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RESEARCH ARTICLE

Virtual endomorphisms of groups

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1. Introduction

A virtual endomorphism of a group G is a homomorphism from a subgroup of finite index $H \leq G$ into G. Similarly a virtual automorphism (an almost automorphism) is an isomorphism between subgroups of finite index.

Virtual automorphisms (commensurations) appear naturaly in theory of lattices of Lie groups (see [Mar91]). Virtual endomorphism of more general sort appear in theory of groups acting on rooted trees. Namely, if an automorphism g of a rooted tree T fixes a vertex v, then it induces an automorphism $g|_v$ of the rooted subtree T_v, "growing" from the vertex v. If the rooted tree T is regular, then the subtree T_v is isomorphic to the whole tree T, and $g|_v$ is identified with an automorphism of the tree T, when we identify T with T_v. It is easy to see that the described map $\phi_v : g \mapsto g|_v$ is a virtual endomorphism of the automorphism group of the tree T (the domain of this virtual endomorphism is the stabilizer of the vertex v).

The described virtual endomorphisms are the main investigation tools of the groups defined by their action on regular rooted trees. Historically the first example of such a group was the Grigorchuk group [Gri80]. Later many other interesting examples where constructed and investigated [GS83a, GS83b, BSV99, SW02]. One of the common feachures of

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these groups is that they are preserved under the virtual endomorphism ϕ_v, i.e., that the restriction $g|_v$ of any element of the group also belongs to the group. Another important property is that the virtual endomorphism ϕ_v contracts the length of the elements of the groups. (Here length of an element of a finitely generated group is the length of the representation of the element as a product of the generators and their inverses.) The groups with the first property are called *self-similar*, or *state-closed*. The groups with the second property are called *contracting*. The contraction property helps to argue by induction on the length of the group elements.

The notion of a self-similar group is very similar to the classical notion of a self-similar set, so that in some cases self-similar groups are called *fractal groups*. See the survey [BGN02], where the analogy and the connections between the notions of self-similar set and self-similar group are studied.

Recentely, the connections became more clear, after the notions of a *limit space* of a contracting group and the notion of an *iterated monodromy group* were defined [Nekc, Nekb, BGN02]. The limit space is a topological space $\mathcal{J}_G$ together with a continuous map $\mathsf{s} : \mathcal{J}_G \to \mathcal{J}_G$, which is naturally associated to the contracting self-similar group. The limit space has often a fractal appearence and the map s is an expanding map on it, which defines a self-similarity structure of the space.

On the other hand, the iterated monodromy groups are groups naturally associated to (branched) self-coverings $s : \mathcal{X} \to \mathcal{X}$ of a topological space. They are always self-similar, and they are contracting if the map s is expanding. In the latter case, the limit space of the iterated monodromy group is homeomorphic to the Julia set of the map s, with the map s on the limit space conjugated with the restriction of s onto the Julia set.

In the present paper we try to collect the basic facts about the virtual endomorphisms of groups. Since the most properties of self-similar groups are related with the dynamics of the associated virtual endomorphism, the main attention is paid to the dynamics of iterations of one virtual endomorphisms.

For a study of iterations of virtual endormorphisms of index 2 and virtual endomorphisms of abelian groups, see also the paper [NS01]. Many results of [NS01] are generalized here.

The structure of the paper is the following. Section "Virtual endomorphisms" introduces the basic definitions and the main examples of virtual endomorphisms. This is the only section, where semigroups of virtual endomorphisms and groups of virtual automorphisms (commensurators) are discussed.

In the next section "Iterations of one virtual endomorphism" we define the main notions related to the dynamics of virtual endomorphisms. This is the *coset tree* and different versions of the notion of invariant subgroup.

Section "Bimodule associated to a virtual endomorphism" is devoted to ring-theoretic aspects of virtual endomorphisms of groups. Every virtual endomorphism of a group defines a bimodule over the group algebra. Many notions related to virtual endomorphisms of groups have their analogs for bimodules over algebras. For instance, in Subsection "Φ-invariant ideals" we study the analogs of the notion of a subgroup invariant under a virtual endomorphism. The rôle of composition of virtual endomorphisms is played by tensor products of bimodules, which are studied in Subsection "Tensor powers of the bimodule". The last subsection introduces, using the language of bimodules, the *standard actions* of a group on a regular tree, defined by a virtual endomorphism. In this way we show that the action of a self-similar group is defined, up to conjugacy, only by the associated virtual endomorphism. More on bimodules, associated to virtual endomorphism is written in [Neka].

The last section is devoted to the the notion of a contracting virtual endomorphism. We give different definitions of the contraction property, define the *contraction coefficient* (or the *spectral radius*) of a virtual endomorphism, and prove the basic properties of groups, posessing a contracting virtual endomorphism. For example, we prove that such groups have an algorithm, solving the word problem in a polynomial time. This was observed for the first time by R. Grigorchuk for a smaller class of groups (see, for example [Gri80, Gri83]), but we show in this paper, that his algorithm works in the general case.

We use the standard terminology and notions from the theory of groups acting on rooted trees. The reader can find it in [GNS00, BGN02, Gri00, Sid98]. We use left actions here, so that the image of a point x under the action of a group element g is denoted $g(x)$. Respectively, in the product $g_1 g_2$, the element g_2 acts first.

2. Virtual endomorphisms

2.1. Definitions and main properties

Definition 2.1. Let G_1 and G_2 be groups. A *virtual homomorphism* $\phi : G_1 \dashrightarrow G_2$ is a homomorphism of groups $\phi : \operatorname{Dom}\phi \to G_2$, where $\operatorname{Dom}\phi \leq G_1$ is a subgroup of finite index, called the *domain* of the virtual homomorphism. A *virtual endomorphism* of a group G is a virtual homomorphism $\phi : G \dashrightarrow G$.

The index $[G_1 : \mathrm{Dom}\,\phi]$ is called the *index of the virtual endomorphism* $\phi : G_1 \dashrightarrow G_2$ and is denoted $\mathrm{ind}\,\phi$.

By $\mathrm{Ran}\,\phi$ we denote the image of $\mathrm{Dom}\,\phi$ under ϕ.

We say that a virtual endomorphism ϕ *is defined* on an element $g \in G$ if $g \in \mathrm{Dom}\,\phi$.

If $H \leq G$ is a subgroup of finite index, then the *identical virtual homomorphism* $id_H : G \dashrightarrow G$ with the domain H is naturally defined.

A composition of two virtual homomorphisms $\phi_1 : G_1 \dashrightarrow G_2$, $\phi_2 : G_2 \dashrightarrow G_3$ is defined on an element $g \in G_1$ if and only if ϕ_1 is defined on g and ϕ_2 is defined on $\phi_1(g)$. Thus, the domain of the composition $\phi_2 \circ \phi_1$ is the subgroup

$$\mathrm{Dom}\,(\phi_2 \circ \phi_1) = \{g \in \mathrm{Dom}\,\phi_1 : \phi_1(g) \in \mathrm{Dom}\,\phi_2\} \leq G_1.$$

Proposition 2.1. *Let $\phi_1 : G_1 \dashrightarrow G_2$ and $\phi_2 : G_2 \dashrightarrow G_3$ be two virtual homomorphisms. Then*

$$[\mathrm{Dom}\,\phi_1 : \mathrm{Dom}\,(\phi_2 \circ \phi_1)] \leq [G_2 : \mathrm{Dom}\,\phi_2] = \mathrm{ind}\,\phi_2.$$

If ϕ_1 is onto, then

$$[\mathrm{Dom}\,\phi_1 : \mathrm{Dom}\,(\phi_2 \circ \phi_1)] = [G_2 : \mathrm{Dom}\,\phi_2]\,.$$

Proof. We have $[\mathrm{Ran}\,\phi_1 : \mathrm{Dom}\,\phi_2 \cap \mathrm{Ran}\,\phi_1] \leq \mathrm{ind}\,\phi_2$ and we have here equality in the case when ϕ_1 is onto. Let $T = \{\phi_1(h_1), \phi_1(h_2), \ldots \phi_1(h_d)\}$ be a left coset transversal for $\mathrm{Dom}\,\phi_2 \cap \mathrm{Ran}\,\phi_1$ in $\mathrm{Ran}\,\phi_1$. Then for every $g \in \mathrm{Dom}\,\phi_1$ there exists a unique $\phi_1(h_i) \in T$ such that $\phi_1(h_j)^{-1}\phi_1(g) = \phi_1(h_i^{-1}g) \in \mathrm{Dom}\,\phi_2$. This is equivalent to $h_i^{-1}g \in \mathrm{Dom}\,(\phi_2 \circ \phi_1)$ and the set $\{h_1, h_2, \ldots, h_d\}$ is a left coset transversal of $\mathrm{Dom}\,(\phi_2 \circ \phi_1)$ in G_1. Thus,

$$[G_2 : \mathrm{Dom}\,\phi_2] = [\mathrm{Ran}\,\phi_1 : \mathrm{Dom}\,\phi_2 \cap \mathrm{Ran}\,\phi_1]\,.$$

□

Corollary 2.2. *A composition of two virtual homomorphisms is again a virtual homomorphism.* □

Consequently, the set of all virtual endomorphisms of a group G is a semigroup under composition. This semigroup is called the *semigroup of virtual endomorphisms* of the group G and is denoted $VE(G)$.

Corollary 2.2 also implies that the class of groups as a class of objects together with the class of virtual homomorphisms as a class of morphisms form a category, which will be called the *category of virtual homomorphisms*.

Commensurability

Let $\phi : G_1 \dashrightarrow G_2$ be a virtual homomorphism. If $H \leq G_2$ is a subgroup, then by $\phi^{-1}(H)$ we denote the set of such elements $g \in \mathrm{Dom}\,\phi$ that $\phi(g) \in H$.

If H is a subgroup of finite index then $\phi^{-1}(H) = \mathrm{Dom}\,(id_H \circ \phi)$, thus $\phi^{-1}(H)$ is a subgroup of finite index in G_1.

Lemma 2.3. *For every virtual homomorphism $\phi : G_1 \dashrightarrow G_2$ and for every subgroup of finite index $H \leq G_2$ the equality*

$$id_H \circ \phi = \phi \circ id_{\phi^{-1}(H)}$$

holds.

Proof. An element $g \in G_1$ belongs to the domain of $id_H \circ \phi$ if and only if $\phi(g) \in H$, i.e., if and only if $g \in \phi^{-1}(H)$. This implies that the domains of the virtual endomorphisms $id_H \circ \phi$ and $\phi \circ id_{\phi^{-1}(H)}$ coincide. They are equal on its domains to the virtual homomorphism ϕ, so they are equal each to the other. □

Definition 2.2. Let $\phi : G_1 \dashrightarrow G_2$ be a virtual homomorphism and let $H \leq G_1$ be a subgroup of finite index. Then the *restriction* of ϕ onto H is the virtual homomorphism $\phi|_H : G_1 \dashrightarrow G_2$ with the domain $\mathrm{Dom}\,\phi \cap H$ such that $\phi|_H(g) = \phi(g)$ for all $g \in \mathrm{Dom}\,\phi \cap H$. In other words, $\phi|_H = \phi \circ id_H$.

Two virtual homomorphisms $\phi_1 : G_1 \dashrightarrow G_2$ and $\phi_2 : G_1 \dashrightarrow G_2$ are said to be *commensurable* (written $\phi_1 \approx \phi_2$) if there exists a subgroup of finite index $H \leq G_1$ such that $\phi_1|_H = \phi_2|_H$.

For example, any two identical virtual endomorphisms id_{H_1} and id_{H_2} are commensurable.

Proposition 2.4. *The relation of commensurability is a congruence on the category of virtual homomorphisms. In particular, it is a congruence on the semigroup $VE(G)$.*

Proof. Let $\phi_1, \phi_2, \psi_1, \psi_2$ be virtual homomorphisms such that $\phi_i \approx \psi_i$ for $i = 1, 2$. Then there exist subgroups of finite index H_i such that $\phi_i \circ id_{H_i} = \psi_i \circ id_{H_i}$. Lemma (2.3) implies:

$$\phi_1 \circ id_{H_1} \circ \phi_2 \circ id_{H_2} = \phi_1 \circ \phi_2 \circ id_{\phi_2^{-1}(H_1)} \circ id_{H_2} = \phi_1 \circ \phi_2 \circ id_{\phi_2^{-1}(H_1) \cap H_2},$$

and

$$\psi_1 \circ id_{H_1} \circ \psi_2 \circ id_{H_2} = \psi_1 \circ \psi_2 \circ id_{\psi_2^{-1}(H_1) \cap H_2}.$$

Thus,

$$\phi_1 \circ \phi_2 \circ id_{\phi_2^{-1}(H_1)\cap H_2} = \psi_1 \circ \psi_2 \circ id_{\psi_2^{-1}(H_1)\cap H_2}.$$

Multiplying the last equality from the right by id_H, where $H = \phi_2^{-1}(H_1)\cap \psi_2^{-1}(H_1)\cap H_2$, we get $\phi_1\circ\phi_2\circ id_H = \psi_1\circ\psi_2\circ id_H$, thus $\phi_1\circ\phi_2 \approx \psi_1\circ\psi_2$. $\square$

We will denote by Commen the category with groups as objects and commensurability classes of virtual homomorphisms as morphisms. Proposition 2.4 shows that this category is well defined.

Definition 2.3. The quotient of the semigroup $VE(G)$ by the congruence "$\approx$" is called the *restricted semigroup of virtual endomorphisms* and is denoted $RVE(G)$.

The semigroup $RVE(G)$ is the endomorphism semigroup of the object G in the category Commen.

Example. It is easy to see that every virtual endomorphism of $\mathbb{Z}^n$ can be extended uniquely to a linear map $\mathbb{Q}\otimes\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ and that two extensions $\mathbb{Q}\otimes\phi_1$ and $\mathbb{Q}\otimes\phi_2$ are equal if and only if the virtual endomorphisms are commensurable. Consequently, the semigroup $RVE(\mathbb{Z}^n)$ is isomorphic to the multiplicative semigroup $\mathrm{End}(\mathbb{Q}^n)$ of rational $n\times n$-matrices.

Let us describe the isomorphisms in the category Commen.

Definition 2.4. A virtual homomorphism $\phi : G_1 \dashrightarrow G_2$ is called *commensuration* if it is injective and $\mathrm{Ran}\,\phi$ is a subgroup of finite index in G_2.

Two groups are said to be *commensurable* if there exists a commensuration between them.

Thus, two groups are commensurable if and only if they have isomorphic subgroups of finite index. The identical virtual endomorphisms id_H are examples of commensurations.

If a virtual homomorphism ϕ is a commensuration, then it has an inverse $\phi^{-1} : G_2 \dashrightarrow G_1$, such that $\phi\circ\phi^{-1} = id_{\mathrm{Ran}\,\phi}$ and $\phi^{-1}\circ\phi = id_{\mathrm{Dom}\,\phi}$.

It is easy to see that two groups are isomorphic in the category Commen if and only if they are commensurable. The respective isomorphism will be the commensuration.

Definition 2.5. *Abstract commensurator* of a group G is the group of commensurability classes of commensurations of the group G with itself.

We denote the abstract commensurator of a group G by $Comm(G)$. From the definitions follows that it is the automorphism group of the object G in the category Commen.

Proposition 2.5. *The abstract commensurator $Comm(G)$ is a group and is isomorphic to the group of invertible elements of the semigroup $RVE(G)$.*
If the groups G_1 and G_2 are commensurable, then the semigroups $RVE(G_1)$ and $RVE(G_2)$ and the groups $Comm(G_1)$ and $Comm(G_2)$ are isomorphic. □

Remarks. If H is a subgroup of a group G, then its *commensurator* is the group of those elements $g \in G$ for which the subgroups H and $g^{-1}Hg$ are commensurable. Two subgroups H_1, H_2 are said to be commensurable if the intersection $H_1 \cap H_2$ has finite index both in H_1 and in H_2.

For applications of the notions of commensurators of subgroups and abstract commensurators of groups in the theory of lattices of Lie groups see the works [Mar91, AB94, BdlH97].

Examples. 1) It is easy to see that the abstract commensurator of the group $\mathbb{Z}^n$ is $\mathrm{GL}(n, \mathbb{Q})$, i.e., the automorphism group of the additive group $\mathbb{Q}^n$.

2) An example very different from the previous is the Grigorchuk group. It is proved by C. Roever [Röv02] that the abstract commensurator of the Grigorchuk group is finitely presented and simple. It is generated by its subgroup isomorphic to the Grigorchuk group and a subgroup, isomorphic to the Higmann-Thompson group.

More on commensurators see the paper [MNS00].

Conjugacy

Definition 2.6. Two virtual homomorphisms $\phi, \psi : G_1 \dashrightarrow G_2$ are said to be *conjugate* if there exist $g \in G_1$ and $h \in G_2$ such that $\mathrm{Dom}\,\phi = g^{-1} \cdot \mathrm{Dom}\,\psi \cdot g$ and $\psi(x) = h^{-1}\phi(g^{-1}xg)h$ for every $x \in \mathrm{Dom}\,\psi$.

If the virtual homomorphism ϕ is onto, then every its conjugate is also onto and is of the form $\psi(x) = h^{-1}\phi(g^{-1}xg)h = \phi(f^{-1}xf)$, where $f = gh'$ for $h' \in \phi^{-1}(h)$.

2.2. Examples of virtual endomorphisms

Self-coverings

Let $\mathcal{M}$ be an arcwise connected and locally arcwise connected topological space, and suppose $\mathcal{M}_0$ is its arcwise connected open subset. Let $F : \mathcal{M}_0 \to \mathcal{M}$ be a d-fold covering map.

Take an arbitrary basepoint $t \in \mathcal{M}$. Let $t' \in \mathcal{M}_0$ be one of its preimages under F and let ℓ be a path, starting at t and ending at t'.

For every loop γ in $\mathcal{M}$, based at t (i.e., for every element γ of the fundamental group $\pi(\mathcal{M}, t)$) there exists a unique path γ', starting at t' and such that $F(\gamma') = \gamma$. The set G_1 of the elements $\gamma \in \pi(\mathcal{M}, t)$ for which γ' is again a loop is a subgroup of index d in $\pi(\mathcal{M}, t)$ and is isomorphic to $\pi(\mathcal{M}_0, t')$.

The *virtual endomorphism, defined by the map* F is the virtual endomorphism of the group $\pi(\mathcal{M}, t)$ with the domain G_1 which is defined as

$$\phi(\gamma) = \ell\gamma'\ell^{-1}.$$

Proposition 2.6. *Up to a conjugacy, the virtual endomorphism ϕ of the group $\pi(\mathcal{M})$ defined by F does not depend on the choice of t, t' and ℓ.*

Proof. Let us take some basepoint r (possibly $r = t$), some its preimage r' under F and some path ℓ' in $\mathcal{M}$, connecting r with r'. Let σ be a path from r to t in $\mathcal{M}$, realizing an isomorphism $\gamma \mapsto \sigma^{-1}\gamma\sigma$ of the group $\pi(\mathcal{M}, r)$ with the group $\pi(\mathcal{M}, t)$. Let ϕ' be the virtual endomorphism of $\pi(\mathcal{M}, r)$ defined by r' and ℓ'. Let $x \in \pi(\mathcal{M})$ be an element, corresponding to a loop γ at r. Then x corresponds to the loop $\sigma^{-1}\gamma\sigma$ at t. Suppose that x belongs to the domain of ϕ'. Then $\phi'(x) = \ell'\gamma'\ell'^{-1}$, where γ' is the F-preimage of γ, starting at r'. The loop at t, representing $\phi'(x)$ is then $\sigma^{-1}\ell'\gamma'\ell'^{-1}\sigma$.

Let σ' be the F-preimage of the path σ, starting at r'. Then its end t'' is an F-preimage of t, possibly different from t'. Take some path ρ in $\mathcal{M}_0$ starting at t' and ending at t''. Then $F(\rho)$ is a loop based at t. We get also the loop $h = \ell\rho\sigma'^{-1}\ell'^{-1}\sigma$ at t. Denote by h the element $F(\rho)^{-1}$ of the fundamental group $\pi(\mathcal{M}, t)$. Then, in $\pi(\mathcal{M}, t)$:

$$\phi(h^{-1}xh) = \phi(F(\rho)\sigma^{-1}\gamma\sigma F(\rho)^{-1}) = \ell\rho\sigma'^{-1}\gamma'\sigma'\rho^{-1}\ell^{-1},$$

since $\rho\sigma'^{-1}\gamma'\sigma'\rho^{-1}$ is a loop, starting at t, whose F-image is

$$F(\rho)\sigma^{-1}\gamma\sigma F(\rho)^{-1}.$$

Therefore (see Figure 1)

$$\begin{aligned}
&\phi'(g) = \sigma^{-1}\ell'\gamma'\ell'^{-1}\sigma = \\
&\left(\sigma^{-1}\ell'\sigma'\rho^{-1}\ell^{-1}\right) \cdot \left(\ell\rho\sigma'^{-1}\ell'^{-1}\sigma\right) \cdot \sigma^{-1}\ell'\gamma'\ell'^{-1}\sigma \cdot \left(\sigma^{-1}\ell'\sigma'\rho^{-1}\ell^{-1}\right) \cdot \\
&\cdot \left(\ell\rho\sigma'^{-1}\ell'^{-1}\sigma\right) = \left(\sigma^{-1}\ell'\sigma'\rho^{-1}\ell^{-1}\right) \cdot \left(\ell\rho\sigma'^{-1}\gamma'\sigma'\rho^{-1}\ell^{-1}\right) \cdot \\
&\cdot \left(\ell\rho\sigma'^{-1}\ell'^{-1}\sigma\right) = g^{-1}\phi(h^{-1}xh)g,
\end{aligned}$$

where $g = \ell\rho\sigma'^{-1}\ell'^{-1}\sigma$, so that the virtual endomorphisms ϕ and ϕ' are conjugate. □

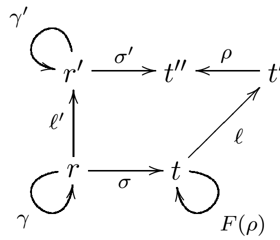

Figure 1:

Example. Take the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and define its double-fold self-covering F, induced by the map $x \mapsto 2x$ on $\mathbb{R}$. Let us take a basepoint $t = 0$. It has two preimages under F: one is 0 and another is 1/2. Take $t' = 0$ and let ℓ be trivial path at 0. The fundamental group of the circle is isomorphic to $\mathbb{Z}$ and is generated by the loop, which is the image of the segment $[0, 1]$ in $\mathbb{T}^1$. It is easy to see that the virtual endomorphism of $\mathbb{Z}$, defined by F is the map $n \mapsto n/2$, defined on the subgroup of even numbers.

The virtual endomorphisms of groups are group-theoretical counterparts of self-coverings of topological spaces. More on relations between dynamics of virtual endomorphisms and dynamics of self-coverings of topological spaces, see the paper [Nekb].

Stabilizers in automorphism groups of graphs

Let Γ be a locally finite graph and let G be a group acting on Γ by automorphisms so that its action on the vertices of Γ is transitive.

Take a vertex v and let G_v be the stabilizer of v in the group G. Let u be another vertex, adjacent to v. Denote by G_{vu} the stabilizer of the vertex u in the group G_v and by G_u the stabilizer of u in G. We obviously have $G_{vu} = G_v \cap G_u$. The group G_{vu} has finite index in G_v and in G_u, not greater than the degree of a vertex in the graph Γ (the degrees of all the vertices of Γ are equal, since G acts on Γ transitively).

Let $g \in G$ be an element such that $g(v) = u$. Then we get a virtual endomorphism $\phi : G_v \dashrightarrow G_v$, with $\mathrm{Dom}\,\phi = G_{vu}$ defined as $\phi(h) = g^{-1}hg$.

This virtual endomorphism is obviously a commensuration. It is proved in [Nek00] that every commensuration can be constructed in such a way.

The proof of the next proposition is straightforward.

Proposition 2.7. *The virtual endomorphism ϕ up to a conjugacy, depends only on the orbit of the edge $\{u, v\}$ with respect to the action of G_v.* $\square$

Self-similar actions

Let X be a finite set, called the *alphabet*. By X^* we denote the set of all finite words over X, i.e., the free monoid, generated by X. We include the empty word $\varnothing$.

Definition 2.7. An action of a group G on the set X^* is *self-similar* if for every $g \in G$ and every $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w) \tag{1}$$

for every $w \in X^*$.

Let us take some faithful self-similar action of G on X^*. Let G_x be the stabilizer of the one-letter word $x \in X^*$. Then there exists a unique $h \in G$ such that $g(xw) = xh(w)$ for all $w \in X^*$. The subgroup G_x has a finite index not greater than $|X|$ in G and the map $\phi_x : G_x \to G : g \mapsto h$ is a homomorphism. In this way we get a virtual endomorphism $\phi_x : G \dashrightarrow G$ of the group G.

The following is straightforward.

Proposition 2.8. *If $x, y \in X$ belong to the same G-orbit, then the virtual endomorphisms ϕ_x and ϕ_y are conjugate.* $\square$

If the self-similar action is faithful then for every $g \in G$ and for every finite word $v \in X^*$ there exist a unique element $h \in G$ such that

$$g(vw) = g(u)h(w)$$

for every $w \in X^*$. The element h is called *restriction of g at v* and is denoted $g|_v$. It is easy to see that the following properties of restriction hold.

$$g|_{v_1 v_2} = (g|_{v_1})|_{v_2} \tag{2}$$
$$(g_1 g_2)|_v = g_1|_{g_2(v)} g_2|_v. \tag{3}$$

We will see later, that in some sense all virtual endomorphisms of groups are associated to self-similar actions.

3. Iterations of one virtual endomorphism

3.1. Coset tree

Let ϕ be a virtual endomorphism of a group G. Denote $d = \operatorname{ind}\phi$. We get a descending sequence of subgroups of finite index in G:

$$\operatorname{Dom}\phi^0 = G \geq \operatorname{Dom}\phi^1 \geq \operatorname{Dom}\phi^2 \geq \operatorname{Dom}\phi^3 \geq \ldots. \tag{4}$$

We have, by Proposition 2.1, an inequality $[\operatorname{Dom}\phi^n : \operatorname{Dom}\phi^{n+1}] \leq d$ for every $n \geq 0$. Consequently, $[G : \operatorname{Dom}\phi^n] \leq d^n$.

Definition 3.1. The virtual endomorphism ϕ is said to be *regular* if

$$\left[\operatorname{Dom}\phi^n : \operatorname{Dom}\phi^{n+1}\right] = d$$

for every $n \geq 0$.

An example of a non-regular virtual endomorphism is the identical endomorphism id_H for H not equal to the whole group.

On the other hand, from Proposition 2.1 follows that if ϕ is onto, then it is regular. Nevertheless, non-surjective virtual endomorphism can be regular, for example the virtual endomorphism $n \mapsto \frac{3}{2}n$ of the group $\mathbb{Z}$, defined on even numbers, is regular.

Definition 3.2. The *coset tree* $T(\phi)$ of a virtual endomorphism ϕ is the rooted tree whose nth level is the set of left cosets $\{g\operatorname{Dom}\phi^n : g \in G\}$ and two cosets $g\operatorname{Dom}\phi^n$ and $h\operatorname{Dom}\phi^{n+1}$ are adjacent if and only if $g\operatorname{Dom}\phi^n \geq h\operatorname{Dom}\phi^{n+1}$. The root of the coset tree is the vertex

$$1 \cdot \operatorname{Dom}\phi^0 = G.$$

The coset tree $T(\phi)$ is a level-homogeneous tree of branch index

$$([G : \operatorname{Dom}\phi], [\operatorname{Dom}\phi : \operatorname{Dom}\phi^2], [\operatorname{Dom}\phi^2 : \operatorname{Dom}\phi^3], \ldots).$$

In particular, it is regular if and only if the virtual endomorphism is regular.

The group G acts on the coset tree by left multiplication:

$$g(h\,\mathrm{Dom}\,\phi^n) = gh\,\mathrm{Dom}\,\phi^n.$$

This action is obviously an action by automorphisms of the rooted tree and is level-transitive.

Directly from the description follows that the stabilizer of the vertex $1 \cdot \mathrm{Dom}\,\phi^n$ in the group G is the subgroup $\mathrm{Dom}\,\phi^n$. The stabilizer of the vertex $g\,\mathrm{Dom}\,\phi^n$ is its conjugate subgroup $g \cdot \mathrm{Dom}\,\phi^n \cdot g^{-1}$.

The *nth level stabilizer* is the subgroup

$$St_n(\phi) = \bigcap_{g \in G} g \cdot \mathrm{Dom}\,\phi^n \cdot g^{-1},$$

equal to the set of all elements of G, fixing every vertex of the nth level of the coset tree.

The nth level stabilizer is a normal subgroup of finite index in G.

3.2. Invariant subgroups

Definition 3.3. Let ϕ be a virtual endomorphism of a group G. A subgroup $H \le G$ is said to be

1. *ϕ-semi-invariant* if $\phi(H \cap \mathrm{Dom}\,\phi) \subseteq H$;
2. *ϕ-invariant* if $H \subseteq \mathrm{Dom}\,\phi$ and $\phi(H) \subseteq H$;
3. *ϕ^{-1}-invariant* if $\phi^{-1}(H) \le H$.

Recall that $\phi^{-1}(H) = \{g \in \mathrm{Dom}\,\phi \;:\; \phi(g) \in H\}$. Note that every ϕ-invariant subgroup is ϕ-semi-invariant.

If a subgroup $H \le G$ is ϕ-invariant, then it is a subgroup of $\mathrm{Dom}\,\phi^n$ for every $n \in \mathbb{N}$. On the other hand, the *parabolic subgroup*

$$P(\phi) = \bigcap_{n \in \mathbb{N}} \mathrm{Dom}\,\phi^n$$

is obviously ϕ-invariant. Thus, the parabolic subgroup is the maximal ϕ-invariant subgroup of G.

Example. Let ϕ be a surjective virtual endomorphism of a group G. Let us show that the center $Z(G)$ of the group G is ϕ-semi-invariant. If $h \in Z(G) \cap \mathrm{Dom}\,\phi$, then $\phi(h)\phi(g) = \phi(g)\phi(h)$ for every $g \in \mathrm{Dom}\,\phi$. But the set of elements of the form $\phi(g)$ is the whole group G. Thus, $\phi(h) \in Z(G)$.

Proposition 3.1. *If $H \leq G$ is a normal ϕ-semi-invariant subgroup, then the formula*

$$\psi(gH) = \phi(g)H$$

for $g \in \mathrm{Dom}\,\phi$ gives a well defined virtual endomorphism ψ of the quotient G/H.

Proof. The domain of the map ψ is the image of the subgroup of finite index $\mathrm{Dom}\,\phi$ under the canonical homomorphism $G \to G/H$ and thus has finite index in G/H. Suppose that $g_1H = g_2H$ for some $g_1, g_2 \in \mathrm{Dom}\,\phi$. Then $g_1^{-1}g_2 \in H \cap \mathrm{Dom}\,\phi$, so $\phi(g_1^{-1}g_2) \in H$, thus $\phi(g_1)H = \phi(g_2)H$. □

The virtual endomorphism ψ is called the *quotient of* ϕ by the subgroup H and is denoted ϕ/H.

Proposition 3.2. *The subgroup*

$$\mathcal{C}(\phi) = \bigcap_{n\in\mathbb{N}} St_n(\phi) = \bigcap_{n\in\mathbb{N}} \bigcap_{g\in G} g^{-1} \cdot \mathrm{Dom}\,\phi^n \cdot g$$

is the maximal among normal ϕ-invariant subgroups of G.

The subgroup $\mathcal{C}(\phi)$ is the kernel of the action of G on the coset tree $T(\phi)$.

Proof. An element $h \in G$ belongs to $\mathcal{C}(\phi)$ if and only if every its conjugate belongs to $\mathrm{Dom}\,\phi^n$ for every $n \in \mathbb{N}$. From this follows that $\mathcal{C}(\phi)$ is normal and ϕ-invariant, since from $h \in \mathcal{C}(\phi)$ follows that all the conjugates of h and $\phi(h)$ belong to $\mathcal{C}(\phi)$.

On the other hand, if N is a normal, ϕ-invariant subgroup of G, then for every $h \in N$ the element $\phi^n(h)$ belongs to N for all $n \in \mathbb{N}$ and thus, $g^{-1}\phi^n(h)g \in N$ for all $g \in G$ and $n \in \mathbb{N}$. This implies that $h \in \mathcal{C}(\phi)$. □

Definition 3.4. The subgroup $\mathcal{C}(\phi)$ is called the *core* of the virtual endomorphism ϕ or the ϕ-*core* of G. The group G is said to be ϕ-*simple* if its ϕ-core is trivial.

Examples. 1) Let ϕ be the virtual endomorphism $n \mapsto n/2$ of $\mathbb{Z}$, with the domain equal to the set of even numbers. Then the group $\mathbb{Z}$ is obviously ϕ-simple.

2) More generally, if ϕ is a virtual endomorphism of the $\mathbb{Z}^n$, then $\mathbb{Z}^n$ is ϕ-simple if and only if no eigenvalue of the respective linear transformation is an algebraic integer (see [NS01]).

3) For examples of virtual endomorphisms of linear groups with trivial core, see the paper [NS01].

4) It is an open question, if the free group of rank 3 with the generators a, b, c is ϕ-simple, where ϕ is defined on the generators of its domain by the equalities

$$\begin{aligned}\phi(a^2) &= cb\\ \phi(b^2) &= bc\\ \phi(ab) &= c^2\\ \phi(c) &= a\\ \phi(a^{-1}ca) &= b^{-1}ab.\end{aligned}$$

This question is equivalent to a question of S. Sidki in [Sid00] and originates from an automaton, defined by S. V. Aleshin in [Ale83].

Proposition 3.3. *Let ϕ be a virtual endomorphism of a group G. If $H \leq G$ is a ϕ-invariant normal subgroup, then*

$$\mathcal{C}(\phi/H) = \mathcal{C}(\phi)/H.$$

Proof. Let K be a normal ϕ/H-invariant subgroup of G/H and let $\tilde{K}$ be its full preimage in G. Then $\tilde{K}$ is also normal. We have $K \leq \mathrm{Dom}\,(\phi/H)$, so every element of $\tilde{K}$ is a product of an element of $\mathrm{Dom}\,\phi$ and an element of H (see the definition of a quotient of a virtual endomorphism by a normal subgroup). But $H \leq \mathrm{Dom}\,\phi$, thus $\tilde{K} \leq \mathrm{Dom}\,\phi$. Let $\tilde{g} \in \tilde{K}$ be an arbitrary element and let g be its image in K. Then, by definition of ϕ/H, $\phi(\tilde{g})H = (\phi/H)(g)$, but $(\phi/H)(g) \in K$, so $\phi(\tilde{g}) \in \tilde{K}$ and the subgroup $\tilde{K}$ is ϕ-invariant.

On the other hand, if $\tilde{K}$ is a normal ϕ-invariant subgroup of G, then its image in G/H is also normal and ϕ-invariant.

This implies that the maximal ϕ/H-invariant normal subgroup of G/H is the image of the maximal ϕ/H-invariant normal subgroup of G. □

Corollary 3.4. *The group $G/\mathcal{C}(\phi)$ is $\phi/\mathcal{C}(\phi)$-simple.* □

In this way new groups can be constructed. We can start from some known group F, define a virtual endomorphism ϕ on it, and get the group $F/\mathcal{C}(\phi)$. If the group F is finitely generated, then the domain of ϕ is also finitely generated, and ϕ is uniquely determined by its value on the generators of its domains.

Example. The Grigorchuk group is the group $F/\mathcal{C}(\phi)$ for F the free group generated by $\{a, b, c, d\}$ and ϕ defined on the generators of its domain as

$$\begin{aligned} \phi(a^2) &= 1 & & \\ \phi(b) &= a & \qquad \phi(a^{-1}ba) &= c \\ \phi(c) &= a & \qquad \phi(a^{-1}ca) &= d \\ \phi(d) &= 1 & \qquad \phi(a^{-1}da) &= b. \end{aligned}$$

The next proposition shows that we can restrict in such constructions to the case when F is a free group.

Proposition 3.5. *Let ϕ be a virtual endomorphism of a finitely generated group G. Then there exist a virtual endomorphism $\tilde{\phi}$ of a finitely generated free group F, a $\tilde{\phi}$-invariant normal subgroup $K \leq F$ and an isomorphism $\rho : F/K \to G$ such that $\rho \circ \left(\tilde{\phi}/K\right) = \phi \circ \rho$. Then the quotient $F/\mathcal{C}\left(\tilde{\phi}\right)$ is isomorphic to $G/\mathcal{C}(\phi)$.*

Proof. Let $\{g_1, g_2, \ldots, g_n\}$ be a finite generating set of the group G. Set F to be the free group of rank n with the free generating set $\{\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_n\}$. Let $\pi : F \to G$ be the canonical epimorphism $\pi\left(\tilde{g}_i\right) = g_i$ and let K be the kernel of π, so that $F/K \cong G$. Denote by ρ the respective isomorphism $\rho : F/K \to G$.

The preimage $\pi^{-1}(\mathrm{Dom}\,\phi)$ is a subgroup of finite index in F, so it is a finitely generated free group. Let $\{h_1, h_2, \ldots, h_m\}$ be a free generating set of $\pi^{-1}(\mathrm{Dom}\,\phi)$. We can define a virtual endomorphism $\tilde{\phi}$ of the group F with the domain $\mathrm{Dom}\,\tilde{\phi} = \pi^{-1}(\mathrm{Dom}\,\phi)$ putting $\tilde{\phi}(h_i)$ to be equal to some of the elements of the set $\pi^{-1}\left(\phi(\pi(h_i))\right)$. Then we have $\pi(\tilde{\phi}(g)) = \phi\left(\pi\left(g\right)\right)$ for all $g \in \{h_1, h_2, \ldots, h_m\}$, and thus for all $g \in \mathrm{Dom}\,\tilde{\phi}$.

Note that $K \leq \pi^{-1}(\mathrm{Dom}\,\phi) = \mathrm{Dom}\,\tilde{\phi}$, and that for every $g \in K$ we have $\pi(\tilde{\phi}(g)) = \phi(\pi(g)) = 1$, so that $\tilde{\phi}(g) \in K$ and K is $\tilde{\phi}$-invariant. Then the equality $\pi \circ \tilde{\phi} = \phi \circ \pi$ is equivalent to the equality $\rho \circ \left(\tilde{\phi}/K\right) = \phi \circ \rho$.

We have, by Proposition 3.3

$$\mathcal{C}(\phi) = \rho\left(\mathcal{C}\left(\tilde{\phi}/K\right)\right) = \rho\left(\mathcal{C}\left(\tilde{\phi}\right)/K\right),$$

thus ρ induces an isomorphism of $F/\mathcal{C}\left(\tilde{\phi}\right)$ with $G/\mathcal{C}(\phi)$. $\square$

Definition 3.5. Let H be a subgroup of G. Define $\Delta_\phi(H)$ to be the set of all elements $g \in G$ such that for every $h \in G$ the element $h^{-1}gh$ belongs to $\mathrm{Dom}\,\phi$ and $\phi(h^{-1}gh) \in H$.

We write Δ_ϕ^n for the nth iteration of the operation Δ_ϕ.

Note that $\Delta_\phi^n(G)$ is the nth level stabilizer $St_n(\phi)$.

Proposition 3.6. *1. For every subgroup $H \le G$ the subgroup $\Delta_\phi(H)$ is normal and is contained in $St_1(\phi) \le \mathrm{Dom}\,\phi$.*

2. $\phi(\Delta_\phi(H)) \le H$.

3. If H is normal, then the virtual endomorphism ϕ induces a well defined virtual homomorphism $\overline{\phi} : G/\Delta_\phi(H) \dashrightarrow G/H$.

4. If the subgroup H is a normal ϕ-invariant subgroup, then $\Delta_\phi(H)$ is a normal ϕ-invariant subgroup.

5. A normal subgroup H is ϕ-invariant if and only if $H \le \Delta_\phi(H)$.

Proof. The first two claims follow directly from the definitions.

If H is normal, then the equality $\overline{\phi}(g\Delta_\phi(H)) = \phi(g)H$ gives a well defined virtual homomorphism $\overline{\phi} : G/\Delta_\phi(H) \dashrightarrow G/H$, since from $g_1^{-1}g_2 \in \Delta_\phi(H)$ follows that $\phi(g_1^{-1}g_2) \in H$.

If H is normal and ϕ-invariant, then $\phi(h^{-1}gh)$ is defined and belongs to H for every $h \in G$, thus $H \le \Delta_\phi(H)$. But then $\phi(\Delta_\phi(H)) \le H \le \Delta_\phi(H)$, so $\Delta_\phi(H)$ is ϕ-invariant.

If $H \le \Delta_\phi(H)$, then for every $g \in H \le \Delta_\phi(H)$ we have $\phi(g) \in H$, thus H is ϕ-invariant. $\square$

Definition 3.6. For any virtual endomorphism ϕ we define

$$\mathcal{E}_n(\phi) = \Delta_\phi^n(\{1\})$$

and $\mathcal{E}_\infty(\phi) = \cup_{n\ge 0}\mathcal{E}_n(\phi)$.

Proposition 3.6 implies that the subgroups $\mathcal{E}_n(\phi)$ are normal and ϕ-invariant for all $n = 0, 1, \ldots, \infty$. It also implies that $\mathcal{E}_n(\phi) \le \mathcal{E}_{n+1}(\phi)$ for all n.

Note also that if $\mathcal{E}_1(\phi) = \{1\}$, then $\mathcal{E}_n(\phi) = \{1\}$ for all $n = 0, 1, \ldots, \infty$. Therefore, $\mathcal{E}_\infty(\phi) = \{1\}$ if and only if $\mathcal{E}_1(\phi) = \{1\}$.

4. Bimodule associated to a virtual endomorphism

4.1. Permutational G-bimodules and the set $\phi(G)G$

Definition 4.1. Let G be a group. A *(permutational) G-bimodule* is a set M with left and right commuting actions of G on M, i.e., with two maps $G \times M \to M : (g, m) \mapsto g \cdot m$ and $M \times G \to M : (m, g) \mapsto m \cdot g$ such that

1. $1 \cdot m = m \cdot 1 = m$ for all $m \in M$;
2. $(g_1 g_2) \cdot m = g_1 \cdot (g_2 \cdot m)$ and $m \cdot (g_1 g_2) = (m \cdot g_1) \cdot g_2$ for all $g_1, g_2 \in G$ and $m \in M$;
3. $(g_1 \cdot m) \cdot g_2 = g_1 \cdot (m \cdot g_2)$ for all $g_1, g_2 \in G$ and $m \in M$.

Two bimodules M_1, M_2 are *isomorphic* if there exists a bijection $f : M_1 \to M_2$, which agrees with the left and the right actions, i.e., such that $g \cdot f(m) \cdot h = f(g \cdot m \cdot h)$ for all $g, h \in G$ and $m \in M_1$.

We say that the right action is *free* if for any $m \in M$ from $m \cdot g = m$ follows that $g = 1$. The right action is *d-dimensional* if the number of the orbits for the right action is d. The bimodule is *irreducible* if for any two elements $m_1, m_2 \in M$ there exist $g, h \in G$ such that $m_2 = g \cdot m_1 \cdot h$.

Proposition 4.1. *Suppose that M is an irreducible G-bimodule with a free d-dimensional right action. Take some $x \in M$. Let G_1 be the subset of all the elements $g \in G$ for which $g \cdot x$ and x belong to the same orbit of the right action. Then G_1 is a subgroup of index d in G and for every $g \in G_1$ there exists a unique $h \in G$ such that $g \cdot x = x \cdot h$. The map $\phi_x : g \mapsto h$ is a virtual endomorphism of the group G.*

The constructed virtual endomorphism ϕ_x is the *endomorphism, associated to the bimodule M (and the element x).*

Proof. The element h is uniquely defined, since the right action is free. The set G_1 is obviously a subgroup. The fact that the map ϕ_x is a homomorphism from G_1 to G follows directly from the definition of a permutational bimodule. The subgroup G_1 has index d in G, since the right action is d-dimensional, and the bimodule is irreducible. □

Proposition 4.2. *Let M be an irreducible G-bimodule with free d-dimensional right action. Then any two associated virtual endomorphisms ϕ_x and ϕ_y are conjugate. If ϕ is conjugate with an associated virtual endomorphism ϕ_x, then it is also associated to M, i.e., $\phi = \phi_y$ for some $y \in M$.*

Proof. Since the bimodule is irreducible, for every $x, y \in M$ there exist $g, h \in G$ such that $y = g \cdot x \cdot h$. Then for every $f \in G$ we have $f \cdot y = y \cdot \phi_y(f)$, what is equivalent to $fg \cdot x \cdot h = g \cdot x \cdot h\phi_y(f)$, i.e., to $g^{-1}fg \cdot x = x \cdot h\phi_y(f)h^{-1}$. It implies that $\phi_y(f) = h^{-1}\phi_x(g^{-1}fg)h$, i.e., that ϕ_y and ϕ_x are conjugate.

Similar arguments show that if $\phi(f) = h^{-1}\phi_x(g^{-1}fg)h$, then ϕ is the virtual endomorphism, associated to M and $g \cdot x \cdot h \in M$. □

Let us show that the bimodule M is uniquely determined, up to an isomorphism, by the associated virtual endomorphism.

Let ϕ be a virtual endomorphism of a group G. We consider the set $\phi(G)G$ of expressions of the form $\phi(g_1)g_0$, where $g_1, g_0 \in G$. Two expressions $\phi(g_1)g_0$ and $\phi(h_1)h_0$ are considered to be equal if and only if $g_1^{-1}h_1 \in \operatorname{Dom}\phi$, and $\phi(g_1^{-1}h_1) = g_0h_0^{-1}$.

Another way to describe this equivalence relation is to say that two expressions $\phi(g_1)g_0$ and $\phi(h_1)h_0$ are equal if and only if there exists an element $s \in G$ such that $sg_1, sh_1 \in \operatorname{Dom}\phi$ and $\phi(sg_1)g_0 = \phi(sh_1)h_0$ in G.

It is not hard to prove that the described relation is an equivalence.

Definition 4.2. Let $v = \phi(g_1)g_0$ be an element of $\phi(G)G$ and $g \in G$ be arbitrary. *Right action* of the group G on $\phi(G)G$ is defined by the rule $v \cdot g = \phi(g_1)g_0g$ and the *left action* is defined by $g \cdot v = \phi(gg_1)g_0$.

The actions are well defined, since from $\phi(g_1)g_0 = \phi(h_1)h_0$ follows that

$$\phi\left(g_1^{-1}h_1\right) = \phi\left(\left(gg_1\right)^{-1}\left(gh_1\right)\right) = g_0h_0^{-1} = (g_0g)(h_0g)^{-1},$$

thus $\phi(gg_1)g_0 = \phi(gh_1)h_0$ and $\phi(g_1)g_0g = \phi(h_1)h_0g$.

From the definition directly follows that the left and the right actions commute, i.e., that $(g \cdot v) \cdot h = g \cdot (v \cdot h)$ for all $g, h \in G$ and $v \in \phi(G)G$.

The set $\phi(G)G$ together with the left and right actions of the group G is called the *G-bimodule, associated to the virtual endomorphism ϕ*.

It is easy to see that the bimodule $\phi(G)G$ is irreducible. The right action is free, since from $\phi(g_1)g_0g = \phi(g_1)g_0$ follows that $\phi(g_1^{-1}g_1) = g_0^{-1}g_0g$, thus $g = 1$. The right action is $(\operatorname{ind}\phi)$-dimensional, since $\phi(g_1)g_0$ and $\phi(h_1)h_0$ belong to one orbit of the right action if and only if $g_1^{-1}h_1 \in \operatorname{Dom}\phi$.

Proposition 4.3. *Let M be an irreducible G-bimodule with free d-dimensional right action and let ϕ be its associated virtual endomorphism. Then the bimodule M is isomorphic to the bimodule $\phi(G)G$.*

Proof. Let us fix some $x_0 \in M$. Let $\phi = \phi_{x_0}$ be the virtual endomorphism associated to M and x_0. Define a map $F : \phi(G)G \to M$ by the rule $\phi(g_1)g_0 = g_1 \cdot x_0 \cdot g_0$.

If $\phi(g_1)g_0 = \phi(h_1)h_0$, then $g_1^{-1}h_1 \cdot x_0 = x_0 \cdot g_0h_0^{-1}$, thus $h_1 \cdot x_0 \cdot h_0 = g_1 \cdot x_0 \cdot g_0$, what implies that the map F is well defined.

On the other hand, if $h_1 \cdot x_0 \cdot h_0 = g_1 \cdot x_0 \cdot g_0$, then $g_1^{-1}h_1 \cdot x_0 = x_0 \cdot g_0h_0^{-1}$, i.e., $\phi(g_1)g_0 = \phi(h_1)h_0$, thus the map F is injective.

Since the bimodule M is irreducible, for every $x \in G$ one can find $g_1, g_0 \in G$ such that $x = g_1 \cdot x_0 \cdot g_0$, so the map F is a bijection.

We have $F(\phi(g \cdot g_1)g_0 \cdot h) = gg_1 \cdot x_0 \cdot g_0 h = g \cdot F(\phi(g_1)g_0) \cdot h$, thus the map F agrees with the right and the left multiplications, so it is an isomorphism of the G-bimodules. □

The next is a corollary of Propositions 4.2 and 4.3.

Corollary 4.4. *The G-bimodules $\phi_1(G)G$ and $\phi_2(G)G$ are isomorphic if and only if the virtual endomorphisms ϕ_1 and ϕ_2 are conjugate.* □

Example. 1. Consider a faithful self-similar action of a group G on the set X^*. Let $M = X \times G$ be a direct product of sets. The right action of the group G on M is the natural one:

$$(x \cdot g) \cdot h = x \cdot gh.$$

We write an element (x, g) of M as $x \cdot g$.

If $x \cdot g \in M$ and $h \in G$ then, by the definition of a self-similar action, there exists $h|_x \in G$ such that $h(xw) = h(x)h|_x(w)$ for all $w \in X^*$. We define the left action of G on M by the formula

$$h \cdot x \cdot g = h(x) \cdot h|_x g.$$

The obtained permutational bimodule M is called the *self-similarity bimodule* of the action. It is easy to see that the right action of the self-similarity bimodule is free and $|X|$-dimensional and that the bimodule is irreducible, if the action is transitive on the set X^1.

The self-similarity bimodule M is isomorphic to the permutational bimodule $\phi(G)G$, where ϕ is the virtual endomorphism, associated to the self-similar action.

Example. 2. Let $F : \mathcal{M}_0 \to \mathcal{M}$ be a d-fold covering map, where $\mathcal{M}$ is an arcwise connected and locally arcwise connected topological space and $\mathcal{M}_0$ is its open arcwise connected subset. Let $t \in \mathcal{M}$ be an arbitrary point.

Let L be the set of homotopy classes of the paths starting at t and ending at a point z such that $F(z) = t$. (We consider only the homotopies, fixing the endpoints.) Then the set L is a permutational $\pi_1(\mathcal{M}, t)$-bimodule for the following actions:

1. For all $\gamma \in \pi_1(\mathcal{M}, t)$ and $\ell \in L$:

$$\gamma \cdot \ell = \ell\gamma',$$

where γ' is the F-preimage of γ, which starts at the endpoint of ℓ.

2. For all $\gamma \in \pi_1(\mathcal{M}, t)$ and $\ell \in L$:

$$\ell \cdot \gamma = \gamma\ell.$$

It is not hard to prove that the described permutational bimodule is irreducible, free and d-dimensional from the right. Consequently, it is of the form $\phi(\pi_1(\mathcal{M}, t))\pi_1(\mathcal{M}, t)$, where ϕ is the associated virtual endomorphism. It is the endomorphism, defined by F, as in Subsection 2.2.

4.2. Quotients of a permutational bimodule

Definition 4.3. Let M_i be a permutational bimodule over a group G_i, $i = 1, 2$. The bimodule M_2 is a *quotient* of the bimodule M_1 if there exists a surjective map $p : M_1 \to M_2$ and a surjective homomorphism $\pi : G_1 \to G_2$ such that

$$\pi(g_1) \cdot p(m) \cdot \pi(g_2) = p(g_1 \cdot m \cdot g_2)$$

for all $g_1, g_2 \in G_1$ and $m \in M_1$.

Proposition 4.5. *Let ϕ_1 and ϕ_2 be virtual endomorphisms of the groups G_1 and G_2 respectively. Then the bimodule $\phi_2(G_2)G_2$ is a quotient of the bimodule $\phi_1(G_1)G_1$ if and only if there exists a normal ϕ_1-semi-invariant subgroup $N \leq G_1$ such that G_2 is isomorphic to G_1/N so that ϕ_2 is conjugate to ϕ_1/N.*

Proof. Suppose that the bimodule $\phi_2(G_2)G_2$ is a quotient of the bimodule $\phi_1(G_1)G_1$. Let $\pi : G_1 \to G_2$ be the respective homomorphism and let $p : \phi_1(G_1)G_1 \to \phi_2(G_2)G_2$ be the surjective map. Denote by N the kernel of the homomorphism π.

Replacing, if necessary ϕ_2 by a conjugate virtual endomorphism (see Proposition 4.2), we may assume that $p(\phi_1(1)1) = \phi_2(1)1$. Then

$$p(\phi_1(g_1)g_0) = p(g_1 \cdot \phi_1(1) \cdot g_0) = \pi(g_1)p(\phi_1(1)1)\phi(g_0) = \phi_2(\pi(g_1))\pi(g_0)$$

for all $g_0, g_1 \in G_1$.

If g is an element of $N \cap \mathrm{Dom}\, \phi_1$, then $\phi_1(g)1 = \phi_1(1)g'$ in $\phi_1(G_1)G_1$, where $g' = \phi(g)$, thus $p(\phi_1(g)1) = \phi_2(\pi(g))1 = \phi_2(1)1 = \phi_2(1)\pi(g')$. Hence, $\pi(g') = 1$, i.e., $\phi(g) \in N$ and the subgroup N is ϕ-semi-invariant. If g is an arbitrary element of $\mathrm{Dom}\, \phi_1$, then again

$$p(\phi_1(g)1) = \phi_2(\pi(g))1 = \phi_2(1)\pi(g')$$

for $g' = \phi_1(g)$. Consequently, $\pi(\phi_1(g)) = \phi_2(\pi(g))$, i.e., $\phi_2 = \phi_1/N$.

Suppose now that N is a normal ϕ_1-semi-invariant subgroup of G_1. Let us introduce an equivalence relation on $\phi_1(G_1)G_1$ by the rule:

$$\phi_1(g_1)g_0 \sim \phi_1(h_1)h_0$$

if and only if

$$g_1^{-1}h_1 \in \operatorname{Dom}\phi_1 \text{ and } \phi_1(g_1^{-1}h_1)h_0g_0^{-1} \in N.$$

It is easy to see that the defined relation is an equivalence and that the quotient of $\phi_1(G_1)G_1$ has a structure of a permutational bimodule over G_1/N, which is isomorphic to the bimodule $\phi_1/N\big(G_1/N\big)G_1/N$. Then Proposition 4.2 finishes the proof. □

4.3. Bimodules over group algebras

Definition 4.4. Let A be an algebra over a field $\Bbbk$. An A-*bimodule* is a $\Bbbk$-space Φ with structures of left and right A-modules such that the left and the right multiplications commute. In other words, two $\Bbbk$-linear maps $A \otimes_{\Bbbk} \Phi \to \Phi : a \otimes v \mapsto a \cdot v$ and $\Phi \otimes \Bbbk A \to \Phi : v \otimes a \mapsto v \cdot a$ are fixed such that

1. $(a_1a_2) \cdot v = a_1 \cdot (a_1 \cdot v)$ and $v \cdot (a_1a_2) = (v \cdot a_1) \cdot a_2$ for all $a_1, a_2 \in A$ and $v \in \Phi$;

2. $(a_1 \cdot v) \cdot a_2 = a_1 \cdot (v \cdot a_2)$ for all $a_1, a_2 \in A$ and $v \in \Phi$.

If M is a permutational G-bimodule, and $\Bbbk$ is a field, than the left and the right actions of G on M extend by linearity to a structure of $\Bbbk G$-bimodule on the linear space $\langle M \rangle_{\Bbbk}$. Here $\langle M \rangle_{\Bbbk}$ denotes the linear space over the field $\Bbbk$ with the basis M, and $\Bbbk G$ is the group algebra of G over the field $\Bbbk$. The $\Bbbk G$-bimodule $\langle M \rangle_{\Bbbk}$ is called *linear span* of the permutational bimodule M.

In particular, if ϕ is a virtual endomorphism of the group G, then the linear span $\Phi = \Phi_{\Bbbk}$ over $\Bbbk$ of the permutational bimodule $\phi(G)G$ is called the *bimodule, associated to* ϕ. By Φ_R and Φ_L we denote the underlying right and left modules, respectively.

We get directly from Corollary 4.4 the next

Proposition 4.6. *Let ϕ_1 and ϕ_2 be conjugate virtual endomorphisms of a group G. Then the respective associated bimodules Φ_1 and Φ_2 are isomorphic.* □

4.4. Inner product

Definition 4.5. Let Φ be the $\mathbb{C}$-span of the permutational bimodule $\phi(G)G$. The group algebra $\mathbb{C}G$ is equipped with the involution $(\alpha g)^* = \overline{\alpha} g^{-1}$, where $\overline{\alpha}$ is the complex conjugation.

The *inner product* on the bimodule Φ, associated to the virtual endomorphism ϕ is the function $\langle \cdot | \cdot \rangle : \Phi \times \Phi \to \mathbb{C}G$, defined by the conditions:

1. the function $\langle \cdot | \cdot \rangle$ is linear over the second variable;
2. $\langle v_1 | v_2 \rangle = \langle v_2 | v_1 \rangle^*$ for all $v_1, v_2 \in \Phi$;
3. $\langle \phi(g_1)h_1 | \phi(g_2)h_2 \rangle = 0$ if $g_1^{-1}g_2 \notin \operatorname{Dom}\phi$ and
$$\langle \phi(g_1)h_1 | \phi(g_2)h_2 \rangle = h_1^{-1}\phi(g_1^{-1}g_2)h_2$$
otherwise.

In general, even if $\Bbbk$ is not equal to $\mathbb{C}$, the last condition of the definition gives a well defined function $\langle \cdot | \cdot \rangle : \phi(G)G \times \phi(G)G \to G \cup \{0\}$, which will be also called *inner product*.

Proposition 4.7. *The equality*
$$\langle v_1 | g \cdot v_2 \rangle = \left\langle g^{-1} \cdot v_1 \middle| v_2 \right\rangle \tag{5}$$
holds for all $v_1, v_2 \in \phi(G)G$ and $g \in G$.

If $\langle v_1 | v_2 \rangle \neq 0$ for $v_1, v_2 \in \phi(G)G$, then
$$v_1 \cdot \langle v_1 | v_2 \rangle = v_2. \tag{6}$$

Proof. Let $v_i = \phi(g_i)h_i$ for $i = 1, 2$. Then, for equality (5):
$$\langle v_1 | gv_2 \rangle = h_1^{-1}\phi(g_1^{-1}gg_2)h_2^{-1} = h_1^{-1}\phi\left(\left(g^{-1}g_1\right)^{-1}g_2\right)h_2 = \left\langle g^{-1} \cdot v_1 \middle| v_2 \right\rangle.$$

For equality (6):
$$v_1 \cdot \langle v_1 | v_2 \rangle = \phi(g_1)h_1 \cdot h_1^{-1}\phi(g_1^{-1}g_2)h_2 = \phi(g_2)h_2.$$
$\square$

As a corollary, we get, that in the case $\Bbbk = \mathbb{C}$ we have
$$\langle v_1 | a \cdot v_2 \rangle = \langle a^* \cdot v_1 | v_2 \rangle \tag{7}$$
for all $v_1, v_2 \in \Phi$ and $a \in \mathbb{C}G$.

4.5. Standard bases and wreath products

Definition 4.6. A *basis* of a permutational G-bimodule M is an orbit transversal of the right action.

A *standard basis* of the bimodule Φ, associated to a virtual endomorphism ϕ is the set of the form

$$\{\phi(r_1)h_1, \phi(r_2)h_2, \ldots, \phi(r_d)h_d\},$$

where $\{r_1, r_2, \ldots, r_d\}$ is a left coset transversal of the subgroup $\operatorname{Dom}\phi$ in G and $\{h_1, h_2, \ldots, h_d\}$ is an arbitrary sequence of elements of the group G.

It is easy to see that the notions of standard basis of the bimodule Φ and standard basis of the permutational bimodule $\phi(G)G$ coincide.

Proposition 4.8. *Every standard basis of the bimodule Φ is a free $\Bbbk G$-basis of the right module Φ_R. In particular, the module Φ_R is a free right $\Bbbk G$-module of dimension* $\operatorname{ind}\phi$. □

Note also, that directly from the definitions follows that the standard basis is *orthonormal*, i.e., that $\langle x_i |\, x_j\rangle$ is 0 for $i \neq j$ and 1 for $i = j$.

Since the left and the right multiplications commute, we get a homomorphism

$$\psi_L : \Bbbk G \to \operatorname{End}_{\Bbbk}(\Phi_R) = M_{d\times d}(\Bbbk G)$$

defined by the rule $\psi_L(a)(v) = a \cdot v$. By Proposition 4.8, the algebra $\operatorname{End}_{\Bbbk}(\Phi_R)$ is isomorphic to the algebra $M_{d\times d}(\Bbbk G)$ of $d \times d$-matrices over $\Bbbk G$. Here, as usual $d = \operatorname{ind}\phi$. We call the homomorphism ψ_L the *linear recursion, associated to ϕ.*

The linear recursion is computed using the formula in the next proposition, which follows directly from the definitions.

Proposition 4.9. *Let $X = \{x_1 = \phi(r_1)h_1, x_2 = \phi(r_2)h_2, \ldots, x_d = \phi(r_d)h_d\}$ be a standard basis of Φ_R. Then for any $g \in G$ and $x_i \in X$ we have*

$$g \cdot x_i = x_j \cdot h_j^{-1}\phi\left(r_j^{-1} g r_i\right) h_i,$$

where j is uniquely defined by the condition $r_j^{-1} g r_i \in \operatorname{Dom}\phi$. □

The formula in Proposition 4.9 can be also interpreted as a homomorphism $\psi : G \to Symm(X) \wr G$, where "$\wr$" is the wreath product and $Symm(X)$ is the symmetric group on X. Let us recall at first the notion of a permutational wreath product.

Definition 4.7. Let G be a group and let H be a permutation group of a set X. Then the *(permutational) wreath product* $H \wr G$ is the semidirect product $H \ltimes G^X$, where H acts on the group G^X by the respective permutations of the direct multiples.

The elements of the wreath product $H \wr G$ are written as products $h \cdot f$, where $f \in G^X$ and h is an element of H. The element f can be considered either as a function from X to G, or as a tuple $(g_1, g_2, \ldots g_d)$, if an indexing $X = \{x_1, x_2, \ldots, x_d\}$ of the set X is fixed. In the last case the multiplication rule for the elements of $H \wr G$ are the following:

$$h'(g'_1, g'_2, \ldots g'_d) \cdot h(g_1, g_2, \ldots g_d) = h'h(g'_{h(1)}g_1, g'_{h(2)}g_2, \ldots, g'_{h(d)}g_d), \quad (8)$$

where $h(i)$ is the index for which $h(x_i) = x_{h(i)}$.

In the case of a standard basis Proposition 4.9 implies that for every $g \in G$ and $x \in X$ there exist $y \in X$ and $h \in G$ such that $g \cdot x = y \cdot h$. It is easy to see that $x \mapsto y$ is a permutation of the set X. Let us denote this permutation by σ_g. In this way we get a homomorphism $g \mapsto \sigma_g$ of G to the symmetric group $Symm(X)$. The kernel of this homomorphism is the first-level stabilizer $St_1(\phi)$.

Proposition 4.10. *The map*

$$\psi : g \mapsto \sigma_g(h_{i_1}^{-1}\phi(r_{i_1}^{-1}gr_1)h_1, h_{i_2}^{-1}\phi(r_{i_2}^{-1}gr_2)h_2, \ldots, h_{i_d}^{-1}\phi(r_{i_d}^{-1}gr_d)h_d),$$

where the sequence $(i_1, i_2, \ldots, i_d)$ *is such that* $r_{i_k}^{-1}gr_k \in \operatorname{Dom}\phi$ *for all* $k = 1, 2, \ldots, k$ *and* σ_g *is the permutation* $k \mapsto i_k$*, is a homomorphism* $\psi : G \to Symm(X) \wr G$.

Proof. If $\psi(g) = \sigma_g(g_1, g_2, \ldots g_d)$ and $\psi(h) = \sigma_h(h_1, h_2, \ldots, h_d)$ then $hg \cdot x_i = h \cdot x_j \cdot g_i = \sigma_h\sigma_g(x_i) \cdot h_j g_i$, where $x_j = \sigma_g(x_i)$. This agrees with the multiplication formula (8), thus $\psi(hg) = \psi(h)\psi(g)$. □

The obtained homomorphism $\psi : G \to Symm(X) \wr G$ is called the *wreath product recursion* associated to the virtual endomorphism ϕ (and the basis X).

On the other hand, any homomorphism $\psi : G \to Symm(X) \wr G$ is associated to some virtual endomorphism. It is the virtual endomorphism ϕ which is defined on $g \in G$ if and only if $\psi(g) = \sigma(g_1, g_2, \ldots, g_d)$, where $\sigma(x_1) = x_1$. If ϕ is defined on g, then $\phi(g) = g_1$. Let us choose a left coset transversal $T = \{r_1, r_2, \ldots, r_d\}$ of $\operatorname{Dom}\phi$ such that $r_i = \sigma_i(r_{i1}, \ldots, r_{id})$, where $\sigma_i(x_1) = x_i$. Then $Y = \{y_1 = \phi(r_1)r_{11}^{-1}, y_2 = \phi(r_2)r_{21}^{-1}, \ldots, y_d = \phi(r_d)r_{d1}^{-1}\}$ is a standard basis of the respective module Φ_R. Then a direct computation shows that the homomorphism ψ is reconstructed back as the wreath product recursion, associated to the virtual endomorphism ϕ and the basis Y.

Example. Let us consider the virtual endomorphism $\phi(n) = n/2$ of the group $\mathbb{Z}$. Its domain is the subgroup of even numbers. The coset transversal is in this case, for example, the set $\{0, 1\}$.

Let us write the elements of the group $\mathbb{Z}$ in a multiplicative notation, so that $\mathbb{Z}$ is identified with the infinite cyclic group, generated by an element τ. Then, the coset transversal is written as $\{1, \tau\}$.

Thus we choose the following standard basis of the permutational bimodule $\phi(\mathbb{Z}) + \mathbb{Z}$:

$$X = \{\mathit{0} = \phi(1)1, \mathit{1} = \phi(\tau)1\}.$$

Then the wreath recursion is

$$\psi(\tau) = \sigma(1, \tau),$$

where σ is the transposition $(\mathit{0}, \mathit{1})$ of the set X.

The respective linear recursion is

$$\psi(\tau) = \left(\begin{array}{cc} 0 & \tau \\ 1 & 0 \end{array} \right).$$

Proposition 4.11. *The kernel of the wreath product recursion associated to a virtual endomorphism ϕ is equal to $\mathcal{E}_1(\phi)$.*

Proof. An element $g \in G$ belongs to the kernel of ψ if and only if $g \cdot x_i = x_i \cdot 1$ for every $x_i \in X$. Hence, $g \in \ker \psi$ if and only if $g \in St_1(\phi)$ and $h_i^{-1}\phi(r_i^{-1} g r_i) h_i = 1$, i.e., $\phi(r_i^{-1} g r_i) = 1$. But $\{r_i\}$ is the left coset representative system, so $g \in \ker \psi$ if and only if for every $h \in G$ the element $h^{-1}gh$ belongs to $\operatorname{Dom} \phi$ and $\phi(h^{-1}gh) = 1$ $\square$

4.6. Φ-invariant ideals

Definition 4.8. Let Φ be a bimodule over a $\Bbbk$-algebra A and let I be a two-sided ideal of A. Denote by $I \cdot \Phi$ the $\Bbbk$-subspace of Φ spanned by the elements of the form $a \cdot v$, where $a \in I$ and $v \in \Phi$. Analogically, denote by $\Phi \cdot I$ the subspace spanned by the elements $v \cdot a$.

If $I \subset A$ is a two-sided ideal in A then its Φ-*preimage* is the set

$$\Phi^{-1}(I) = \{a \in A \ : \ a \cdot v \in \Phi \cdot I \text{ for all } v \in \Phi\}.$$

Proposition 4.12. *For every two-sided ideal $I \subset A$ the sets $I \cdot \Phi$ and $\Phi \cdot I$ are sub-bimodules of Φ and the set $\Phi^{-1}(I)$ is a two-sided ideal of A.*

Proof. Let $a \in A$ and $v \in I \cdot \Phi$ be arbitrary. Then v is a linear combination over $\Bbbk$ of the elements of the form $b \cdot u$, where $b \in I$ and $v \in \Phi$. Hence, $a \cdot v$ and $v \cdot a$ are linear combinations of the elements of the form $ab \cdot u$ and $b \cdot (u \cdot a)$, respectively. But $ab \in I$, so that $ab \cdot u \in I \cdot \Phi$. The element $b \cdot (u \cdot a)$ belongs to $I \cdot \Phi$ by definition. Thus, $a \cdot v$ and $v \cdot a$ belong to the set $I \cdot \Phi$ and it is a sub-bimodule. The fact that $\Phi \cdot I$ is a sub-bimodule is proved in the same way.

Let $a_1, a_2 \in \Phi^{-1}(I)$ and $a \in A$ be arbitrary. Then for every $v \in \Phi$ we have $a_1 \cdot v, a_2 \cdot v \in \Phi \cdot I$, thus $(a_1 + a_2) \cdot v = a_1 \cdot v + a_2 \cdot v \in \Phi \cdot I$, since $\Phi \cdot I$ is closed under addition. We also have $aa_1 \cdot v = a \cdot (a_1 \cdot v) \in \Phi \cdot I$, since $\Phi \cdot I$ is a left-submodule of Φ; and $a_1 a \cdot v = a_1 \cdot (a \cdot v) \in \Phi \cdot I$, since $a_1 \in \Phi^{-1}(I)$. ☐

Definition 4.9. An ideal I is said to be Φ-*invariant* if $I \subseteq \Phi^{-1}(I)$. The algebra A is said to be Φ-*simple* if it has no proper Φ-invariant two-sided ideals.

An ideal I is Φ-invariant if and only if $I \cdot \Phi \subseteq \Phi \cdot I$.

Suppose that the ideal I is Φ-invariant. Denote by Φ/I the quotient of the $\Bbbk$-spaces $\Phi/(\Phi \cdot I)$. Then Φ/I has a structure of an A/I-bimodule, defined as

$$(a+I)\cdot(v + \Phi \cdot I) = a\cdot v+\Phi\cdot I, \qquad (v + \Phi \cdot I)\cdot(a+I) = v\cdot a+\Phi\cdot I. \quad (9)$$

It is easy to prove, using Proposition 4.12, that multiplications (9) are well defined.

Example. If Φ is associated to a virtual endomorphism ϕ of a group G and N is a normal ϕ-invariant subgroup of G, then the ideal of $\Bbbk G$ generated by $1 - N$ is Φ-invariant, since

$$\begin{aligned}\phi((1-g)g_1)g_0 &= \phi(g_1)g_0 - \phi(gg_1)g_0 = \\ &\phi(g_1)g_0 - \phi(g_1)g_0 \cdot \left(g_0^{-1}\phi\left(g_1^{-1}gg_1\right)g_0\right) = \\ &\qquad \phi(g_1)g_0\left(1 - g_0^{-1}\phi\left(g_1^{-1}gg_1\right)g_0\right).\end{aligned}$$

Consequently, if G is not ϕ-simple, then $\Bbbk G$ is not Φ-simple.

In fact, the operation Φ^{-1} on ideals is an exact analog of the operation Δ_ϕ on the normal subgroups of the group G. Namely, the above formula shows that if H is a normal subgroup, then $\Phi^{-1}\big((1-H)\big) = (1-\Delta_\phi(H))$, where (A) denotes the two-sided ideal of $\Bbbk G$ generated by the set A.

The algebra $\Bbbk G$ needs not to be Φ-simple even if the group G is ϕ-simple. But a Φ-simple quotient of the algebra $\Bbbk G$ can be constructed from a ϕ-simple group by the following construction, which is essentially due to S. Sidki.

Let us define a sequence of ideals in $\Bbbk G$:

$$\mathcal{I}_0 = \{0\}, \quad \mathcal{I}_n = \Phi^{-1}\left(\mathcal{I}_{n-1}\right) \text{ for } n \geq 1, \quad \mathcal{I}_\infty = \bigcup_{n\geq 0} \mathcal{I}_n \tag{10}$$

It is easy to see that $\mathcal{I}_{n+1} \supseteq \mathcal{I}_n$, that $\mathcal{I}_n$ are Φ-invariant and that $\Phi^{-1}(\mathcal{I}_\infty) = \mathcal{I}_\infty$. The ideals $\mathcal{I}_n$ and $\mathcal{I}_\infty$ are analogs of the ϕ-invariant subgroups $\mathcal{E}_n(\phi)$, $\mathcal{E}_\infty(\phi)$, defined before. In particular, the ideal $\mathcal{I}_1$ is exactly the kernel of the linear recursion $\psi_L : \Bbbk G \to \mathrm{End}_{\Bbbk G}(\Phi_R)$, which parallels Proposition 4.11.

Theorem 4.13. *Let G be a ϕ-simple group and let I be a proper Φ-invariant ideal of $\Bbbk G$. Then $I \subseteq \mathcal{I}_\infty$. In particular, the algebra $\Bbbk G/\mathcal{I}_\infty$ is $\Phi/\mathcal{I}_\infty$-simple.*

Let us prove at first the following lemmas.

Lemma 4.14. *Let I be a Φ-invariant ideal of A and let J be a Φ/I-invariant ideal of A/I. Then the full preimage $\tilde{J}$ of J in A is Φ-invariant.*

Proof. Let a belong to $\tilde{J}$. This means that $a + I$ belongs to J. Then for every $v \in \Phi$ the element $(a + I)(v + \Phi \cdot I)$ belongs to $(\Phi/I) \cdot J$, since it belongs to $J \cdot (\Phi/I)$ and J is Φ/I-invariant. But $(a + I)(v + \Phi \cdot I) = a \cdot v + I \cdot \Phi \cdot I \subseteq a \cdot v + \Phi \cdot I$, since I is Φ-invariant. Thus the coset $a \cdot v + \Phi \cdot I$ is a subset of $\Phi \cdot \tilde{J}$, which is the preimage of $(\Phi/I) \cdot J$. In particular, $a \cdot v \in \Phi \cdot \tilde{J}$, and the ideal $\tilde{J}$ is Φ-invariant. □

Lemma 4.15. *Let $\{r_1, r_2, \ldots, r_d\} \subset G$ be a left coset transversal of $\mathrm{Dom}\,\phi$ in G. Let I be an ideal of $\Bbbk G$. Then $a = \alpha_1 g_1 + \alpha_2 g_2 + \cdots \alpha_m g_m \in \Bbbk G$, were $\alpha_i \in \Bbbk$ and $g_i \in G$, belongs to $\Phi^{-1}(I)$ if and only if for every $i = 1, 2, \ldots, d$ the sum*

$$a_i = \sum_{g_j r_i \in \mathrm{Dom}\,\phi} \alpha_j \phi(g_j r_i)$$

belongs to I.

Proof. The set $v_i = \{\phi(r_i) \cdot 1\}_{i=1,\ldots,d}$ is a $\Bbbk G$-basis of the right module Φ_R. Consequently, $v \in \Phi$ is an element of $\Phi \cdot I$ if and only if $v = \sum_{i=1}^{d} v_i \cdot b_i$, where $b_i \in I$. We also obviously have that $a \in \Phi^{-1}(I)$ if and only if $a \cdot v_i \in \Phi \cdot I$ for every $i = 1, \ldots, d$. But

$$a \cdot v_i = \sum_{j=1}^{d} v_j \cdot a_j,$$

where the elements a_j are defined as in the proposition. □

Proof of Theorem 4.13. Choose a left coset transversal

$$\{r_1 = 1, r_2, \dots, r_d\} \subset G$$

of $\mathrm{Dom}\,\phi$ in G. Suppose that I is not a subset of $\mathcal{I}_\infty$. Let $\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_m g_m$ be an element of I not belonging to $\mathcal{I}_\infty$ with the minimal possible m. By Lemma 4.15 the elements $a_i = \sum_{g_j r_i \in \mathrm{Dom}\,\phi} \alpha_i \phi(g_j r_i)$ belong to I. There exists i such that $a_i \notin \mathcal{I}_\infty$, otherwise all $a_i \in \mathcal{I}_n$ for some n and thus $a \in \Phi^{-1}(\mathcal{I}_n) = \mathcal{I}_{n+1} \subseteq \mathcal{I}$. But a was chosen to be the shortest element of $I \setminus \mathcal{I}_\infty$. Thus, the only possibility is that one a_{i_0} is equal to $\sum_{j=1}^m \alpha_j \phi(g_j r_{i_0}) \in I \setminus \mathcal{I}_\infty$ and for all the other i we have $a_i = 0$. Then a_{i_0} is again a minimal element of $I \setminus \mathcal{I}_\infty$ and we can repeat the considerations.

It follows that for any two indices $1 \le i, j \le m$ we have $g_i g_j^{-1} \in \mathrm{Dom}\,\phi$. On the next step we get $\phi(g_i r_{i_0})\phi(g_j r_{i_0})^{-1} = \phi(g_i g_j^{-1}) \in \mathrm{Dom}\,\phi$ and then by induction, that $g_i g_j^{-1} \in \mathrm{Dom}\,\phi^n$ for all $1 \le i, j \le m$ and $n \in \mathbb{N}$. Considering $ga = \sum_{i=1}^m \alpha_i g g_i$ we prove that $g(g_i g_j^{-1})g^{-1} \in \mathrm{Dom}\,\phi^n$. Hence, $g_i g_j^{-1}$ belongs to the core $\mathcal{C}(\phi)$ of virtual endomorphism, which is trivial. Consequently, all g_i are equal, i.e., $m = 1$ and $a = \alpha_1 g_1$ for some $g_1 \in G$. But then $1 \in I$ and $I = \Bbbk G$. Contradiction.

The $\Phi/\mathcal{I}_\infty$-simplicity of $\Bbbk G/\mathcal{I}_\infty$ follows now directly from Lemma 4.14. □

Example. The first paper, where the algebra $\Bbbk G/\mathcal{I}_\infty$ was considered is [Sid97]. It is investigated there for the case of the Gupta-Sidki group [GS83a] and the field $\mathbb{F}_3$.

The Gupta-Sidki group can be defined as the group $G = F/\mathcal{C}(\phi)$, where F is the free group generated by two elements a, b and ϕ is its virtual endomorphism

$$\phi(a^3) = 1, \quad \phi(b) = b, \quad \phi(a^{-1}ba) = a, \quad \phi(a^{-2}ba^2) = a^{-1}.$$

It is proved in [GS83a] that G is a torsion 3-group, i.e., that every its element is of order 3^k. S. Sidki proved that the ring $\Bbbk G/\mathcal{I}_\infty$ for $\Bbbk = \mathbb{F}_3$ is primitive and is just-infinite, i.e., that every its proper quotient is finite-dimensional.

4.7. Tensor powers of the bimodule

We define the set $\phi^n(G)\phi^{n-1}(G)\dots\phi(G)G$, analogically to the set $\phi(G)G$, as the set of formal expressions of the form

$$\phi^n(g_n)\phi^{n-1}(g_{n-1})\cdots g_0,$$

where an expression $\phi^n(g_n)\phi^{n-1}(g_{n-1})\dots g_0$ is identified with an expression

$$\phi^n(h_{n-1})\phi^{n-1}(h_{n-1})\dots h_0$$

if and only if there exists $s \in G$ such that

$$\phi\left(\phi\left(\phi\left(sg_n\right)g_{n-1}\right)\dots g_1\right)g_0 = \phi\left(\phi\left(\phi\left(sh_n\right)h_{n-1}\right)\dots h_1\right)h_0$$

in G.

The group G acts on the set $\phi^n(G)\phi^{n-1}(G)\dots G$ on the left by

$$g : \phi^n(g_n)\phi^{n-1}(g_{n-1})\dots g_0 \mapsto \phi^n(gg_1)\phi^{n-1}(g_{n-1})\dots g_0$$

and on the right by

$$g : \phi^n(g_n)\phi^{n-1}(g_{n-1})\dots g_0 \mapsto \phi^n(g_n)\phi^{n-1}(g_{n-1})\dots g_0 g.$$

It is easy to see that these actions are well defined.

We have the following natural interpretation of the set

$$\phi^n(G)\phi^{n-1}(G)\dots G$$

in terms of the associated bimodule.

Recall, that if Φ_1 and Φ_2 are two A-bimodules, then their tensor product is the bimodule $\Phi_1 \otimes_A \Phi_2$ which, as a $\Bbbk$-space is the quotient of the $\Bbbk$-tensor product $\Phi_1 \otimes_{\Bbbk} \Phi_2$ by the $\Bbbk$-subspace, spanned by the elements

$$(v_1 \cdot a) \otimes v_2 - v_1 \otimes (a \cdot v_2),$$

for all $v_1 \in \Phi_1, v_2 \in \Phi_2, a \in A$. The left and the right multiplications are defined by the rules $a_1 \cdot (v_1 \otimes v_2) \cdot a_2 = (a_1 \cdot v_1) \otimes (v_2 \cdot a_2)$. We will denote in the sequel the tensor product $\Phi_1 \otimes_A \Phi_2$ just as $\Phi_1 \otimes \Phi_2$.

Proposition 4.16. *The linear span over the field $\Bbbk$ of the permutational bimodule $\phi^n(G)\dots\phi(G)G$ is isomorphic to the nth tensor power $\Phi^{\otimes n} = \underbrace{\Phi \otimes \Phi \otimes \dots \otimes \Phi}_{n \text{ times}}$ of the bimodule Φ associated to the virtual endomorphism ϕ.*

Proof. Consider the map $F_1 : \phi^n(G)\phi^{n-1}(G)\dots G \to \Phi^{\otimes n}$ defined as

$$F_1\bigl(\phi^n(g_n)\phi^{n-1}(g_{n-1})\dots g_0\bigr) = \phi(g_n)1 \otimes \phi(g_{n-1})1 \otimes \dots \otimes \phi(g_2)1 \otimes \phi(g_1)g_0.$$

It is easy to see that the map F_1 preserves the left and the right multiplications by the elements of G, thus, it can be extended to a morphism of $\Bbbk G$-bimodules.

On the other hand, the map $F_2 : \Phi^{\otimes n} \to \phi^n(G)\phi^{n-1}(G)\dots G$ defined as

$$F_2\big(\phi(g_n)h_n \otimes \phi(g_{n-1})h_{n-1} \otimes \dots \otimes \phi(g_1)h_1\big) = \\ = \phi^n(g_n)\phi^{n-1}(h_n g_{n-1})\phi^{n-2}(h_{n-1}g_{n-2})\dots\phi(h_2 g_1)h_1$$

also can be extended to a morphism of $\Bbbk G$-bimodules and is inverse to the map F_1. Thus, the maps F_1 and F_2 are isomorphisms of the bimodules. $\square$

4.8. Standard actions

Proposition 4.17. *Let $X = \{x_i = \phi(r_i)h_i\}_{i=1,\dots,d}$ be a standard basis of the right module Φ_R. Then the set*

$$X^n = \{x_{i_1} \otimes x_{i_2} \otimes \dots \otimes x_{i_n} \ : \ x_{i_k} \in X\}$$

is a basis of the right module of the bimodule $\Phi^{\otimes n}$.

Proof. The set $\{x \cdot g \ : \ x \in X, g \in G\}$ is a $\Bbbk$-basis of the space Φ. Consequently, the set M_n of the elements of the form $x_{i_1} \cdot g_1 \otimes x_{i_2} \cdot g_2 \otimes \dots \otimes x_{i_n} \cdot g_n$ is a $\Bbbk$-basis of the tensor power $\Phi^{\otimes_{\Bbbk} n}$. By Proposition 4.9, every element of M_n can be reduced to the form $x_{j_1} \otimes x_{j_2} \otimes \dots \otimes x_{j_n} \cdot h$, where h is some element of G. It is easy to see that such reduction is unique, and that two elements of M_n are equal if and only if the respective reductions coincide. From this follows that the set X^n is a basis of the right $\Bbbk G$-module of $\Phi^{\otimes n}$. $\square$

For every $n \geq 1$ we get a homomorphism $\psi^{\otimes n} : \Bbbk G \to \operatorname{End} \Phi_R^{\otimes n}$ coming from the left multiplications seen as endomorphisms of the right module $\Phi_R^{\otimes n}$. For every standard basis X, the set X^n is a free basis of the right module $\Phi_R^{\otimes n}$, and thus, the module $\Phi^{\otimes n}$ is free $|X|^n$-dimensional, and the algebra $\operatorname{End} \Phi_R^{\otimes n}$ is isomorphic to the algebra of $|X|^n \times |X|^n$-matrices over the algebra $\Bbbk G$. The homomorphisms maps $\psi^{\otimes n}$ are called the *iterated linear recursions.*

More generally, the bimodule structure defines natural homomorphisms $\psi_n : \operatorname{End} \Phi_R^{\otimes n} \to \operatorname{End} \Phi_R^{\otimes(n+1)}$. Namely, if g is an endomorphism of the right module $\Phi_R^{\otimes n}$, then its image in $\operatorname{End} \Phi_R^{\otimes(n+1)}$ is the endomorphism $\psi_n(g)$ defined as

$$\psi_n(g)(v_1 \otimes v) = g(v_1) \otimes v,$$

where $v_1 \in \Phi^{\otimes n}$ and $v \in \Phi$.

The defined homomorphism ψ_n agrees with the introduced linear recursions, i.e., $\psi_n \circ \psi^{\otimes n} = \psi^{\otimes(n+1)}$.

Note that the kernel of the iterated linear recursion $\psi^{\otimes n}$ is the ideal $\mathcal{I}_n$.

We will write in many cases the element $x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n} \in X^n$ as a word $x_{i_1} x_{i_2} \ldots x_{i_n} \in X^*$. Then the set X^n is identified with the set of the words of length n over the alphabet X.

It follows from Proposition 4.9 that for every $v \in X^n$ and for every $g \in G$ there exists a unique pair (u, h), where $u \in X^n$ and $h \in G$, such that

$$g \cdot v = u \cdot h. \tag{11}$$

It is easy to see that the map $v \mapsto u$ is a permutation of the set X^n and that in this way we get an action of the group G on the set X^n. Taking union we get an action of G on the set X^*. We will call this action the *standard action of G with respect to the basis X*.

It follows from Equation (11) that the standard actions are self-similar in sense of Definition 2.7.

The element h in (11) is called the *restriction* of g at v and is denoted $g|_v$. The image of a word v under the action of $g \in G$ and the restriction $g|_v$ can be computed inductively using Proposition 4.9. This notion of a restriction is a generalization of the previously defined notion for self-similar actions. In particular, the properties (2) and (3) hold, and if the action is faithful, then the restriction is defined uniquely by the condition that $g(vu) = g(v)g|_v(u)$ for all $u \in X^*$.

Proposition 4.18. *Take any faithful self-similar action of a group G over the alphabet $X = \{x_1, x_2, \ldots, x_d\}$. Let $\phi = \phi_{x_1}$ be the associated virtual endomorphism. Take elements r_i for $i = 1, 2, \ldots, d$ such that $r_i(x_1) = x_i$. Let $h_i = r_i|_{x_1}$. Then $\tilde{X} = \{\tilde{x}_1 = \phi(r_1)h_1^{-1}, \tilde{x}_2 = \phi(r_2)h_2^{-1}, \ldots, \tilde{x}_d = \phi(r_d)h_d^{-1}\}$ is a standard basis of the bimodule Φ, associated to the virtual endomorphism ϕ and the original action of G on X^* coincides with the standard action of G with respect to the basis $\tilde{X}$, i.e.,*

$$g(x_{i_1} x_{i_2} \ldots x_{i_n}) = g(\tilde{x}_{i_1} \tilde{x}_{i_2} \ldots \tilde{x}_{i_n})$$

for every $g \in G$ and $x_{i_1} x_{i_2} \ldots x_{i_n} \in X^*$.

Proof. Let g be an arbitrary element of the group G and let $x_i \in X$ be an arbitrary letter. Let $g(x_i) = x_j$. Then $s_j^{-1} g s_i(x_1) = x_1$, so that $s_j^{-1} g s_i \in \operatorname{Dom}\phi$. For every $v \in X^*$ we have $s_j^{-1} g s_i(x_1 v) = x_1 \phi(s_j^{-1} g s_i)(v)$, by definition of ϕ. Then

$$\begin{aligned} g(x_i v) = g s_i(x_1 h_i^{-1}(v)) = s_j(s_j^{-1} g s_i)(x_1 h_i^{-1}(v)) = \\ = s_j(x_1 \phi(s_j^{-1} g s_i) h_i^{-1}(v)) = x_j h_j \phi(s_j^{-1} g s_i) h_i^{-1}(v) \end{aligned}$$

and the proof is finished by induction on the length of the word v. □

Proposition 4.19. *Let $X = \{x_i = \phi(r_i)h_i\}$ and $Y = \{y_i = \phi(s_i)g_i\}$ be two standard bases of the bimodule Φ. Then the respective standard actions of the group G on X^* and Y^* are conjugate, i.e., there exists a bijection $\alpha : X^* \to Y^*$ such that the equality $\alpha^{-1}g\alpha(v) = g(v)$ holds for every $g \in G$ and $v \in X^*$.*

Proof. Let $v = x_{i_1}x_{i_2}\dots x_{i_n}$ be an arbitrary element of X^*. It follows from Proposition 4.9 that there exists a unique $\alpha(v) \in Y^*$ such that $v = \alpha(v) \cdot h$ for some $h \in G$.

We have $g \cdot v = g(v) \cdot g|_v$ for the standard action on X^*, so that $g \cdot v = \alpha(g(v)) \cdot hg|_v$ for some $h \in G$. On the other hand

$$g \cdot v = g \cdot \alpha(v) \cdot h(v) = g(\alpha(v)) \cdot g|_{\alpha(v)}h(v).$$

Consequently, $\alpha(g(v)) = g(\alpha(v))$. □

Recall, that due to Proposition 4.7, we have for every $x_i \in X$ the equality $x_i = y_j \cdot \langle y_j|\, x_i\rangle$, where y_j is such that $\langle y_j|\, x_i\rangle \neq 0$, i.e., $s_j^{-1}r_i \in \mathrm{Dom}\,\phi$. Therefore, $v = y_{j_1}\cdot\langle y_{j_1}|\, x_{i_1}\rangle \otimes y_{j_2}\cdot\langle y_{j_2}|\, x_{i_2}\rangle \otimes\cdots\otimes y_{j_n}\cdot\langle y_{j_n}|\, x_{i_n}\rangle$ for some $y_{j_1}y_{j_2}\dots y_{j_n} \in Y^*$, and the map $\alpha : X^* \to Y^*$ can be more explicitely defined by the recurrent formula

$$\alpha(x_i \otimes v) = y_j \otimes \langle y_j|\, x_i\rangle\,(\alpha\,(v))\,, \tag{12}$$

where $v \in X^*$, $y_j \in Y$ is such that $\langle y_j|\, x_i\rangle \neq 0$, and $\langle y_i|\, x_i\rangle \in G$ acts on $\alpha(v)$ by the standard action of G on Y^*.

Proposition 4.20. *The virtual endomorphism ϕ is regular if and only if the respective standard action is transitive on the sets X^n (is* level transitive*).*

If the virtual endomorphism ϕ is regular, then the standard action is conjugate with the action of the group G on the coset tree of ϕ, i.e., there exists an isomorphism of rooted trees $\Lambda : X^* \to T(\phi)$ such that $\Lambda(g(v)) = g(\Lambda(v))$ for all $v \in X^*$.

Proof. It follows from Proposition 4.19 that if one standard action is level-transitive, then all the other standard actions are level-transitive. Therefore, it is sufficient to prove the proposition for one standard basis, so we can assume that our standard basis contains the element $x_0 = \phi(1)1$. The standard action is level transitive if and only if the index of the stabilizer of the word $x_0x_0\dots x_0 = x_0^n$ is equal to d^n, where $d = |X| = \mathrm{ind}\,\phi$. But the stabilizer of the word $x_0x_0\dots x_0 = x_0^n$ is equal to $\mathrm{Dom}\,\phi^n$.

If the virtual endomorphism ϕ is regular, then the isomorphism $\Lambda : X^* \to T(\phi)$ may be defined as $\Lambda(v) = g \cdot \mathrm{Dom}\,\phi^{|v|}$, where g is such that $g(x_0^n) = v$. □

5. Contracting virtual endomorphisms

5.1. Definitions and basic properties

Let ϕ be a virtual endomorphism of a group G. Choose some standard basis $X = \{x_1 = \phi(r_1)h_1, \dots, X_d = \phi(r_d)h_d\}$ of the right module Φ_R and consider the standard action of the group G on the space X^*.

Definition 5.1. The standard action is said to be *contracting* if there exists a finite set $\mathcal{N}$ such that for every $g \in G$ there exists $n_0 \in \mathbb{N}$ such that

$$g|_v \in \mathcal{N},$$

for all $v \in X^n$, $n \geq n_0$.

The minimal set $\mathcal{N}$ with the above property is called the *nucleus* of the standard action.

It is easy to see that if the standard action is contracting then it is *finite state*, i.e., for every $g \in G$ the set $\{g|_v \ : \ v \in X^*\}$ is finite.

We will use the following notation. If A and B are two subsets of a group G, then AB is the set of products ab, where $a \in A$ and $b \in B$. The power A^n is a short notation for $\underbrace{A \cdot A \cdots A}_{n \text{ times}}$. If $A \subset G$ and $W \subset X^*$, then $A|_W$ is the set of restrictions $a|_w$, where $a \in A$ and $w \in W$.

Lemma 5.1. *Suppose that the group G is generated by a finite set $S = S^{-1} \ni 1$. Then a standard action of G is contracting if and only if there exists a finite set $\mathcal{N} \subset G$ and a number n such that*

$$(S \cup \mathcal{N})^2|_{X^n} \subseteq \mathcal{N}.$$

Proof. If the action is contracting, then the above condition holds for $\mathcal{N}$ equal to the nucleus. In the other direction, by induction on the length of a group element we prove that for every $g \in G$ there exists $k_0 \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all $v \in X^{nk}$, where $k \geq k_0$. Then the nucleus of the action is a subset of $\mathcal{N}|_{\cup_{0 \leq m \leq n-1} X^m}$. □

Proposition 5.2. *Suppose that the virtual endomorphism ϕ is contracting with respect to the standard basis X. Let $A \subset G$ be a finite set. Then the set of all possible $h \in G$ such that*

$$g_1 \cdot x_{i_1} \otimes g_2 \cdot x_{i_2} \otimes \cdots \otimes g_m \cdot x_{i_m} = v \cdot h, \tag{13}$$

for some $g_i \in A$, $x_{i_k} \in X$ and $v \in X^m$, is finite.

Proof. It is sufficient to prove the proposition for some set $A' \supseteq A$, so we assume that the set A contains the nucleus $\mathcal{N}$ of the action and that it is *state-closed*, i.e., that for every $g \in A$ and $v \in X^*$ the restriction $g|_v$ also belongs to A. We can do this, since the action is finite-state.

There exists a number k such that $A^2|_v \subseteq \mathcal{N} \subseteq A$ for every word $v \in X^*$ of length greater or equal to k. It is easy to see that then $A^{2n}|_v \subseteq A^n$ for every $v \in X^k$ and every $n \in \mathbb{N}$.

It is sufficient to find a finite set B such that it contains all h, which appear in Equation (13) for numbers m divisible by k.

We can write

$$g_1 \cdot x_{i_1} \otimes g_2 \cdot x_{i_2} \otimes \cdots \otimes g_m \cdot x_{i_m} = v_1 \cdot h_1 \otimes v_2 \cdot h_2 \otimes \cdots \otimes v_{m/k} \cdot h_{m/k},$$

where $h_i \in G$ and $v_i \in X^k$ for all i. From the fact that A is state-closed follows that $h_i \in A^k$. But then $h_1 \cdot v_2 = h_1(v_2) \cdot h_1|_{v_2}$ and $h_1|_{v_2}$ also belongs to A^k, so $(h_1|_{v_2} h_2)|_{v_3} \in A^{2k}|_{v_3} \subseteq A^k$, and we get an inductive proof of the fact that $v_1 \cdot h_1 \otimes v_2 \cdot h_2 \otimes \cdots \otimes v_{m/k} \cdot h_{m/k} = u \cdot h$ for some $h \in A^k$. □

Directly from Proposition 5.2 we get

Corollary 5.3. *If the standard action is contracting then for any finite set $A \subset G$ there exists a finite set $\Sigma_A \subset G$ such that $A \subseteq \Sigma_A$ and*

$$\Sigma_A|_X \cdot A \subseteq \Sigma_A.$$

Now we are ready to prove that the property of an action to be contracting does not depend on the particular choice of the standard basis.

Proposition 5.4. *If some standard action for a virtual endomorphism ϕ is contracting, then any other standard action for ϕ is contracting.*

Proof. Let $X = \{x_1, x_2, \ldots, x_d\}$ and $Y = \{y_1, y_2, \ldots, y_d\}$ be two standard bases. Then we can permute the vectors in the basis so that there exist $r_i \in G$ such that $y_i = x_i \cdot r_i$. Take $A = \{r_i\}_{i=1,\ldots,d}$. Let Σ_A be as in Corollary 5.3 with respect to the standard action over the alphabet X.

Let $g \in G$ and $y_i \in Y$ be arbitrary. Then $g|_{y_i}$ is defined by the condition $g \cdot x_i \cdot r_i = x_j \cdot r_j g|_{y_i}$. Thus, $g|_{y_i} = r_j^{-1} g|_{x_i} r_i$. It is easy to prove now by induction on n that for every $v \in Y^n$ the restriction $g|_v$ belongs to the set $\Sigma_A^{-1} \cdot g|_u \cdot \Sigma_A$ for some $u \in X^n$. Consequently, the standard action with respect to Y is also contracting with the nucleus a subset of $\Sigma_A^{-1} \cdot \mathcal{N} \cdot \Sigma_A$, where $\mathcal{N}$ is the nucleus of the action on X^*. □

Proposition 5.4 justifies the following notion.

Definition 5.2. A virtual endomorphism ϕ is *contracting* if some (equivalently, if all) respective standard actions are contracting.

The next proposition shows that the contraction can be detected by a finite number of group relation.

Proposition 5.5. *Suppose that the virtual endomorphism ϕ of a finitely generated group G is contracting. Then there exist a finitely presented group F, a contracting virtual endomorphism $\tilde{\phi}$ of F, a normal $\tilde{\phi}$-invariant subgroup N of F and an isomorphism $\rho : G \to F/N$ such that $\rho \circ \phi = \tilde{\phi}/N \circ \rho$.*

Proof. Let us fix some standard basis $X = \{x_i\}_{i=1,\dots,d}$, where $x_1 = \phi(1)1$ and consider the respective standard action of the group G. Let $\mathcal{N}$ be its nucleus, and let S be a finite symmetric generating set of G, which includes the identity. Since the action is contracting, we may suppose that the set S is state-closed, i.e., that for every $s \in S$ and $x \in X$ the restriction $s|_x$ also belongs to S. We may also suppose that S contains the nucleus $\mathcal{N}$. Let $\tilde{S}$ be a set, which is in a bijective correspondence $\tilde{S} \to S : \tilde{s} \mapsto s$ with the set S. Take the group F generated by the set $\tilde{S}$ and defined by all relations of the form $\tilde{s}_1\tilde{s}_2 = \tilde{s}_3$, where $\tilde{s}_i$ are such that $s_1s_2 = s_3$ in the group G. In other words, the group F is the group defined by all the relations of the length 3, which hold for the generators S of the group G.

Let us define a permutational bimodule M over the group F with the standard basis X by the natural rules:

$$\tilde{s}_1 \cdot x = y \cdot \tilde{s}_2, \text{ if and only if } s_1 \cdot x = y \cdot s_2.$$

Another way to interpret the above construction is to say that we define the wreath product recursion $F \to Symm(X) \wr F$ on the generators of F in the same way as was defined the recursion $G \to Symm(X) \wr G$ on the generators of G.

The only thing to check in order to prove that the bimodule M is well defined, is to prove that if g is a word in generators $\tilde{S}$, representing the trivial element, then $g \cdot x = x \cdot 1$ in M for every $x \in X$. But this follows from the fact that if $s_1s_2 = s_3$ in G, $s_2 \cdot x = y \cdot s_2'$, $s_1 \cdot y = z \cdot s_1'$, for some $s_1, s_2, s_3, s_1' = s_1|_y, s_2' = s_2|_x \in S$ and $x, y, z \in X$, then $s_3 \cdot x = z \cdot s_3'$, $s_1s_2 \cdot x = z \cdot s_1's_2'$, so that $s_3' = s_1's_2'$, where $s_3' = s_3|_x \in S$.

Let $\tilde{\phi}$ be the virtual endomorphism of F, associated to the bimodule M and the element x_1 (recall that x_1 corresponds to $\phi(1)1$).

Directly from the definitions follows that the permutational bimodule $\phi(G)G$ is a quotient of the bimodule M with the natural quotient map $\pi : \tilde{s} \mapsto s : F \to G$ and the map $p : M \to \phi(G)G$ defined as $p(x \cdot g) =$

$x \cdot \pi(g)$. Then by Proposition 4.5, the kernel N of the map π is a $\tilde{\phi}$-semi-invariant subgroup such that $\tilde{\phi}/N$ is conjugated with ϕ. But from the choice of $\tilde{\phi}$ follows that in fact we have $\tilde{\phi}/N = \phi$.

If $g \in N$, then $g \cdot x = x \cdot g|_x$ for every x, since $\pi(g) \cdot x = x \cdot \pi(g)|_x$. Thus, $N \leq \operatorname{Dom} \tilde{\phi}$, and N is a normal $\tilde{\phi}$-invariant subgroup.

It remains to prove that the virtual endomorphism $\tilde{\phi}$ is contracting. Since the action of G is contracting, by Lemma 5.1 there exists n such that $S^2|_{X^n} \subseteq S$. But we have included all the relations of the form $s_1 s_2 = s_3, s_i \in S$ into the relations of F and the restrictions of the words in generators of F are computed by the same rules as the restrictions of the words in generators of G. Thus, $\tilde{S}^2|_{X^n} \subseteq \tilde{S}$, and by Lemma 5.1, the action of F is contracting. $\square$

Proposition 5.6. *If a virtual endomorphism ϕ of a group G is contracting and the nucleus of a standard action does not contain non-trivial elements of $\mathcal{C}(\phi)$ then $\mathcal{C}(\phi) = \mathcal{E}_\infty(\phi)$.*

Proof. Let that $g \in \mathcal{C}(\phi)$ be arbitrary. Then there exists $n \in \mathbb{N}$ such that $g|_v$ belongs to the nucleus for every $v \in X^n$. But then $g \cdot v = v \cdot g|_v$ and $g|_v \in \mathcal{C}(\phi)$, hence $g|_v = 1$ and $g \in \mathcal{E}_n(\phi)$. $\square$

The following is a direct corollary of Proposition 4.5.

Proposition 5.7. *If a virtual endomorphism ϕ of a group G is contracting, and N is a normal ϕ-semi-invariant subgroup of G, then the virtual endomorphism ϕ/N of the group G/N is also contracting.* $\square$

The next easy fact is proved in [Nekc].

Proposition 5.8. *If a virtual endomorphism ϕ of a group G is contracting and onto, then the group G is generated by the nucleus of the standard action.* $\square$

5.2. Contraction coefficient

If the group is finitely generated, then the contractivity of a virtual endomorphism can be established using a more intuitive definition.

If the group G is finitely generated, then we denote by $l(g)$ the length of the shortest representation of g in a product of the generators and the inverses, for a fixed finite generating set of the group.

Definition 5.3. Let G be a finitely generated group, let ϕ be its virtual endomorphism. Let us fix also a standard self-similar action of G on X^*. The number

$$\rho = \lim_{n\to\infty} \sqrt[n]{\limsup_{l(g)\to\infty} \max_{v\in X^n} \frac{l(g|_v)}{l(g)}}$$

is called the *contraction coefficient* of the action.

The number

$$\rho_\phi = \lim_{n\to\infty} \sqrt[n]{\limsup_{g\in \mathrm{Dom}\,\phi^n, l(g)\to\infty} \frac{l\left(\phi^n\left(g\right)\right)}{l(g)}}, \tag{14}$$

is called the *contraction coefficient* (or the *spectral radius*) of the virtual endomorphism ϕ.

Note that the function $\rho_\phi(n) = \limsup_{g\in \mathrm{Dom}\,\phi^n, l(g)\to\infty} \frac{l(\phi^n(g))}{l(g)}$ is sub-multiplicative, i.e., $\rho_\phi(n+m) \le \rho_\phi(n)\rho(m)$, since

$$\frac{l\left(\phi^{n+m}\left(g\right)\right)}{l(g)} = \frac{l\left(\phi^{n+m}\left(g\right)\right)}{l(\phi^n(g))} \cdot \frac{l\left(\phi^{n}\left(g\right)\right)}{l(g)},$$

$$\limsup_{g\in \mathrm{Dom}\,\phi^{n+m}, l(g)\to\infty} \frac{l\left(\phi^{n+m}\left(g\right)\right)}{l(\phi^n(g))} \le \limsup_{g\in \mathrm{Dom}\,\phi^{m}, l(g)\to\infty} \frac{l\left(\phi^{m}\left(g\right)\right)}{l(g)}$$

and

$$\limsup_{g\in \mathrm{Dom}\,\phi^{n+m}, l(g)\to\infty} \frac{l\left(\phi^{n}\left(g\right)\right)}{l(g)} \le \limsup_{g\in \mathrm{Dom}\,\phi^{n}, l(g)\to\infty} \frac{l\left(\phi^{n}\left(g\right)\right)}{l(g)}.$$

Therefore, from the well-known Polya Lemma, the limit in (14) exists. Similar arguments show that the contraction coefficient ρ of the standard action also exists. Both coefficients are finite, since $\rho_\phi \le \rho$ and ρ is not greater than $\max_{g\in S, x\in X} l(g|_x)$, where S is the generating set.

Note also, that if l_1 and l_2 are the length functions computed with respect to different finite generating sets, then there exists a number $C > 0$ such that $C^{-1}l_2(g) \le l_1(g) \le Cl_2(g)$ for all $g \in G$. From this easily follows that the contraction coefficients computed with respect to l_1 will be the same as the coefficients, computed with respect to l_2.

The following proposition is proved in [Nekc].

Proposition 5.9. *A standard action is contracting if and only if its contraction coefficient is less than one.*

Suppose that the virtual endomorphism ϕ is regular. Then it is contracting if and only if its contraction coefficient is less than one. If it is contracting, then ρ_ϕ is equal to the contraction coefficient of every associated standard action. □

Let w be an infinite word in the alphabet X, i.e., a sequence $x_1x_2\ldots$, $x_i \in X$. If $g \in G$, then by $g(x_1x_2\ldots)$ we denote the word $y_1y_2\ldots$ such that $g(x_1x_2\ldots x_n) = y_1y_2\ldots y_n$ for every n. From the definition of a self-similar action follows that the word $g(x_1x_2\ldots)$ is well defined and that we get in this way an action of G on the set X^ω of infinite words.

Definition 5.4. Let G be a finitely generated group, acting on a set A. *Growth degree* of the G-action is the number

$$\gamma = \sup_{w\in A} \limsup_{r\to\infty} \frac{\log |\{g(w) \ : \ l(g) \le r\}|}{\log r}$$

where $l(g)$ is the length of a group element with respect to some fixed finite generating set of G.

One can show, in the same way as before, that the growth degree γ does not depend on the choice of the generating set of G.

Proposition 5.10. *Suppose that a standard action of a group G on X^* is contracting. Then the growth degree of the action on X^ω is not greater than $\frac{\log|X|}{-\log\rho}$, where ρ is the contraction coefficient of the action on X^*.*

Proof. The statement is more or less classical. See, for instance the similar statements in [Gro81, BG00, Fra70].

Let ρ_1 be such that $\rho < \rho_1 < 1$. Then there exists $C > 0$ and $n \in \mathbb{N}$ such that for all $g \in G$ we have $l(g|_{x_1x_2\ldots x_n}) < \rho_1^n \cdot l(g) + C$.

Then cardinality of the set $B(w,r) = \{g(w) \ : \ l(g) \le r\}$, where $w = x_1x_2\ldots \in X^\omega$ is not greater than

$$|X|^n \cdot |\{B\left(x_{n+1}x_{n+2}\ldots, \rho_1^n \cdot r + C\right)|,$$

since the map $\sigma^n : x_1x_2\ldots \mapsto x_{n+1}x_{n+2}\ldots$ maps $B(w,r)$ into

$$B\left(x_{n+1}x_{n+2}\ldots, \rho_1^n \cdot r + C\right)$$

and every point of X^ω has exactly $|X|^n$ preimages under σ^n. The map σ^n is the nth iteration of the shift map $\sigma(x_1x_2\ldots) = x_2x_3\ldots$.

Let $k = \left[\frac{\log r}{-n\log\rho_1}\right] + 1$. Then $\rho_1^{nk}\cdot r < 1$ and the number of the points in the ball $B(w,r)$ is not greater than

$$|X|^{nk} \cdot \left|B\left(\sigma^{nk}(w), R\right)\right|,$$

where

$$R = \rho_1^{nk}\cdot r + \rho_1^{n(k-1)}\cdot C + \rho_1^{n(k-2)}\cdot C + \cdots + \rho_1^n \cdot C + C < 1 + \frac{C}{1-\rho_1^n}.$$

But $|B(u,R)|$ for all $u \in X^\omega$ is less than $K_1 = |S|^R$, where S is the generating set of G (we assume that $S = S^{-1} \ni 1$). Hence,

$$\begin{aligned} &|B(w,r)| < K_1 \cdot |X|^{n\left(\frac{\log r}{-n\log\rho_1}+1\right)} = \\ &= K_1 \cdot \exp\left(\frac{\log|X|\log r}{-\log\rho_1} + n\log|X|\right) = K_2 \cdot r^{\frac{\log|X|}{-\log\rho_1}}, \end{aligned}$$

where $K_2 = K_1 \cdot |X|^n$. Thus, the growth degree is not greater than $\frac{\log|X|}{-\log\rho_1}$ for every $\rho_1 \in (\rho, 1)$, so it is not greater than $\frac{\log|X|}{-\log\rho}$. ∎

Lemma 5.11. *Let ϕ be a contracting virtual endomorphism of a ϕ-simple infinite finitely generated group G. Then the contraction coefficient of its standard action is greater or equal to $1/\operatorname{ind}\phi$.*

Proof. Consider the standard action on the set X^* for a standard basis X, containing the element $x_0 = \phi(1)1$. Then the parabolic subgroup $P(\phi) = \cap_{n\geq 0} \operatorname{Dom}\phi^n$ is the stabilizer of the word $w = x_0x_0x_0\ldots \in X^\omega$. The subgroup $P(\phi)$ has infinite index in G, otherwise $\cap_{g\in G} g^{-1}Pg = \mathcal{C}(\phi)$ will have finite index, and G will be not ϕ-simple. Consequently, the G-orbit of w is infinite. Then there exists an infinite sequence of generators $s_1, s_2, \ldots$ of the group G such that the elements of the sequence

$$w, s_1(w), s_2s_1(w), s_3s_2s_1(w), \ldots$$

are pairwise different. This implies that the growth degree of the orbit Gw

$$\gamma = \limsup_{r\to\infty} \frac{|\{g(w) \ : \ l(g) \leq r\}|}{\log r}$$

is greater or equal to 1, thus the growth degree of the action of G on X^ω is not less than 1, and by Proposition 5.10, $1 \leq \frac{\log|X|}{-\log\rho}$. □

Proposition 5.12. *If there exists a faithful contracting action of a finitely-generated group G then for any $\epsilon > 0$ there exists an algorithm of polynomial complexity of degree not greater than $\frac{\log|X|}{-\log\rho} + \epsilon$ solving the word problem in G.*

Proof. We assume that the generating set S is symmetric (i.e., that $S = S^{-1}$) and contains all the restrictions of all its elements, so that always $l(g|_v)$ is not greater than $l(g)$.

We will denote by F the free group generated by S and for every $g \in F$ by $\hat{g}$ we denote the canonical image of g in G.

Let $1 > \rho_1 > \rho$. Then $\rho_1 \cdot |X| > 1$, since by Lemma 5.11, $\rho \cdot |X| \geq 1$. There exist n_0 and l_0 such that for every word $v \in X^*$ of the length n_0 and every $g \in G$ of the length $\geq l_0$ we have

$$l\left(g|_v\right) < \rho_1^n l(g).$$

Assume that we know for every $g \in F$ of the length less than l_0 if $\hat{g}$ is trivial or not. Assume also that we know all the relations $g \cdot v = u \cdot h$ for all $g, l(g) \leq l_0$ and $v \in X^{n_0}$.

Then we can compute in $l(\hat{g})$ steps, for any $g \in F$ and $v \in X^n$, the element $h \in F$ and the word $u \in X^{n_0}$ such that $\hat{g} \cdot v = u \cdot \hat{h}$. If $v \neq u$ then we conclude that $\hat{g}$ is not trivial and stop the algorithm. If for all $v \in X^{n_0}$ we have $v = u$, then $\hat{g}$ is trivial if and only if all the obtained restrictions $\hat{h} = \hat{g}|_v$ are trivial. We know, whether $\hat{h}$ is trivial if $l(h) < l_0$. We proceed further, applying the above computations for those h, which have the length not less than l_0.

But $l(h) < \rho_1^n l(g)$, if $l(g) \geq l_0$. So on each step the length of the elements becomes smaller, and the algorithm stops in not more than $-\log l(g)/\log \rho_1$ steps. On each step the algorithm branches into $|X|$ algorithms. Thus, since $\rho_1 \cdot |X| > 1$, the total time is bounded by

$$\begin{aligned}
&l(g)\left(1 + \rho_1 \cdot |X| + (\rho_1 \cdot |X|)^2 + \cdots + (\rho_1 \cdot |X|)^{[-\log l(g)/\log \rho_1]}\right) < \\
&\tfrac{l(g)}{\rho_1 \cdot |X| - 1}\left((\rho_1 \cdot |X|)^{1 - \log l(g)/\log \rho_1} - 1\right) = \\
&\tfrac{l(g)\rho_1 \cdot |X|}{\rho_1 \cdot |X| - 1}\left((\rho_1 \cdot |X|)^{-\log l(g)/\log \rho_1} - (\rho_1 \cdot |X|)^{-1}\right) = \\
&C_1 l(g)\left(\exp\left(\log l(g)\left(\tfrac{\log |X|}{-\log \rho_1} - 1\right)\right) - C_2\right) = \\
&= C_1 l(g)^{-\log |X|/\log \rho_1} - C_1 C_2 l(g),
\end{aligned}$$

where $C_1 = \frac{\rho_1 \cdot |X|}{\rho_1 \cdot |X| - 1}$ and $C_2 = (\rho_1 \cdot |X|)^{-1}$. □

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