambda, u) \in \mathbb{R} \times B$ s.t.

$$F(\lambda, u) = 0, \tag{1.1}$$

λ is an eigenvalue, u is an eigenfunction corresponding to the eigenvalue λ. An eigenvalue may have multiple eigenfunctions. It leads to

Bifurcation Problem: Find λ_0 s.t. its multiplicity changes as λ crosses λ_0.

Nonlinear Eigenvalue Problem: Find $(\lambda, u) \in \mathbb{R} \times (B \setminus \{0\})$ s.t. $F(\lambda)u = 0$ where $F(\lambda) : B \to V$, e.g., the quadratic eigenvalue problem:

$$F(\lambda) = \lambda^2 A + \lambda B + C.$$

Nonlinear Eigenfunction Problem: Find $(\lambda, u) \in \mathbb{R} \times B$ s.t. $F(u) = \lambda G(u)$ where F and G are some operators from B into V.

Variational Multiple Solution Problem: find all $u \in B$ s.t.

$$A(u) \equiv J'(u) = 0 \quad \text{(Euler-Lagrange equation)}$$

for some $A : B \to B^*$, $J \in C^1(B, \mathbb{R})$ and J' its Frechet derivative.
It leads to compute multiple critical points u.
The most well-studied critical points of J are the local extrema.
The classical critical point theory (Calculus of Variations) and traditional numerical methods focus on solving for such stable solutions.
Critical points u^* that are not local extrema are called saddle points, i.e., for any $u^* \in \mathcal{N}(u^*) \subset B$, there exist $v, w \in \mathcal{N}(u^*)$ such that

$$J(v) < J(u^*) < J(w).$$

In physical systems, critical points are equilibrium states and saddles appear as excited transient equilibrium states, thus unstable solutions.

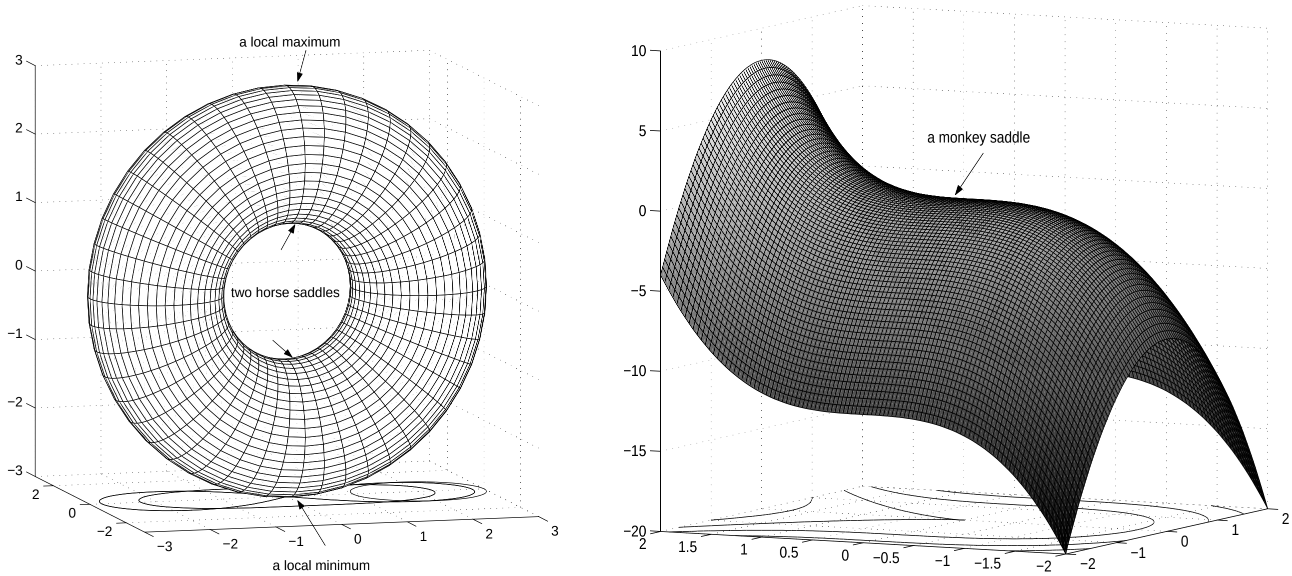


Figure 1: A local maximum, a local minimum and two horse saddles (minimax type) (left) and a monkey saddle (non-minimax type) (right).

Various Constrained Critical Points:

Let $M \subset B$, $F : D(F) \subset B \to \mathbb{R}$. $u_0 \in M$ is a **critical point** of F in M, if $D(F)$ contains $\mathcal{N}(u_0)$ of u_0 s.t. (Euler-Lagrange equation)

$$\frac{d}{dt}F(u(t))|_{t=0} = 0,\ \forall u(t) \in M,\ t \in (-\varepsilon, \varepsilon),\ u(0) = u_0,\ u'(0) \text{ exists.}$$

If M has a tangent space TM_{u_0} at u_0, then $F'(u_0)h = 0 \quad \forall h \in \mathrm{TM}_{u_0}$.

If $M = \{u \in B | G(u) = 0\}$ where $G \in C^1(B, \mathbb{R})$, then $\exists \lambda \in \mathbb{R}$ s.t.

$$\mathcal{L}'(u_0) \equiv F'(u_0) - \lambda G'(u_0) = 0, \tag{1.2}$$

where $\mathcal{L}(u) \equiv F(u) - \lambda G(u)$ is the Lagrange functional. If $u_0 \in \mathrm{int}(M)$ then u_0 is called a **(free) critical point** of F, i.e.,

$$F'(u_0) = 0. \tag{1.3}$$

Generalized derivative (in the sense of Clarke): $0 \in \partial F(u_0)$.

(Geometric, topological, shape derivatives.)

An Application. Consider a semilinear Schrodinger equation

$$iw_t + \Delta w + \kappa f(|w|)w = 0. \tag{1.4}$$

To finding the solitary wave solutions $w(x,t) = u(x)e^{-i\lambda t}$, and the steady-state solutions $w(x,t) = u(x)$, (1.4) leads to semilinear elliptic PDEs

$$\lambda u(x) + \Delta u(x) + \kappa f(|u(x)|)u(x) = 0, \quad u \in W^{1,2}(\Omega); \tag{1.5}$$

$$-\Delta u(x) = \kappa f(|u(x)|)u(x), \quad u \in W^{1,2}(\Omega). \tag{1.6}$$

When a fluid/material is non Darcian/Newtonian, the Darcy's law is replaced by others. One is to replace Δu by $\Delta_p u(x) = \nabla \cdot (|\nabla u(x)|^{p-2}\nabla u(x))$, (1.5) and (1.6) are generalized to two quasilinear elliptic PDEs on $W^{1,p}(\Omega)$

$$\lambda u(x) + \Delta_p u(x) + \kappa f(|u(x)|)u(x) = 0; \tag{1.7}$$

$$-\Delta_p u(x) = \kappa f(|u(x)|)u(x). \tag{1.8}$$

When $p = 2, p < 2, p > 2$, the fluid/material is called, Darcian/Newtonian, pseudo-plastic, dilatant, respectively.
People want to know for what values of λ and κ, (1.7) and (1.8) have solutions.

When $f(|t|)t = \frac{d}{dt}F(t)$ for some function F, (1.7) and (1.8) are the Euler-Lagrange equations of the (energy) functionals on $W_0^{1,p}(\Omega)$

$$J(u,\lambda) := \int_\Omega \Big[\frac{1}{p}|\nabla u(x)|^p - \kappa F(u(x)) - \frac{\lambda}{2}(|u(x)|^2 - \alpha)\Big]dx; \tag{1.9}$$

$$J(u) := \int_\Omega \Big[\frac{1}{p}|\nabla u(x)|^p - \kappa F(u(x))\Big]dx. \tag{1.10}$$

Global Theory and Methods:

Most results in the literature focus mainly on the existence issue (L-S, mountain pass, various linking) and characterize a saddle point as a solution to a two-level global minimax problem

$$\min_{A \in \mathcal{A}} \max_{u \in A} J(u) \tag{1.11}$$

where $\mathcal{A}$ is a collection of certain compact sets A (e.g., a k-D simplex), max and min are global. **Thus not for algorithm implementation**.

The Mountain Pass Lemma of Ambrosetti-Rabinowitz (1973) uses global min in the outer loop and global max along each continuous path connecting two given points in the inner loop.

A mountain pass solution is the solution characterized by the mountain pass lemma. Thus finding a mountain pass solution is equivalent to solving a two-level global optimization problem. So far no such algorithm exists.

The mountain pass method of Choi-McKenna (1993) uses a 1-D simplex along each direction in the inner loop and a local min in the outer loop. (MI=1)
The high-linking method of Ding-Costa-Chen (1999) uses a 2-D simplex in the inner loop and a local min in the outer loop. (MI=1,2)
Though there are gaps in their mathematical justification and no convergence verifications due to no stepsize rules in their algorithms, they opened new doors to numerical computations of multiple critical points.
Nehari (1960) proved that a global min of J on the Nehari manifold

$$\mathcal{N} = \{t_u u : u \in H, \|u\| = 1, t_u > 0, \langle J'(t_u u), u\rangle = 0\}$$

yields a saddle point.

Motivated by the above works and the Morse theory, we developed an algorithm implementable local minimax method for finding multiple critical points, in which the Nehari manifold $\mathcal{N}$ is generalized to a solution set $\mathcal{M}$.

Basic idea: Let L be a closed subspace of H, called a support, spanned by trivial/known solutions, from which an algorithm search wants to stay away. Define a composite function $J(p(u))$ s.t. $p(u) \in [L,u] \setminus L,\ J'(p(u)) \perp [L,u]$. Then design an algorithm to search for u^* s.t. $J'(p(u^*)) \perp L^\perp$.
Consequently $w^* = p(u^*)$ is a saddle point not in L.
There are many ways to do so depending on a specific problem, e.g.,
local min/max solutions $p(u)$ of J on $[L,u]$ satisfy $J'(p(u)) \perp [L,u]$,
a solution $u^* \in L^\perp$ of a local min/max of $J(p(u))$ satisfies $J'(p(u^*)) \perp L^\perp$.
It leads to a two-level local optimization problem: a local minimax method, a very powerful method and can be modified/generalized in many directions.

Since all min and max are local, we only need some local structure. The mountain pass lemma requires a global mountain pass structure:

$$0 < t_u = \arg\max_{t>0} J(tu) < +\infty, \quad \forall u \in B, \|u\| = 1. \tag{1.12}$$

We need only a local mountain pass structure, i.e., (1.12) holds only in an open set $U \subset S_B$ and a barrier forms on ∂U, e.g.,

$$\max_{t>0} J(tu) = +\infty, \quad \forall u \in \partial U.$$

But analysis becomes much more complicated.

2 A Local Minimax Method [Li-Zhou, 2001, 2002, SISC]

Let H be a Hilbert space, $J \in C^1(H, \mathbb{R})$, $L \subset H$ a closed subspace.
Denote $S_{L^\perp} = \{v \in L^\perp : \|v\| = 1\}$ and $[L, v] = \text{span}\{L, v\}, v \in S_{L^\perp}$.
All min and max are in local sense.

Definition 1. *The peak mapping* $P : S_{L^\perp} \to 2^H$ *s.t.*

$$P(v) := \left\{ v^* := \arg \max_{u \in [L,v]} J(u) \right\}, \quad \forall v \in S_{L^\perp}.$$

A peak selection $p : S_{L^\perp} \to H$ *if* $p(v) \in P(v), \forall v \in S_{L^\perp}$.
If p *is locally defined, then* p *is called a local peak selection. (Only local structure)*

Remark 1. We have $p(v) = t_v v + v_L$ for some $t_v \neq 0, v_L \in L$ if $p(v) \notin L$, and $J'(p(v)) \perp [L, v],\ J'(p(v)) \perp p(v),\ \forall v \in S_{L^\perp}$.
In many cases, such a selection p is unique. If it is not unique, then strategies in an algorithm will be designed to consistently track a peak selection.

Lemma 1. *If p is a local peak selection of J near $v_0 \in S_{L^\perp}$ s.t. $J'(p(v_0)) \neq 0$, p is continuous at v_0 and $p(v_0) = t_0 v_0 + v_0^L \notin L$, then there is $s_0 > 0$ s.t. when $0 < s < s_0$,*

$$J(p(v(s))) - J(p(v_0)) < -\frac{t_0 s}{2} \|J'(p(v_0))\|^2 \quad \text{(a stepsize rule)} \tag{2.1}$$

where $v(s) := \dfrac{v_0 - sJ'(p(v_0))}{\|v_0 - sJ'(p(v_0))\|}$. ∎

Theorem 1. (A local minimax characterization) *If p is a local peak selection of J near $v_0 \in S_{L^\perp}$ s.t. p is continuous at $v_0, p(v_0) \notin L$ and*

$$v_0 := \arg \min_{v \in S_{L^\perp}} J(p(v)) = \arg \min_{v \in S_{L^\perp}} \max_{u \in [L,v]} J(u),$$

then $u_0 = p(v_0)$ is a saddle point of J. ∎

PS+$J(p(\cdot))$ bounded below$\Longrightarrow$an existence theorem by Ekeland's VP.

Denote a solution set (a generalization of Nehari's manifold when $L = \{0\}$)

$$\mathcal{M} := \{p(v) : v \in S_{L^\perp}\}.$$

Then $\min_{u \in \mathcal{M}} J(u)$ yields a saddle $u^* = p(v^*)$ that can be approximated by, e.g., a steepest descent method. Inequality (2.1) defines a stepsize rule in the algorithm (to prove convergence).

2.1 A Local Minimax Algorithm

Let $w_1, ..., w_{n-1}$ be n-1 previously found critical points, $L = [w_1, ..., w_{n-1}]$. Given $\varepsilon > 0, \lambda > 0$ and $v^0 \in S_{L^\perp}$ be an ascent-descent direction at w_{n-1}.

Step 1: Let $t_0^0 = 1$, $v_L^0 = 0$ and set $k = 0$;

Step 2: Using the initial guess $w = t_0^k v^k + v_L^k$, solve for

$$w^k \equiv p(v^k) = \arg \max_{u \in [L, v^k]} J(u), \ \text{denote } t_0^k v^k + v_L^k = w^k \equiv p(v^k);$$

Step 3: Compute the steepest descent vector $d^k := -J'(w^k)$;

Step 4: If $\|d^k\| \leq \varepsilon$ then output $w_n = w^k$, stop; else goto Step 5;

Step 5: Set $v^k(s) := \dfrac{v^k + sd^k}{\|v^k + sd^k\|} \in S_{L^\perp}$ and find

$$s^k := \max_{m \in \mathbb{N}} \left\{ \frac{\lambda}{2^m} : 2^m > \|d^k\|, J(p(v^k(\frac{\lambda}{2^m}))) - J(w^k) \leq -\frac{t_0^k \lambda}{2^{m+1}} \|d^k\|^2 \right\}.$$

Initial guess $u = t_0^k v^k(\frac{\lambda}{2^m}) + v_L^k$ is used to find $p(v^k(\frac{\lambda}{2^m}))$ where t_0^k and v_L^k are found in **Step 2**. (track a peak selection)

Step 6: Set $v^{k+1} := v^k(s^k)$ and update $k = k + 1$ then goto **Step 2**.

Some remarks for the algorithm.

2.2 Some Numerical Examples

Consider solving the Henon equation

$$\begin{cases} -\Delta u(x) = |x|^r |u(x)|^{q-1} u(x) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Set $r = 0$, $q = 3$.

Constructing initial guesses v^0 for a desired profile is flexible, one may use sin / cos functions or solve

$$\begin{cases} -\Delta v^0(x) = cv(x) & x \in \Omega, \\ v^0(x) = 0 & x \in \partial\Omega \end{cases}$$

where $\mathrm{cv}(x) = +/-\ (1)$ if we want v^0 to be concave up/down at x and $\mathrm{cv}(x) = 0$ if its concavity at x is not of concern.

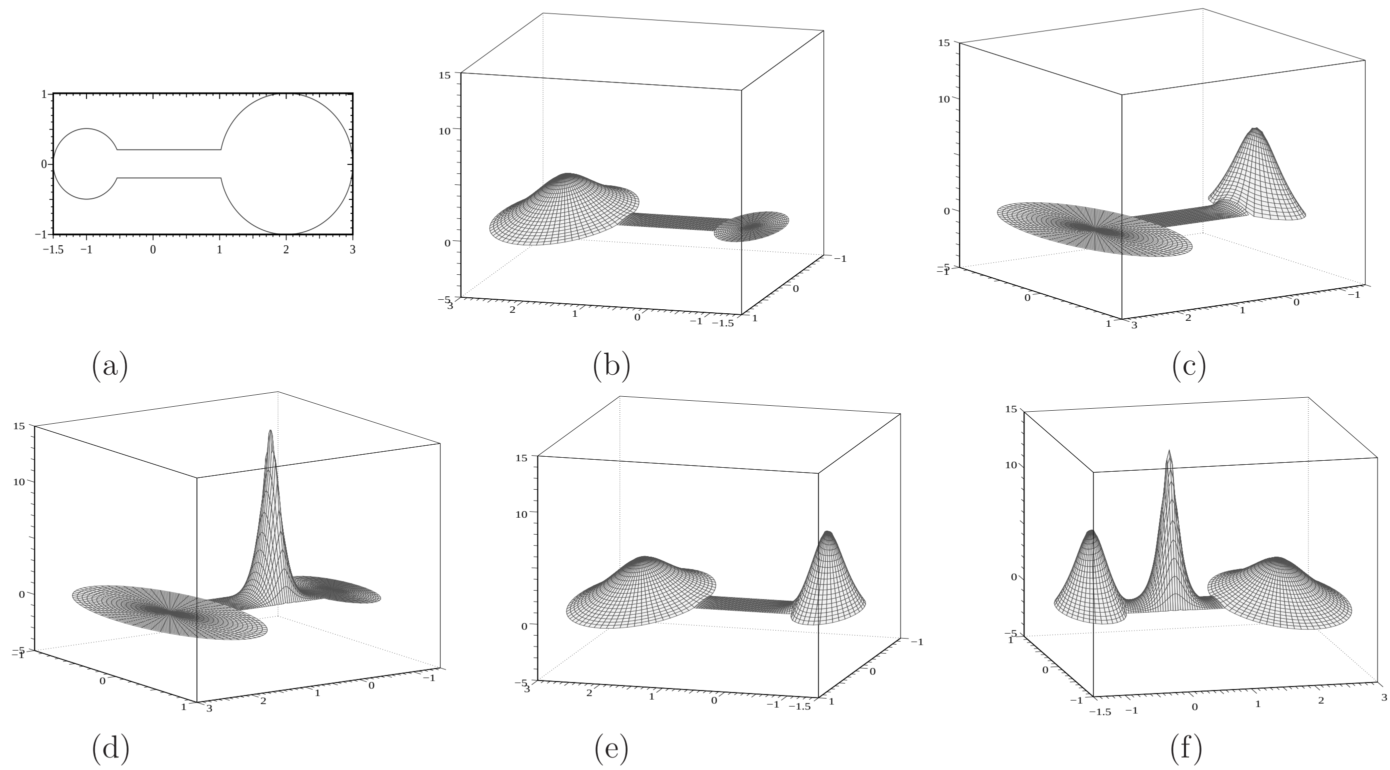


Figure 2: (a) The dumbbell-shaped domain. (b) the ground state with $J = 10.90, u_{\max}^{(1)} = 3.652$. (c) The second one-peak positive solution with $J = 42.22, u_{\max}^{(2)} = 7.037$. (d) The third one-peak positive solution with $J = 159.0, u_{\max}^{(3)} = 13.63$. (e) A two-peak positive solution with $J = 53.12, u_{\max}^{(4)} = 7.037$. (f) A three-peak positive solution with $J = 212.5, u_{\max}^{(5)} = 13.78$.

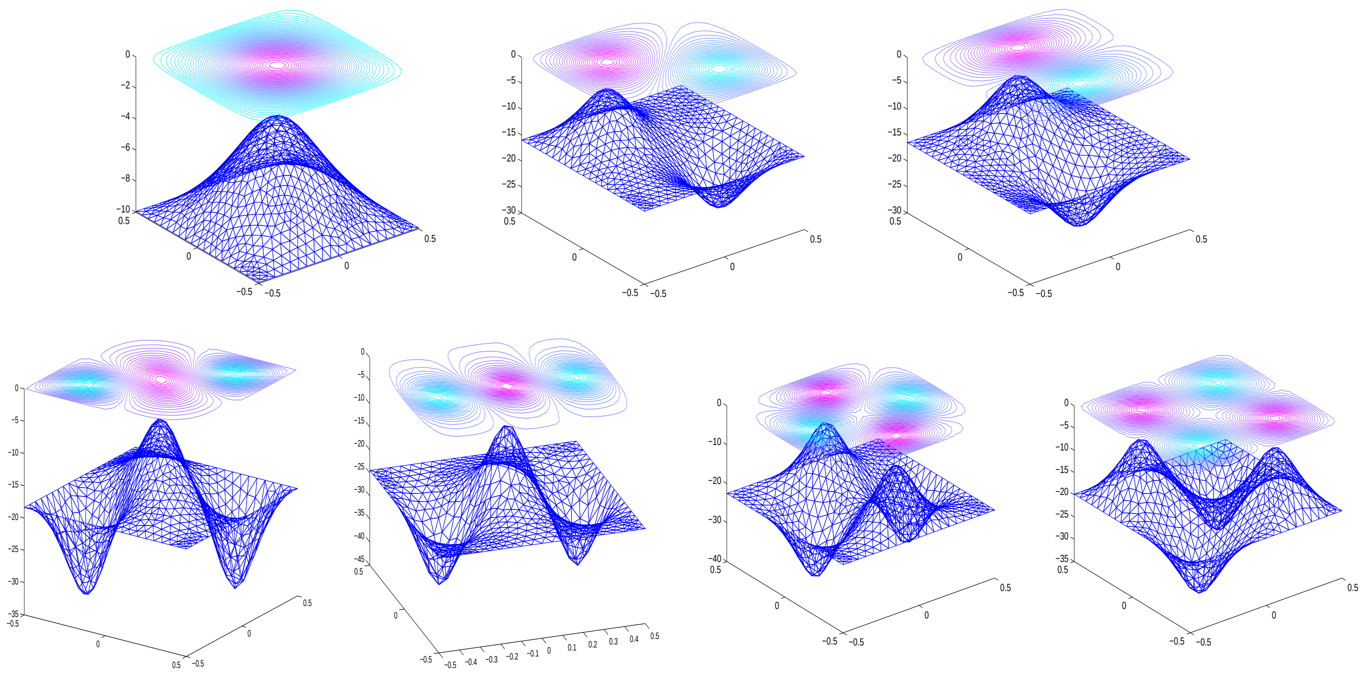


Figure 3: The first 7 solutions $u^{(1)} - u^{(7)}$.

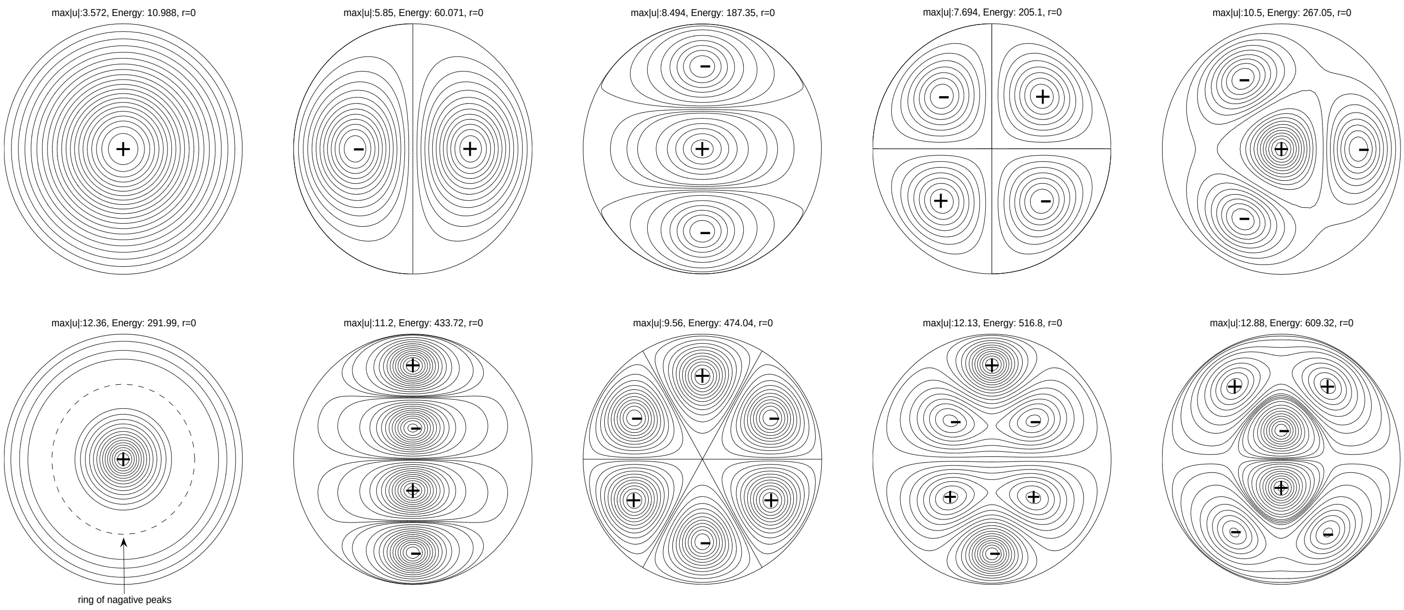


Figure 4: The contours of the first 10 solutions to (2.2) with $r = 0$ and $q = 3$. The sign "+/-" represents a positive/negative peak. The 6th contour plot shows a radial sign-changing solution, where the dashed circle denotes the ring of negative peaks (a 1-dimensional peak set).

2.3 Convergence Results

Assume $w^k = p(v^k)$ be generated by the algorithm and PS condition.

Theorem 2. *If (1) p is continuous, (2) $d(L, w^k) > \alpha > 0$ and (3) $\inf_{v \in S_{L^\perp}} J(p(v)) > -\infty$, then*
(a) $\{v^k\}$ has a subsequence $\{v^{k_i}\}$ s.t. $w^{k_i} = p(v^{k_i}) \to$ a saddle of J;
(b) any convergent subsequence of $\{w^k\}$ converges to a saddle of J. ∎

A point-to-set convergence results is also established.

Theorem 3. *If $\bar{v} = \arg\min_{v \in S_{L^\perp}} J(p(v))$ where p is continuous and $p(\bar{v}) \notin L$ is an isolated critical point, then there exists an open set V in H, $\bar{v} \in V \cap S_{L^\perp}$, s.t. $\forall v^0 \in V \cap S_{L^\perp}$, $w^k \to p(\bar{v})$.* ∎

2.4 Instability Analysis of Saddles [Zhou, 2005, Math. Comp.]

Saddle points are unstable. Can we measure their instability?

Morse Index. If $J''(u^*)$ is a self-adjoint Fredholm operator at u^*,

$$H = H^- \oplus H^0 \oplus H^+ \tag{2.3}$$

where H^-, H^0 and H^+ are respectively m.n.d., the null and m.p.d. subspaces of $J''(u^*)$ in H with $\dim(H^0) < \infty$.

The *Morse index* of the critical point u^* is $\mathrm{MI}(u^*) = \dim(H^-)$.

If u^* is nondegenerate, i.e., $H^0 = \{0\}$, then $\mathrm{MI}(u^*)$ can be used to measure local instability of a critical point. But MI is very expensive to compute, not useful to degenerate cases, and not defined in Banach spaces.

Order of Saddles is of particular interests in computational chemistry/physics, since the subspace B^D corresponds to the reaction coordinates.

Definition 2. *Let $J \in C^1(B, R)$ and $u^* \in B$. If $B = B^I \oplus B^D$ for some subspaces B^I, B^D in B and for each $u_1 \in B^I$ and $u_2 \in B^D$ with $\|u_1\| = 1$ and $\|u_2\| = 1$ there exist $r_1 > 0$ and $r_2 > 0$ s.t.*

$$J(u^* + tu_1) \; > \; J(u^*), \; \forall \, 0 < |t| \leq r_1, \tag{2.4}$$

$$J(u^* + tu_2) \; < \; J(u^*), \; \forall \, 0 < |t| \leq r_2. \tag{2.5}$$

Then u^* is a saddle point of J and $\dim(B^D)$ = *order of saddle* u^*.

Since (2.4) and (2.5) lack of characterization and robustness, and are difficult to apply in a Banach space. Thus we replace (2.5) by

$$J(u^* + tu_2 + o(t)) < J(u^*), \; \forall \, 0 < |t| \leq r_2 \tag{2.6}$$

and define *local saddle index* (LSI)=dim(H^D).

Then in a Hilbert space H, we have

$$\dim(L)+1 \le \mathrm{MI}\ (u^*)+\dim(H^0 \cap [L, v^*]) \le \mathrm{MI}\ (u^*)+\dim(H^0 \cap H^D) = \mathrm{LSI}(u^*).$$

If u^* is a nondegenerate saddle, then $\mathrm{LSI}(u^*) = \mathrm{MI}(u^*)$.

But (2.4) and (2.6) do not concern degeneracy and thus are more general.

Theorem 4. *If p is a local peak selection differentiable at $v^* \in S_{L^\perp}$, $u^* = p(v^*) \notin L$ and $v^* = \arg\min_{v \in S_{L'}} J(p(v))$, then u^* is a critical point with*

$$\dim(L) + 1 = MI(u^*) + \dim(H^0 \cap [L, v^*]); \tag{2.7}$$

If the max with p is strict, then u^* is a saddle of LSI$=\dim(L) + 1$. ∎

The number $\dim(L) + 1$ is known before u^* is computed and called local minimax index (LMI) of u^*.

Remark 2. Usually $\mathrm{MI}(u^*)$ is computed with two steps. 1st find u^* then compute $\mathrm{MI}(u^*)$, thus very expensive.
Here we reverse the process, LMM utilizes the geometric and topological structure of LMI to numerically find a saddle point with such LMI.

Question: How to check if p is continuous?
$u^* = \lim_{k\to\infty} p(v^k)$ need not be a local maximum. On the other hand, non-minimax saddle points, e.g., the Monkey saddles, do exist.
Minimax principle cannot cover them. Need more general framework.
Note that we have $\quad J'(p(v))\perp[L,v], \quad \forall v \in S_{L^\perp}.$

2.5 A Local Min-$\perp$ Method [Zhou, 2004, JMAA]

Definition 3. *$P\colon S_{L^\perp} \to 2^H$ is an L-$\perp$ mapping if*

$$P(v) := \{u \in [L, v] : J'(u) \perp [L, v]\} \quad \forall v \in S_{L^\perp}.$$

$p\colon S_{L^\perp} \to H$ is an L-$\perp$ selection if $p(v) \in P(v)$ $\forall v \in S_{L^\perp}$.
If p is locally defined then p is a local L-$\perp$ selection. ∎

p is a peak selection $\Longrightarrow$ p is an L-$\perp$ selection.
Assume p is an L-$\perp$ selection. All the previous results remain true.

Lemma 2. *If J is C^1, then $G = \{(u, v) : v \in S_{L^\perp}, u \in P(v)\}$ is closed.*

Theorem 5. *Let p be continuous at $v^* \in S_{L^\perp}$ and $p(v^*) \notin L$, then $u^* = p(v^*)$ is a critical point of J iff there is $\mathcal{N}(v^*)$ s.t.*

$$J'(p(v^*)) \perp p(v) - p(v^*), \quad \forall v \in \mathcal{N}(v^*) \cap S_{L^\perp}. \tag{2.8}$$

Remark 3. No $J(\cdot)$ but only $A(\cdot) = J'(\cdot)$ is involved. A potentially useful result to solve non-variational problems $A(u) = 0$ for multiple solutions.

For a composite function $\mathcal{J}(v) = J(p(v))$. Design a two-level method: inner-loop: $p(v) \notin L$, $J'(p(v)) \perp [L, v]$, outer-loop: find v^* s.t. $J'(p(v^*)) \perp L^\perp$ through min or max. LMM is a steepest descent method for a continuous function $J(p(v))$ due to the orthogonal properties of p.

2.6 Differentiability of an L-$\perp$ Selection p

Check if p is differentiable or not. Let $L = [w_1, w_2, .., w_n]$ and $v \in S_{L^\perp}$.

By the definition of p, $u^* = t_0 v + t_1 w_1 + ... + t_n w_n = p(v)$ is solved from (n+1) orthogonal conditions, for $j = 1, ..., n$,

$$\begin{aligned} F_0(v, t_0, t_1, ..., t_n) &:= \langle J'(t_0 v + t_1 w_1 + ... + t_n w_n), v \rangle = 0, \\ F_j(v, t_0, t_1, ..., t_n) &:= \langle J'(t_0 v + t_1 w_1 + ... + t_n w_n), w_j \rangle = 0. \end{aligned}$$

Then we have $(n+1)\times(n+1)$ terms

$$\begin{aligned}
\frac{\partial F_0}{\partial t_0} &= \langle J''(t_0 v + t_1 w_1 + ... + t_n w_n)v, v\rangle, \\
\frac{\partial F_0}{\partial t_i} &= \langle J''(t_0 v + t_1 w_1 + ... + t_n w_n)w_i, v\rangle, \quad i = 1, 2, ..., n. \\
\frac{\partial F_j}{\partial t_0} &= \langle J''(t_0 v + t_1 w_1 + ... + t_n w_n)v, w_j\rangle, \quad j = 1, 2, ..., n. \\
\frac{\partial F_j}{\partial t_i} &= \langle J''(t_0 v + t_1 w_1 + ... + t_n w_n)w_i, w_j\rangle, \quad i, j = 1, 2, ..., n.
\end{aligned}$$

By the implicit function theorem, if the $(n+1)\times(n+1)$ matrix

$$Q'' := \begin{bmatrix} \langle J''(u^*)v, v\rangle, & \langle J''(u^*)w_1, v\rangle, & ... & \langle J''(u^*)w_n, v\rangle \\ \langle J''(u^*)v, w_1\rangle, & \langle J''(u^*)w_1, w_1\rangle, & ... & \langle J''(u^*)w_n, w_1\rangle \\ ... & & ... & \\ \langle J''(u^*)v, w_n\rangle, & \langle J''(u^*)w_1, w_n\rangle, & ... & \langle J''(u^*)w_n, w_n\rangle \end{bmatrix} \tag{2.9}$$

is invertible or $|Q''| \neq 0$, where $u^* = t_0 v + t_1 w_1 + ... + t_n w_n = p(v)$, then p is differentiable at and near v. This can be easily and numerically checked.

2.7 On Symmetry Invariance [Wang-Z,04,05,SINUM,Chen-Z,07,NMPDE]

Newton's method is fast but does not use the variational structure, cannot handle degenerate cases, strongly depends on an initial guess. Thus it is blind to the instability order (may miss the ground states). On the other hand, in [36, 37], we proved that both LMM and Newton's methods are invariant to symmetries. But the LMM's is sensitive to numerical errors, i.e., when the numerical errors dominate the magnitude of the gradient, a symmetry may be broken depending if the support L is sufficient or not. (Haar projection)
The Newton's is insensitive to numerical errors. It traps a symmetry.
The symmetry of an initial guess must match that of the final solution.
It is good and bad depending on if or not you know the symmetries.
The dependence of LMM is very loose. But only a first order convergence can be expected. The best way is LMM+Newton method

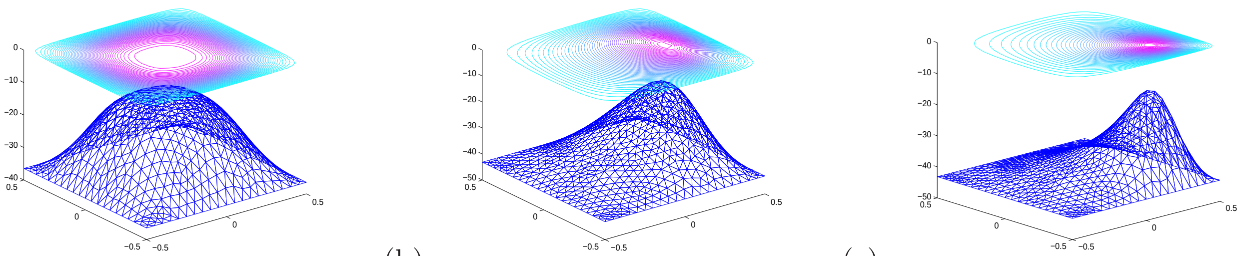
Example. The Henon equation, $r = 2$ in (2.2).

Gidas-Ni-Nirenberg theorem for symmetry does not apply.

Newton or LMM+Newton? $\mathrm{It}_{\mathrm{LMM}} = 4, 14, 20$, or $\varepsilon < 10^{-2}$ (why?).

LMM and Newton's method are invariant to symmetries. But LMM's is sensitive to numerical errors, thus the symmetry may be broken depending on if or not the errors dominate and/or the support L is sufficient.

The Newton's is insensitive to numerical errors. It traps a symmetry. Thus symmetries of an initial guess must match or be less than that of the solution.



(a) (b) (c)

Figure 5: (a) a positive symmetric solution (MI$\geq$ 9) (b) an asymmetric positive solution (MI$\geq$ 2) and (c) an asymmetric ground state. Positive solutions with more peaks may appear if r increases.

3 Several Important Directions

(1) Find multiple saddles in Banach spaces: [Yao-Zhou,05,SISC,07,SINUM]

When non Newton/Darcian fluids/materials are considered, Darcian's law is replaced by other laws. One of them is to replace Δ by Δ_p $(p > 1)$. Then solitary wave and steady-state solutions to a "Schrodinger flow" lead to solve quasilinear elliptic (Eigen) PDE on $W^{1,p}(\Omega)$

$$\lambda u(x) + \Delta_p u(x) + \kappa f(|u(x)|)u(x) = 0; \tag{3.1}$$

$$-\Delta_p u(x) = \kappa f(|u(x)|)u(x). \tag{3.2}$$

Need a pseudo-gradient to replace the gradient.

Definition 4. *Let $u \in X$ be a point s.t. $J'(u) \neq 0$. For given $\theta \in (0,1]$, a point $\Psi(u) \in X$ is a pseudo-gradient of J at u w.r.t. θ if*

$$\|\Psi(u)\| \leq 1, \quad \langle J'(u), \Psi(u)\rangle \geq \theta \|J'(u)\|.$$

Pseudo-gradients are used to find a local minimum of a C^1 functional in a Banach space. However, when saddle points are concerned, such pseudo-gradients do not work, since they will lead to a local minimum.

To prevent such a degeneracy, we need a projected pseudo-gradient. Thanks to p. A projected pseudo-gradient can be constructed at $p(u)$.

Lemma 3. *Let $0 < \theta < 1$ be given. For $v_0 \in S_{L'}$, if p is a local peak selection of J w.r.t. L at v_0 s.t. $J'(p(v_0)) \neq 0$, then there exists a projected pseudo-gradient $\Psi(p(v_0))$ of J at $p(v_0)$ w.r.t. θ s.t.*

(a) $\Psi(p(v_0)) \in L'$, $0 < \|\Psi(p(v_0))\| \leq M$ where $M \geq 1$ is the bound of the linear projection $\mathcal{P}$ from X to L';

(b) $\langle J'(p(v_0)), \Psi(p(v_0)) \rangle \geq \theta \|J'(p(v_0))\|$.

Lemma 4. *([40]) For $v_0 \in S_{L'}$, if there is a local peak selection p of J w.r.t. L at v_0 satisfying (1) p is continuous at v_0, (2) $d(p(v_0), L) > 0$ and (3) $d = J'(p(v_0)) \neq 0$, then there exists $s_0 > 0$ s.t. as $0 < s < s_0$*

$$J(p(v(s))) - J(p(v_0)) < -\frac{s\theta|t_s|\|d\|}{4} \leq -\frac{\theta d(p(v_0), L)\|d\|}{8M}\|v(s) - v_0\| \quad (3.3)$$

where $p(v_0) = t_0 v_0 + w_0$, $p(v_s) = t_s v_s + w_s$ with $t_0, t_s \neq 0$ and $w_0, w_s \in L$,

$$v(s) = \frac{v_0 - sign(t_0)s\Psi(p(v_0))}{\|v_0 - sign(t_0)s\Psi(p(v_0))\|}$$

and $\Psi(p(v_0))$ is a projected pseudo-gradient of J at $p(v_0)$. Thus if $v^* = \arg\min_{v \in S_{L'}} J(p(v))$, then $J'(p(v^*)) = 0$.

Brave to use $\Psi(p(v)) = \frac{-\nabla J(p(v))}{\|\nabla J(p(v))\|_p}$ and cautious to show it is a PPG if $\frac{\|\nabla J(p(v))\|_2^2}{\|\nabla J(p(v))\|_p \|\nabla J(p(v))\|_q} > \theta > 0$. See [40].

(2) Constrained nonlinear eigen-function problem:

$$F'(u) = \lambda G'(u), \quad \text{s.t. } G(u) = \alpha.$$

Case 1. Iso-homogeneous cases $m = \ell$. (Yao-Zhou,07,SISC)

Define the Raleigh quotient $\mathcal{R}(u) = \dfrac{F(u)}{G(u)}$ (degenerate everywhere). Then

$$\mathcal{R}'(u) = 0 \Longleftrightarrow F'(u) = \lambda G'(u), \ \left(\lambda = \mathcal{R}(u) = \frac{F(u)}{G(u)}\right).$$

Order of eigenfunctions coincides with order of their LMI=dim(L).

Case 2. Non iso-hom, non hom cases. (Yao-Zhou,08,SISC)

Its Lagrange functional

$$\mathcal{L}(\lambda, u) = F(u) - \lambda(G(u) - \alpha)$$

may fail to have a mountain-pass structure.

Introduce a new active Lagrange functional

$$\mathcal{L}(\lambda, u) = F(u) - \ell(\lambda)(G(u) - \alpha)$$

where $\ell(\lambda)$ is an active Lagrange multiplier whose selection is to let $\mathcal{L}(\lambda, u)$ have a mountain-pass structure, e.g., $\ell(\lambda) = |\lambda|^k, k = 2$ and to make the algorithm converge. See [44].

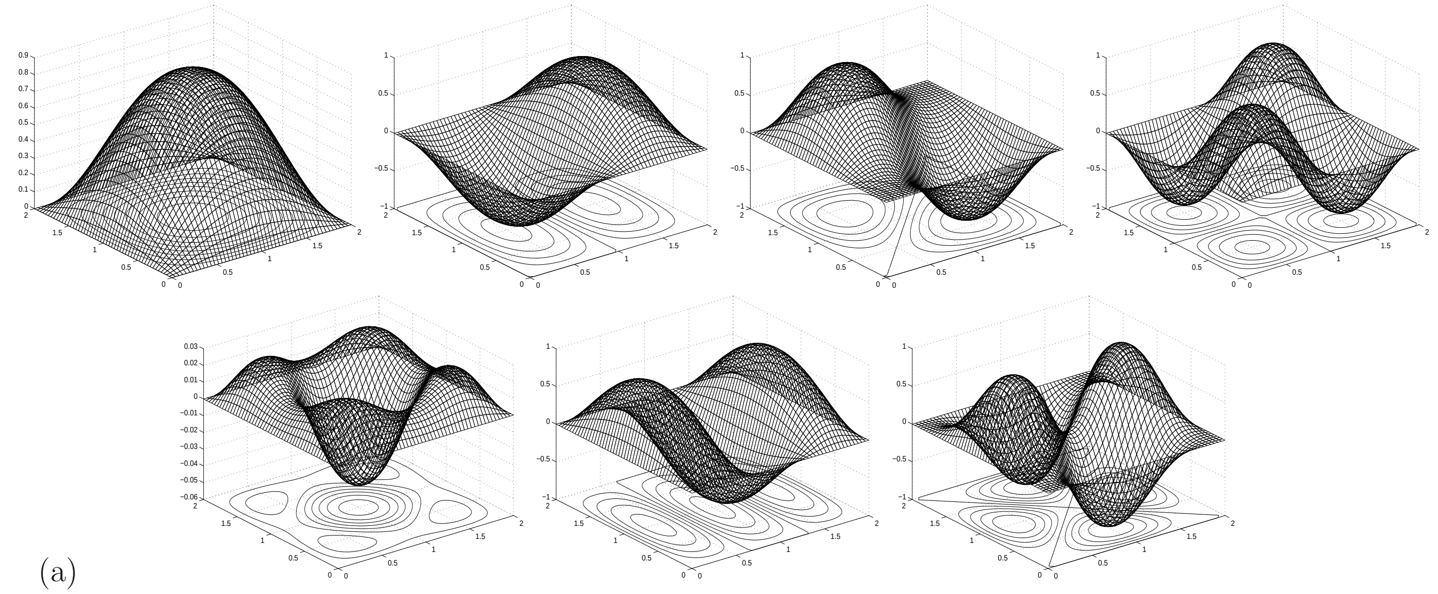
Eigenpairs of the p-Laplacian: $-\Delta_p u(x) = \lambda |u(x)|^{q-2} u(x), u \in W_0^{1,p}(\Omega)$.

Newton's method cannot be applied due to the quasi-nonlinearity.

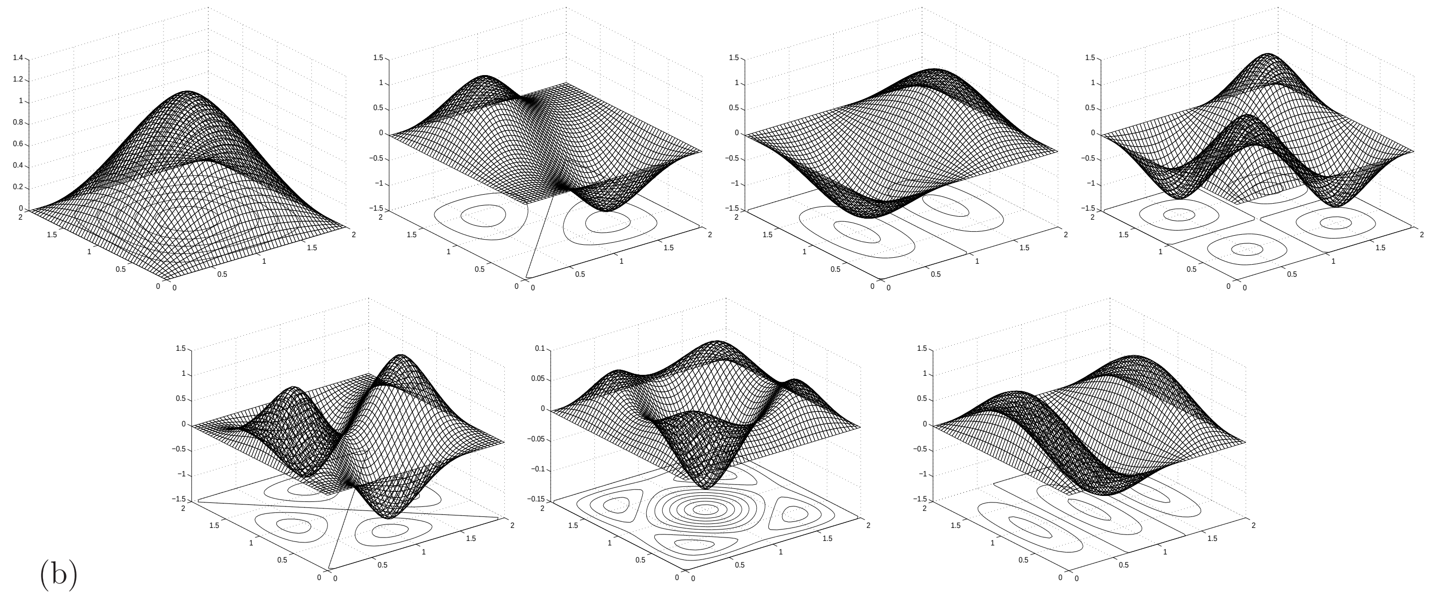
Case a. $p = q = 1.75 < 2, \lambda_1 = 4.2458, \lambda_2 = 9.3173, \lambda_3 = 9.4078,$
$\lambda_4 = 14.2805, \lambda_5 = 16.8378, \lambda_6 = 17.2546, \lambda_7 = 23.3660.$

Case b. $p = q = 2.5 > 2, \lambda_1 = 6.3547, \lambda_2 = 20.2896, \lambda_3 = 20.79854,$
$\lambda_4 = 35.9448, \lambda_5 = 48.2598, \lambda_6 = 49.6794, \lambda_7 = 51.1048.$

Their solutions profiles are listed below. Pay attention to the pattern order changes of the second-third and the 5-7th solutions to the two cases.



(a)



(b)

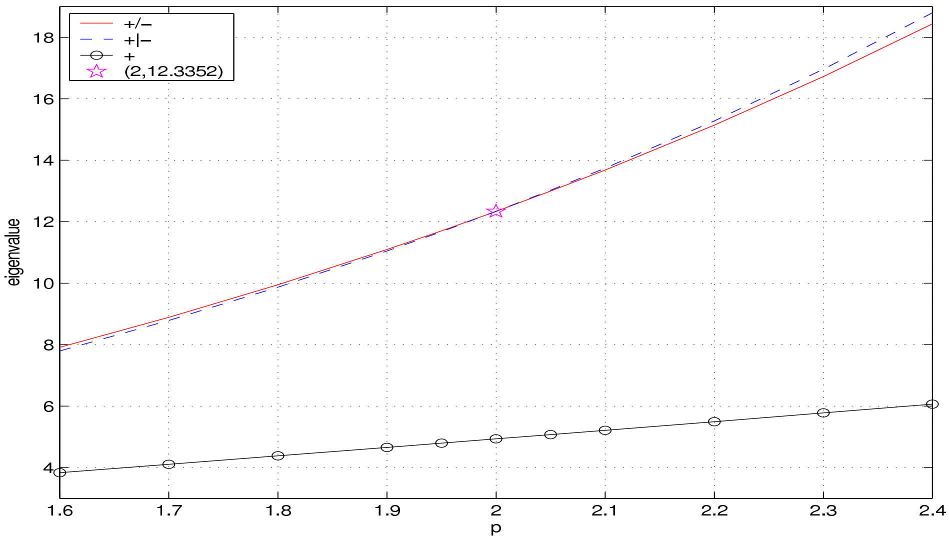


Figure 6: The first three eigenvalues of $-\Delta_p$ for $1.6 \leq p \leq 2.4$.

It is interesting to point out that from the above figure we see that repeated eigenvalues of $-\Delta$ can be separated. E.g., the second eigenvalue of $-\Delta$ is doubled and can be separated by changing $p = 2$ to $p \neq 2$.

Gross-Pitaevskii equation in Bose-Einstein condensates: [Yao-Zhou,08,SISC]

$$iw_t = -\frac{1}{2}\Delta w + V(x)w + \beta|w|^2 w, \quad t > 0, x \in \Omega \subseteq \mathbb{R}^d, \tag{3.4}$$

$$w(x,t) = 0, \quad x \in \Gamma = \partial\Omega, t \geq 0, \tag{3.5}$$

where w is the macroscopic wave function of the condensate,
$V(x) = \frac{1}{2}(\gamma_1^2 x_1^2 + \cdots + \gamma_d^2 x_d^2)$ with $\gamma_1, ..., \gamma_d > 0$ is a trapping potential,
$\beta > 0$ measures a repulsive nonlinearity.
An important invariant is the *normalization* of the wave-function

$$\int_\Omega |w(x,t)|^2 dx = 1. \tag{3.6}$$

Finding the solitary wave solutions $w(x,t) = u(x)e^{-i\mu t}$ to (3.4) leads to

$$\mu u(x) = -\frac{1}{2}\Delta u(x) + V(x)u(x) + \beta|u(x)|^2 u(x), \; x \in \Omega \text{ s.t. } \int_\Omega |u(x)|^2 dx = 1, \tag{3.7}$$

a non homogeneous NEP for eigensolutions $(\mu, u) \in \mathbb{R} \times W_0^{1,2}(\Omega)$.

The active Lagrange functional of (3.7) is

$$J(\lambda,u)=\frac{1}{4}\int_{\Omega}\Big[|\nabla u(x)|^2+2V(x)u^2(x)+u^4(x)\Big]dx-\frac{|\lambda|^k}{2}\Big(\int_{\Omega}u^2(x)dx-1\Big). \tag{3.8}$$

Set $k=2.1$, $\Omega=(-10,10)\times(-10,10)\subset\mathbb{R}^2$. Note "localized" property.

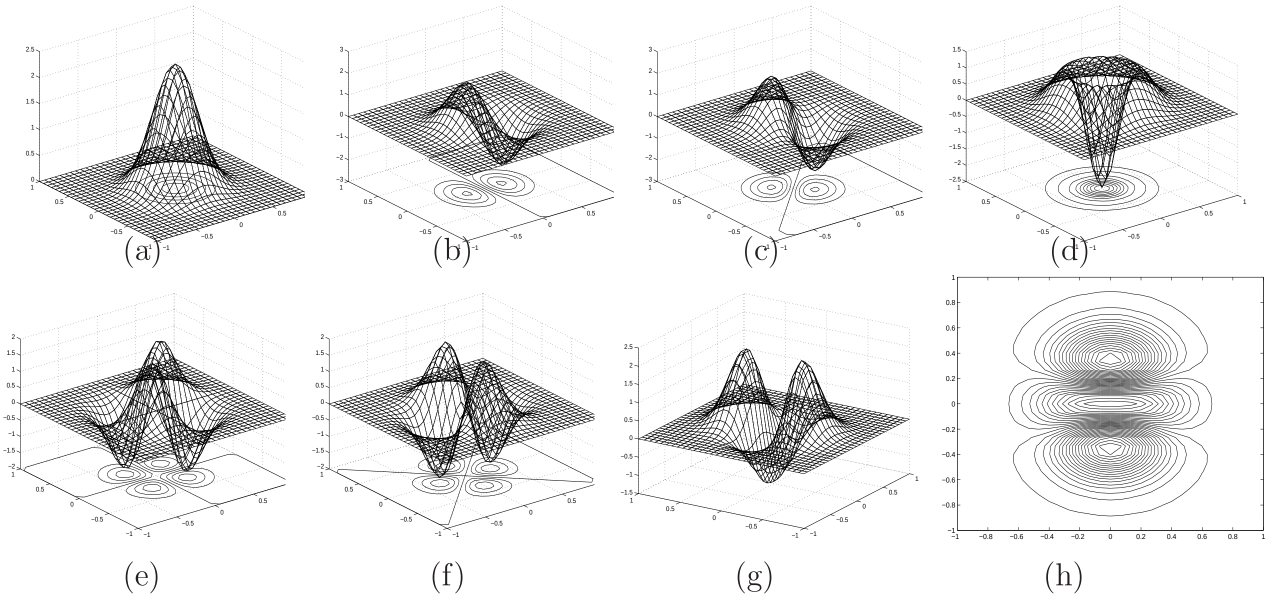


Figure 7: The first 7 eigenfunctions w(x)=10u(10x) of (3.7) with $\beta=1,\gamma_1=\gamma_2=0.2$ and their eigenvalues (a) 0.229817, (b) 0.422992, (c) 0.422995, (d) 0.615675, (e) 0.617554, (f) 0.617599 and (g) 0.618999. (h) The contours of w in (g).

(3) Find multiple nonsmooth critical points. [Yao-Zhou,04, Math. Prog.(B)]

Let $J : B \to R$ be a locally Lipschitz continuous functional and $\partial J(u)$ be the generalized gradient of J at $u \in B$ in the sense of Clarke [5]. $0 \in \partial J(u^*)$

Lemma 5. *Let $B = H$ and p be a local peak selection of J w.r.t. L at $v \in S_{L^\perp}$ s.t. p is continuous at v, dis$(p(v), L) > 0$ and $z \in \partial J(p(v))$ with $\|z\| = \min\{\|w\| : w \in \partial J(p(v))\} > 0$. Then when $s > 0$ is small*

$$J(p(v(s))) - J(p(v)) < -\frac{1}{4}|t_v|\|z\|^2, \tag{3.9}$$

where $v(s) = \dfrac{v - sign(t_v)sz_{L^\perp}}{\|v - sign(t_v)sz_{L^\perp}\|}$, $p(v) = t_v v + w_v, w_v \in L$ *and* $z = z_L + z_{L^\perp}, z_L \in L, z_{L^\perp} \in L^\perp$.

Theorem 6. *Let $B = H$ and p be a local peak selection of J w.r.t. L at $v \in S_{L^\perp}$ s.t. p is continuous at v, dis$(p(v), L) > 0$ and* $J(p(v)) = local\text{-}\min_{u \in S_{L^\perp}} J(p(u))$. *Then $p(v)$ is a critical point of J.*

Next let $B = L \oplus L'$ be a reflexive Banach space.

Definition 5. *For $u_0 \in B$, let $\mu = \min\{\|z\|_{B^*} : z \in \partial J(u_0)\}$. The pseudo-generalized-gradient (PGG) of J at u_0 is the set*

$$\Psi J(u_0) = \{z^* \in B : \|z^*\| = \mu, \inf_{w \in \partial J(u_0)} \langle w, z^* \rangle \geq \langle z, z^* \rangle = \mu^2, \\ z \in \partial J(u_0), \|z\|_{B^*} = \mu\}.$$

Lemma 6. *The PGG $\Psi J(u_0)$ of J at $u_0 \in B$ is a nonempty, convex set in B. If in addition, B^* is locally uniformly convex and $\|\cdot\|_{B^*}$ is Frechet differentiable on $B^* \setminus \{0\}$, then $\Psi J(u_0) = \{\|z\|_{B^*}\|z\|'_{B^*}\}$ where z is the unique point of minimum norm in $\partial J(u_0)$.*

Lemma 7. *Let p be a local peak selection of J w.r.t L at $v \in S_{L'}$ s.t.*

(1) *p is continuous at v and $dis(p(v), L) > 0$,*

(2) *$z^* \in B$ is the PGG of J at $p(v)$ with $\|z^*\| > 0$. If s is small then*

$$J(p(v(s))) - J(p(v)) < -\frac{1}{4}s|t_v|\|z\|^2_{B^*}$$

where $v(s) = \dfrac{v - sign(t_v)sz^*_{L'}}{\|v - sign(t_v)sz^*_{L'}\|}$, $p(v) = t_v v + w_v, w_v \in L, z^* = z^*_L + z^*_{L'}$, $z^*_L \in L$, $z^*_{L'} \in L'$ *and z is a point of minimum norm in $\partial J(p(v))$.*

Theorem 7. *Let p be a local peak selection of J w.r.t L at $v \in S_{L'}$ s.t.*

(1) *p is continuous at v and $dis(p(v), L) > 0$,*

(2) $J(p(v)) = local\text{-}\min_{u \in S_{L'}} J(p(u))$.

Then $p(v)$ is a critical point of J, i.e., $0 \in \partial J(p(v))$.

Multiple Solutions to Systems

When a nonlinear process involves multi-bodies (particles, molecules, species, etc.), it leads to a system, e.g., a semilinear Schrödinger system

$$\begin{cases} i\bar{u}_t(t,x) + \Delta\bar{u}(t,x) + \kappa g(|\bar{u}(t,x)|, |\bar{v}(t,x)|)\bar{u}(t,x) &= 0, \\ i\bar{v}_t(t,x) + \Delta\bar{v}(t,x) + \rho h(|\bar{u}(t,x)|, |\bar{v}(t,x)|)\bar{v}(t,x) &= 0 \end{cases} \tag{3.10}$$

for some physical parameters κ, ρ. Standing wave/ and steady state solutions

$$(\bar{u}, \bar{v}) = (e^{-i\lambda t}u(x), e^{-i\mu t}v(x)), \quad (\bar{u}, \bar{v}) = (u(x), v(x))$$

lead to study semilinear elliptic (eigen) systems on Ω

$$\begin{cases} \lambda u(x) + \Delta u(x) + \kappa g(|u(x)|, |v(x)|)u(x) &= 0, \\ \mu v(x) + \Delta v(x) + \rho h(|u(x)|, |v(x)|)v(x) &= 0, \end{cases}$$

$$\begin{cases} \Delta u(x) + \kappa g(|u(x)|, |v(x)|)u(x) &= 0, \\ \Delta v(x) + \rho h(|u(x)|, |v(x)|)v(x) &= 0, \end{cases}$$

with zero Dirichlet or Neumann boundary condition (B.C.).

Comparing to their single equation counterparts, nonlinear systems are much richer in varieties and complexities, and can be classified in many different ways. We focus on developing theory and methods for finding multiple solutions to a class of semilinear elliptic systems on Ω:

$$\begin{cases} -\Delta u(x) &= G_1'(u(x), v(x), x), \\ -\Delta v(x) &= H_2'(u(x), v(x), x), \end{cases}$$

$$J_1(u,v) = \int_\Omega [\tfrac{1}{2}|\nabla u(x)|^2 - G(u(x), v(x), x)]dx,$$
$$J_2(u,v) = \int_\Omega [\tfrac{1}{2}|\nabla v(x)|^2 - H(u(x), v(x), x)]dx,$$

with zero Dirichlet or Neumann B.C., where $G, H \in C^1(\mathbb{R}^2 \times \Omega, \mathbb{R})$ usually contain higher order terms satisfying some growth conditions, and G_1' and H_2' are their partial Frechet derivatives w.r.t u and v variables, respectively.

u and v interact each other in many different ways on energy profiles J_1, J_2, e.g., strongly coupled, weakly coupled, cooperative, noncooperative.

(4) Multiple co-existing solutions to variational systems [Chen-Z-Yao,07,JANM]

View a solution of the form $(0, v)$ or $(u, 0)$ as trivial.

A Cooperative system and its variational functional

$$-\Delta u(x) - \lambda u(x) = F'_u(u, v), \ -\Delta v(x) - \mu v(x) = F'_v(u, v).$$

$$J(u, v) = \int_\Omega \Big[\frac{1}{2}(|\nabla u(x)|^2 - \lambda u^2(x) + |\nabla v(x)|^2 - \mu v^2(x)) - F(u(x), v(x))\Big] dx.$$

A Noncooperative system and its variational functional (strongly indefinite)

$$-\Delta u(x) - \lambda u(x) = F'_u(u, v), \ -\Delta v(x) - \mu v(x) = -F'_v(u, v).$$

$$J(u, v) = \int_\Omega \Big[\frac{1}{2}(|\nabla u(x)|^2 - \lambda u^2(x) - |\nabla v(x)|^2 + \mu v^2(x)) - F(u(x), v(x))\Big] dx,$$

where F satisfies certain growth conditions.

So $(0, 0)$ is a saddle (critical) point of J.

Let H_1 and H_2 be Hilbert spaces, $H = H_1 \times H_2$, $J \in C^1(H, \mathbb{R})$, $L_1 \subset H_1$ and $L_2 \subset H_2$ be closed subspaces. Denote $L = L_1 \times L_2$.

Definition 6. *A set-valued mapping P: $S_{L^\perp} \to 2^H$ is called an L-$\perp$ mapping of J if* $\forall\, \bar{w} = (\bar{u}, \bar{v}) \in S_{L^\perp}$,

$$P(\bar{w}) = \Big\{(u, v) \in [L_1, \bar{u}] \times [L_2, \bar{v}] : J_1'(u, v) \perp [L_1, \bar{u}], J_2'(u, v) \perp [L_2, \bar{v}].\Big\}.$$

A mapping $p : S_{L^\perp} \to H$ is an L-$\perp$ selection of J if $p(w) \in P(w)$, $\forall w \in S_{L^\perp}$. For $w \in S_{L^\perp}$, if p is locally defined near w, then p is called a local L-$\perp$ selection of J at w.

Remark 4. **(a)** Definition 6 is bi-$\perp$ and stronger than the original one, since

$$J_1' \perp [L_1, u], \ J_2' \perp [L_2, v] \Rightarrow J' = (J_1', J_2') \perp [L, (u, v)], \ \forall (u, v).$$

It enables us to identify and capture the co-existing states, also gives us more flexibility in solving other nonlinear systems.

(b) The bi-$\perp$ condition $(u^*, v^*) = p(\bar{u}, \bar{v})$ can be solved simultaneously from

$$u^* = \arg \max_{u \in [L_1, \bar{u}]} (\text{or } \min) J(u, v^*), \quad v^* = \arg \min_{v \in [L_2, \bar{v}]} (\text{or } \max) J(u^*, v).$$

(c) Definition 6 can be easily extended to a multicomponent system.

(d) Due to two components, how to define instability order?

Let $E_1^+, E_1^- \subset H_1, E_2^+, E_2^- \subset H_2$ where $E_1^+ \times E_2^+, E_1^- \times E_2^-$ are resp. m.p.d. and m.n.d. subspaces of $J''(0,0)$ where E_1^- is finite dimensional.

Cooperative Case: $\dim(E_2^-) < \infty$. Let $L_1 \subset H_1$ contain E_1^- and $L_2 \subset H_2$ contain E_2^- be finite dimensional subspaces. Solve

$$u^* = \arg\min_{u \in S_{L_1^\perp}} J(p(\frac{(u,v^*)}{\|(u,v^*)\|})) \text{ and } v^* = \arg\min_{v \in S_{L_2^\perp}} J(p(\frac{(u^*,v)}{\|(u^*,v)\|})).$$

Then
$$J_1'(p(\frac{(u^*,v^*)}{\|(u^*,v^*)\|}))\perp L_1^\perp \text{ and } J_2'(p(\frac{(u^*,v^*)}{\|(u^*,v^*)\|}))\perp L_2^\perp.$$

Thus $J'(p(\frac{(u^*,v^*)}{\|(u^*,v^*)\|})) = 0$. For each $(u,v) \in S_{L^\perp}$,

$$u(s) = u - sJ_1'(p(u,v)) \text{ and } v(t) = v - tJ_2'(p(u,v))$$

are used to update an iteration and (s,t) are determined by the stepsize rule

$$J(p(\frac{(u(s),v)}{\|(u(s),v)\|})) - J(p(u,v)) \le -\frac{1}{2}s_u s\|J_1'(p(u,v))\|^2$$

$$J(p(\frac{(u,v(t))}{\|(u,v(t))\|})) - J(p(u,v)) \le -\frac{1}{2}t_v t\|J_2'(p(u,v))\|^2.$$

Noncooperative Case: $\dim(E_2^+) < \infty$. Let $L_1 \subset H_1$ contain E_1^-, $L_2 \subset H_2$ contain E_2^+ be finite dimensional subspaces. Solve

$$u^* = \arg\min_{u \in S_{L_1^\perp}} J(p(\frac{(u, v^*)}{\|(u, v^*)\|})) \text{ and } v^* = \arg\max_{v \in S_{L_2^\perp}} J(p(\frac{(u^*, v)}{\|(u^*, v)\|}))$$

or

$$J(p(\frac{(u^*, v)}{\|(u^*, v)\|})) \leq J(p(\frac{(u^*, v^*)}{\|(u^*, v^*)\|})) \leq J(p(\frac{(u, v^*)}{\|(u, v^*)\|}))$$

for all $(u, v) \in \mathcal{N}(u^*, v^*) \cap S_{L^\perp}$. Then

$$J_1'(p(\frac{(u^*, v^*)}{\|(u^*, v^*)\|})) \perp L_1^\perp \text{ and } J_2'(p(\frac{(u^*, v^*)}{\|(u^*, v^*)\|})) \perp L_2^\perp.$$

Thus $J'(p(\frac{(u^*,v^*)}{\|(u^*,v^*)\|})) = 0$. For each $(u, v) \in S_{L^\perp}$,

$$u(s) = u - sJ_1'(p(u, v)) \text{ and } v(t) = v + tJ_2'(p(u, v))$$

are used to update an iteration and (s, t) are determined by the stepsize rule

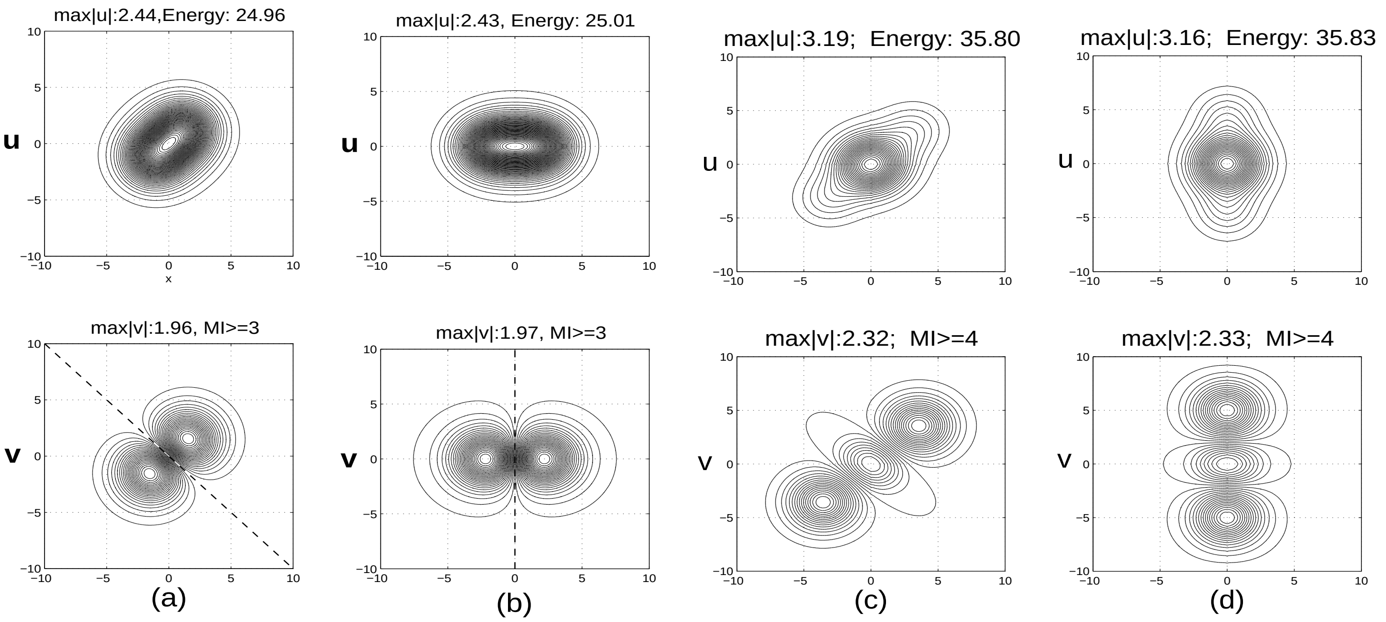
$$J(p(\frac{(u(s), v)}{\|(u(s), v)\|})) - J(p(u, v)) \leq -\frac{1}{2} s_u s \|J_1'(p(u, v))\|^2$$

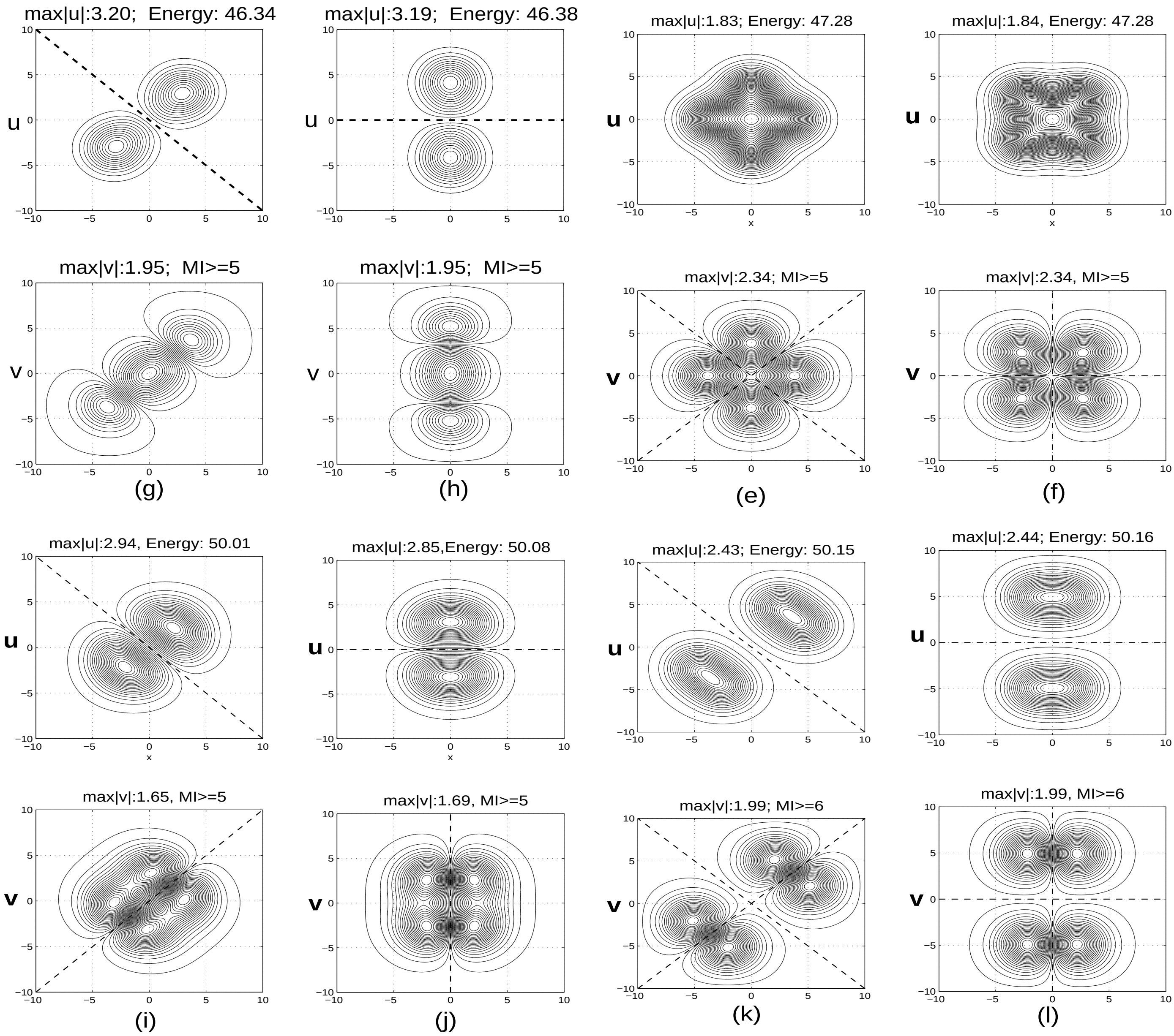
$$J(p(\frac{(u, v(t))}{\|(u, v(t))\|})) - J(p(u, v)) \geq \frac{1}{2} t_v t \|J_2'(p(u, v))\|^2.$$

A Cooperative System in nonlinear optics (A multiple vector soliton problem):
[Chen-Zhou-Yao,07,JANM]

$$\begin{cases} -\Delta u(x,y) = -u(x,y) + \frac{u^2(x,y)+v^2(x,y)}{1+\mu(u^2(x,y)+v^2(x,y))}u(x,y), \\ -\Delta v(x,y) = -\gamma v(x,y) + \frac{u^2(x,y)+v^2(x,y)}{1+\mu(u^2(x,y)+v^2(x,y))}v(x,y), \end{cases} \tag{3.11}$$

with $u = v = 0$ on $\partial\Omega$ where $\Omega = (-10,10) \times (-10,10)$, $\gamma = 0.65, \mu = 0.5$.

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max|v|:1.95; MI>=5
max|v|:1.95; MI>=5
max|v|:2.34; MI>=5
max|v|:2.34, MI>=5
(g)
(h)
(e)
(f)
max|u|:2.94, Energy: 50.01
max|u|:2.85,Energy: 50.08
max|u|:2.43; Energy: 50.15
max|u|:2.44; Energy: 50.16
max|v|:1.65, MI>=5
max|v|:1.69, MI>=5
max|v|:1.99; MI>=6
max|v|:1.99, MI>=6
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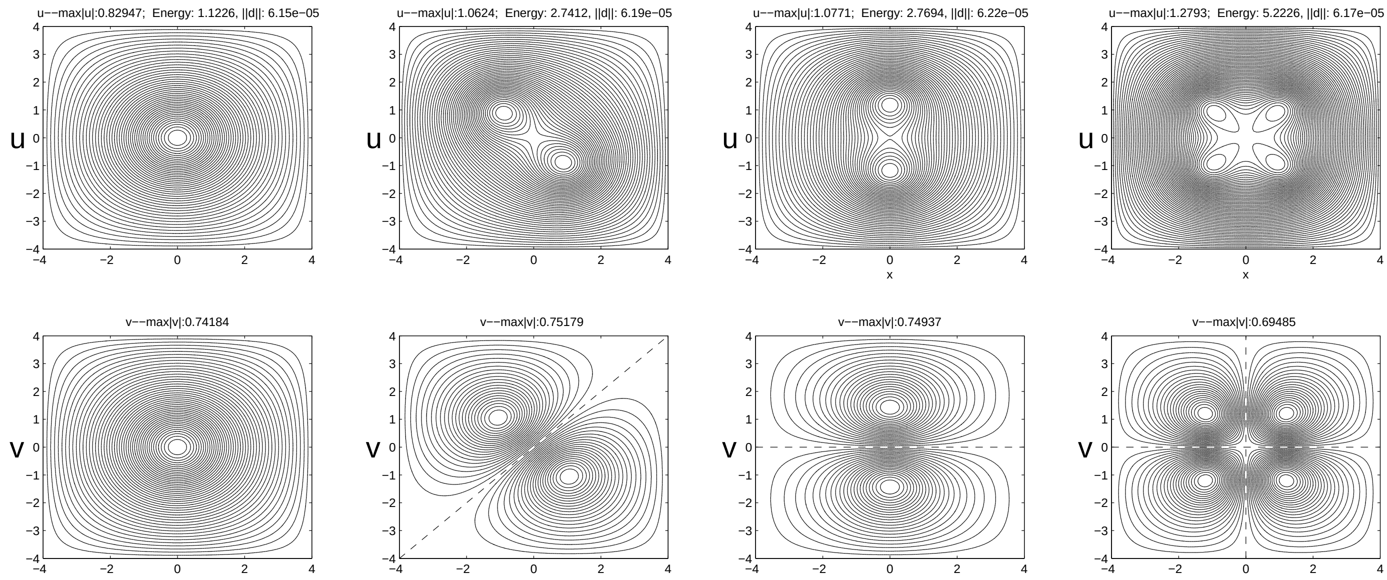
A Noncooperative System: [Chen-Zhou]

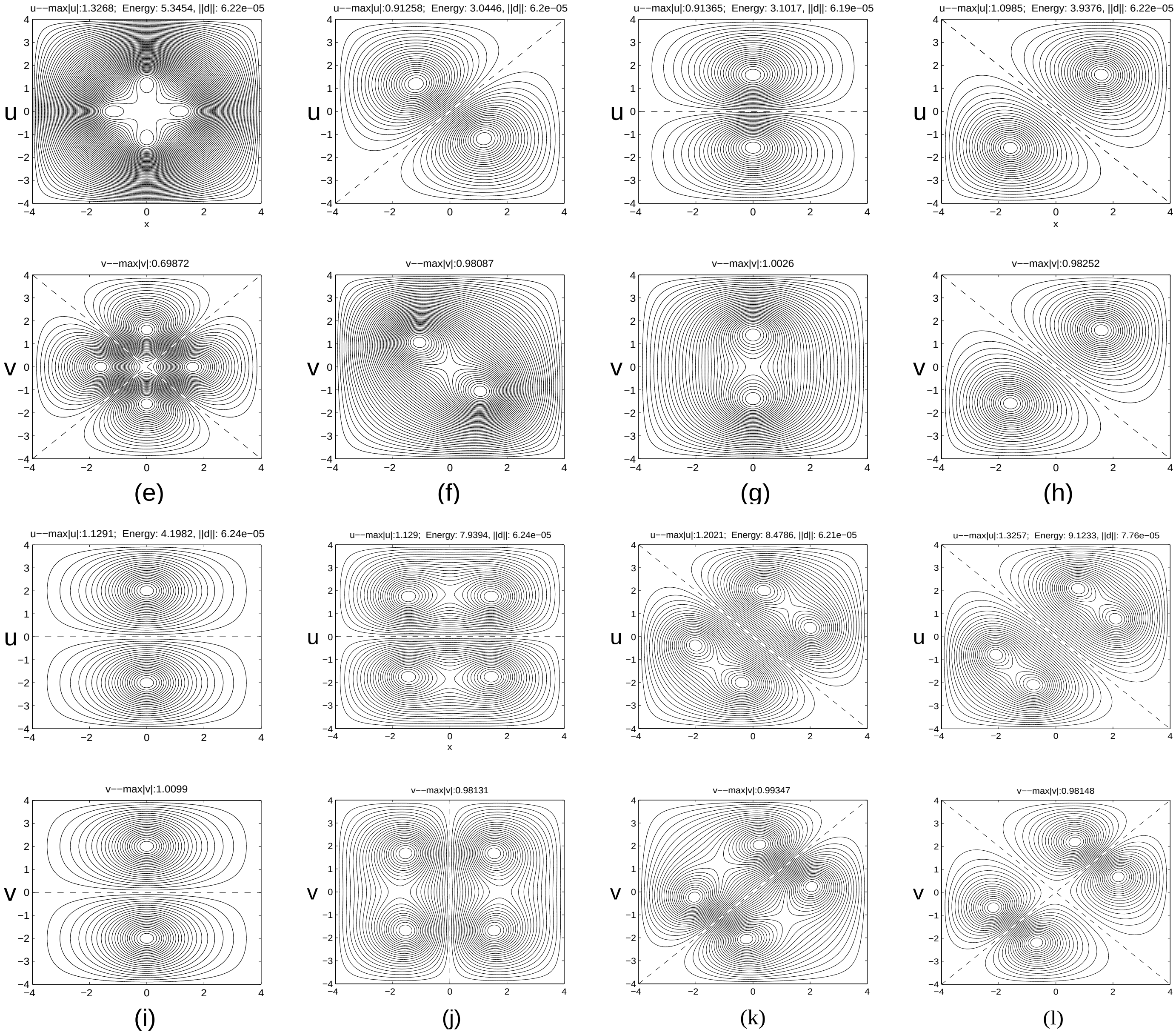
$$\begin{cases} -\Delta u(x,y) = \lambda u(x,y) - \delta v(x,y) + |u(x,y)|^{p-1}u(x,y), \\ -\Delta v(x,y) = \delta u(x,y) + \gamma v(x,y) - |v(x,y)|^{q-1}v(x,y), \end{cases} \tag{3.12}$$

with $u = v = 0$ on $\partial\Omega$ where $\Omega = (-1,1) \times (-1,1)$.

We choose $p = q = 3, \lambda = \gamma = -0.5, \delta = 5$.



(a) (b) (c) (d)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v−−max|v|:0.69872
v−−max|v|:0.98087
v−−max|v|:1.0026
v−−max|v|:0.98252
(e)
(f)
(g)
(h)
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u−−max|u|:1.3257; Energy: 9.1233, ||d||: 7.76e−05
v−−max|v|:1.0099
v−−max|v|:0.98131
v−−max|v|:0.99347
v−−max|v|:0.98148
(i)
(j)
(k)
(l)

(5) 0 is a saddle. $H = H^+ \oplus H^-$, H^+, H^- m.p.d., m.n.d. subspaces of $J''(0)$. Case (a) $\dim(H^-) < +\infty$. Then we let $L = H^-, L^\perp = H^+$ and apply LMM to find a saddle point $u_1^* = p(v_1^*)$. We have $\mathrm{MI}(u_1^*) = \dim(L) + 1$ and $\mathrm{MI}(u_1^*)$ relative to 0 is $\dim(L) + 1 - \dim(H^-) = 1$. Next let $L = [H^-, u_1^*]$, use LMM to find $u_2^* = p(v_2^*)$. $\mathrm{MI}(u_2^*) = \dim(L) + 1$ and $\mathrm{MI}(u_2^*)$ relative to 0 is $\dim(L) + 1 - \dim(H^-) = 2$, etc. For example, when

$$J(u) = \int_\Omega \left[\frac{1}{2}\Big(|\nabla u(x)|^2 - \lambda |u(x)|^2\Big) - \frac{1}{4}|u(x)|^4\right] dx$$

where $\lambda_i < \lambda < \lambda_{i+1}$ and $\lambda_1 < \lambda_2 < \ldots < \lambda_i < \ldots$ are the eigenvalues of $-\Delta$ on Ω. Let $u_1, u_2, \ldots$ be the corresponding eigenfunctions of $-\Delta$. Then we have $H^- = [u_1, \ldots, u_i]$ and 0 is a saddle point of J with $\mathrm{MI}(0) = i$.

Case (b) $\dim(H^-) = +\infty$. Have some practical difficulty to evaluate $p(v)$. Let $L_1 \subset H^+, L_2 \subset H^-$ be finite-d subspaces, $p(u,v) \in [L_1, u] \oplus [L_2, v]$ s.t.

$$J'(p(u,v)) \perp [L_1, u], J'(p(u,v)) \perp [L_2, v], \quad \forall (u,v) \in S_{(H^+ \cap L_1^\perp) \times (H^- \cap L_2^\perp)}.$$

Then

$$u^* = \arg \min_{u \in S_{H^+ \cap L_1^\perp}} J(p(\frac{(u,v^*)}{\|(u,v^*)\|})) \text{ and } v^* = \arg \max_{v \in S_{H^- \cap L_2^\perp}} J(p(\frac{(u^*,v)}{\|(u^*,v)\|}))$$

lead to

$$J'(p(\frac{(u^*,v^*)}{\|(u^*,v^*)\|})) \perp H^+ \cap L_1^\perp \text{ and } J'(p(\frac{(u^*,v^*)}{\|(u^*,v^*)\|})) \perp H^- \cap L_2^\perp.$$

Thus $J'(p(\frac{(u^*,v^*)}{\|(u^*,v^*)\|})) = 0$. Where

$$u(s) = u - sJ'(p(u,v))_{H^+} \text{ and } v(t) = v + tJ'(p(u,v))_{H^-}$$

are used to update an iteration and (s,t) is determined by the stepsize rule

$$J(p(\frac{(u(s),v)}{\|(u(s),v)\|})) - J(p(u,v)) \leq -\frac{1}{2} s_u s \|J'(p(u,v))_{H^+}\|^2$$
$$J(p(\frac{(u,v(t))}{\|(u,v(t))\|})) - J(p(u,v)) \geq \frac{1}{2} t_v t \|J'(p(u,v))_{H^-}\|^2.$$

E.g., to find periodic solutions to a semilinear Schrödinger equation

$$-\Delta u(x) + V(x)u(x) = f(x,u), \quad u \in H^1(\mathbb{R}^n), \tag{3.13}$$

where V and f are periodic w.r.t. to x and 0 lies in a gap of the spectrum of $-\Delta + V$. Its functional is

$$J(u) = \int_{\mathbb{R}^n} \Big[\frac{1}{2}(|\nabla u(x)|^2 + V(x)u^2(x)) - F(x,u)\Big]dx,$$

where $F(x,t) = \int_0^t f(x,s)ds$ satisfies certain growth conditions. Thus we have a spectrum decomposition $H = H^+ \oplus H^-$, where H^+ and H^- are resp., infinite dimensional m.p.d. and m.n.d. subspaces of $J''(0)$.

(6) Non Variational Problems: to be continue.