Contracting Groups

Volodymyr Nekrashevych

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Outline

1. Self-similar contracting groups: definitions, examples and elementary properties.
2. Absence of free subgroups.
3. Schreier graphs and limit spaces of contracting groups.
4. Bounded automata. Amenability of a class of contracting groups.
5. Operator algebras associated with contracting groups.

Self-similar groups

Definition

A *self-similar group* (G, X) is a faithful action of a group G on X^* such that for every $g \in G$ and every $x \in \mathsf{X}$ there exist $h \in G$ and $y \in \mathsf{X}$ such that

$$g(xw) = yh(w)$$

for all $w \in \mathsf{X}^*$.

Example: take $G \cong \mathbb{Z}$ generated by a. Define an action of G on $\{0, 1\}^*$ by the rule

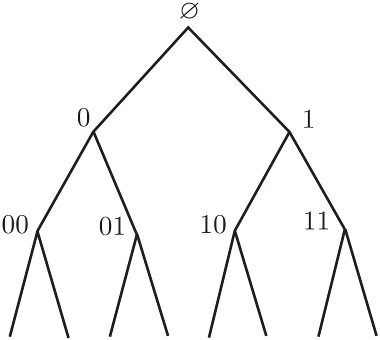
$$a(0w) = 1w, \qquad a(1w) = 0a(w).$$

It describes the process of adding 1 to a diadic integer.

For every $v \in \mathsf{X}^*$ and every $g \in G$ there exists $h \in G$ such that

$$g(vw) = g(v)h(w)$$

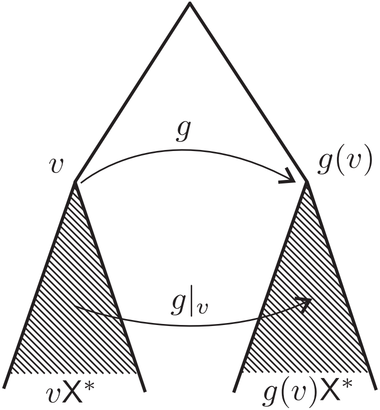
for all $w \in \mathsf{X}^*$.
It follows that G acts on X^* by the automorphisms of the associated rooted tree (the right Cayley graph of the free monoid X^*).



If $g(vw) = g(v)h(w)$ for all w, then we denote

$$h = g|_v$$

and call h the *section* (or *restriction*) of g at v.



We have the following properties of sections:

$$g|_{v_1 v_2} = g|_{v_1}|_{v_2}, \qquad (g_1 g_2)|_v = g_1|_{g_2(v)} g_2|_v.$$

A self-similar group (G, X) is *self-replicating* if it is level-transitive and for every $h \in G$ and $x \in \mathsf{X}$ there exists $g \in G$ such that

$$g(x) = x, \quad g|_x = h.$$

Actions of $\mathbb{Z}^n$

Let A be an $n \times n$ matrix with integral entries. Let $d = \det A$, then $A(\mathbb{Z}^n)$ is a subgroup of index d of $\mathbb{Z}^n$.

Choose a coset transversal $r_1, r_2, \ldots, r_d$ of $\mathbb{Z}^n / A(\mathbb{Z}^n)$. We can represent then every element $g \in \mathbb{Z}^n$ as a formal series

$$g = r_{i_0} + A(r_{i_1}) + A^2(r_{i_2}) + \cdots$$

in a unique way.

The group $\mathbb{Z}^n$ acts on the series

$$r_{i_0} + A(r_{i_1}) + A^2(r_{i_2}) + \cdots$$

self-similarly:

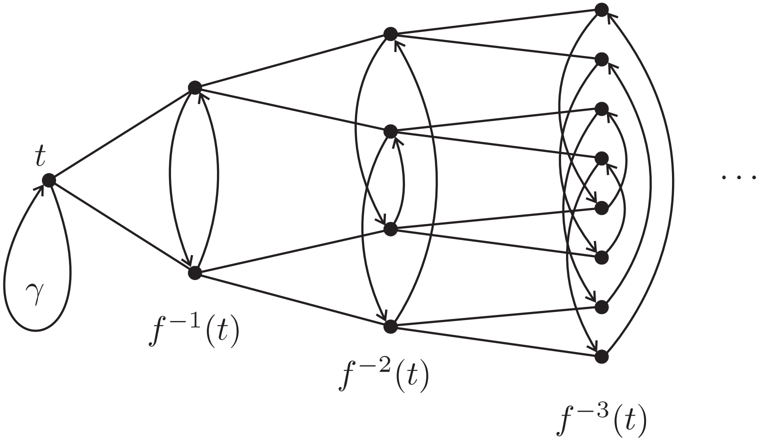
$$g + \left(r_{i_0} + A(r_{i_1}) + A^2(r_{i_2}) + \cdots\right) = r_{j_0} + A\left(h + r_{i_1} + A(r_{i_2}) + A^2(r_{i_3}) + \cdots\right),$$

where r_{j_0} and $h \in \mathbb{Z}^n$ are uniquely determined by

$$g + r_{i_0} = r_{j_0} + A(h).$$

Iterated monodromy groups

Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a covering of a space by a subset.



Choose an alphabet X, $|\mathsf{X}| = \deg f$, a bijection $\Lambda : \mathsf{X} \to f^{-1}(t)$, and a path $\ell(x)$ from t to $\Lambda(x)$ for every $x \in \mathsf{X}$.

Define the map $\Lambda : \mathsf{X}^* \longrightarrow T_t$ inductively by the rule:

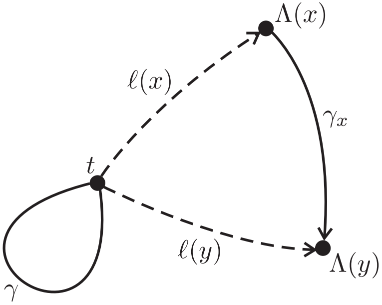
$\Lambda(xv)$ is the end of the $f^{|v|}$-lift of $\ell(x)$ starting at $\Lambda(v)$.

The map $\Lambda : \mathsf{X}^* \longrightarrow T_t$ is an isomorphism of rooted trees.

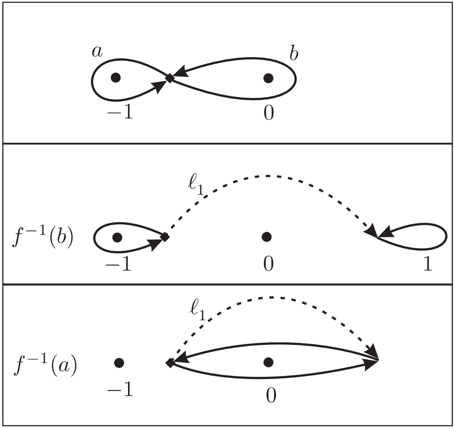
The isomorphism $\Lambda : \mathsf{X}^* \longrightarrow T_t$ conjugates the iterated monodromy action of $\pi_1(\mathcal{M}, t)$ on T_t to a self-similar action

$$\gamma(xv) = y\delta(v),$$

where $v \in \mathsf{X}^*, x \in \mathsf{X}$ and δ is the loop $\ell(x)\gamma_x\ell(y)^{-1}$



IMG $\left(z^2 - 1\right)$



Contracting groups

Definition

A self-similar group G is called *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists n such that $g|_v \in \mathcal{N}$ whenever $|v| \geq n$.

The smallest set $\mathcal{N}$ satisfying this property is called the *nucleus* of the group.

Let (G, X) be a finitely generated self-similar group. The number

$$\rho = \limsup_{n\to\infty} \sqrt[n]{\limsup_{l(g)\to\infty} \max_{v\in\mathsf{X}^n} \frac{l\left(g|_v\right)}{l(g)}}$$

is the *contraction coefficient*.

Proposition

The action is contracting if and only if its contraction coefficient ρ is less than 1*.*

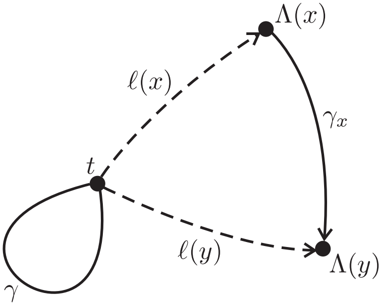
Recall that a self-similar action of $\mathbb{Z}^n$ defined by a matrix A and a coset transversal $r_1, r_2, \ldots, r_d$ is given by the rule

$$g + (r_{i_0} + A(r_{i_1}) + A^2(r_{i_2}) + \cdots) = r_{j_0} + A(h + r_{i_1} + A(r_{i_2}) + \cdots),$$

where $h = A^{-1}(g + r_{i_0} - r_{j_0})$.

It follows that this action is contracting if and only if the spectral radius of A^{-1} is less than one.

If $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is an expanding covering, then $\mathrm{IMG}(f)$ is contracting.



In particular, if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a *post-critically finite* rational function, then $\mathrm{IMG}(f)$ is contracting.

The Grigorchuk group

$$\begin{array}{ll} a(0w) = 1w & a(1w) = 0w \\ b(0w) = 0a(w) & b(1w) = 1c(w) \\ c(0w) = 0a(w) & c(1w) = 1d(w) \\ d(0w) = 0w & d(1w) = 1b(w). \end{array}$$

is contracting with $\rho = 1/2$.

Let $A \leq Symm(\mathsf{X})$ be a 2-transitive permutation group on X, $|\mathsf{X}| > 2$. Fix $x_1, x_2 \in \mathsf{X}$.

The group $\mathcal{W}(A)$ is generated by A acting on the first letter of words from X^* and by the set $\overline{A}$ of the transformations

$$\overline{\alpha}(x_1 w) = x_1 \overline{\alpha}(w), \quad \overline{\alpha}(x_2 w) = x_2 \alpha(w), \quad \overline{\alpha}(x_i w) = x_i w.$$

L. Bartholdi showed that $\mathcal{W}(A)$ for $A = \mathrm{PSL}(3,2)$ acting on $P^2\mathbb{F}_2$ has non-uniform exponential growth.

Germs

Definition

Let G be a group acting on a locally finite rooted tree T. For $w \in \partial T$ the *group of germs* is

$$G_{(w)} = G_w / \{g \in G_w \ : \ g \text{ acts trivially on a nbhd of } w\}.$$

If G is contracting, then $|G_{(w)}| \leq |\mathcal{N}|$ for all $w \in \partial \mathsf{X}^* = \mathsf{X}^\omega$.

Free groups acting on rooted trees

Theorem

Let G be a group acting faithfully on a locally finite rooted tree T. Then one of the following is true

1. *G has no free subgroups,*
2. *$G_{(w)}$ has a free subgroup for some $w \in \partial T$,*
3. *there is a free subgroup $F < G$ and $w \in \partial T$ such that $F_w = \{1\}$.*

Let G be a finitely generated group acting on a set A. *Growth degree* of the G-action is

$$\gamma = \sup_{w \in A} \limsup_{r \to \infty} \frac{\log |\{g(w) \ : \ l(g) \leq r\}|}{\log r}.$$

Proposition

Let (G, X) be contracting. Then the growth degree of the action of G on X^ω is $\leq \frac{\log |\mathsf{X}|}{-\log \rho}$.

Corollary

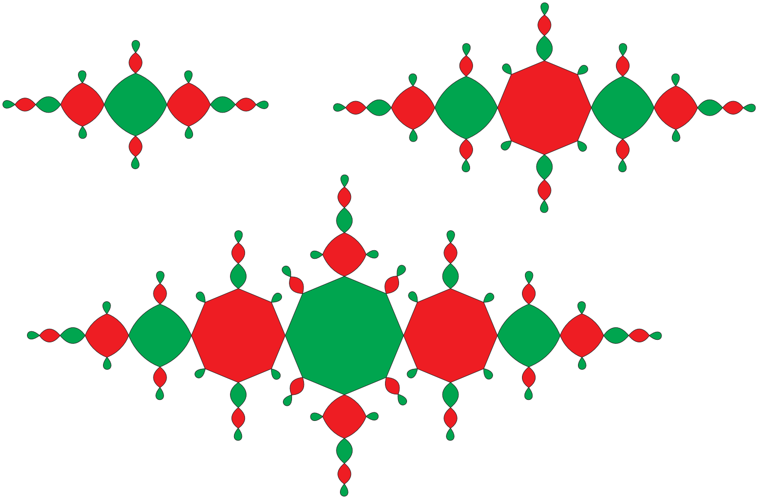
Contracting groups have no free subgroups.

Schreier graphs

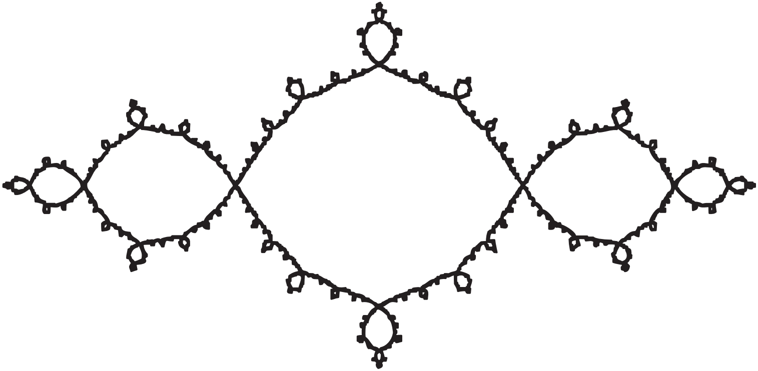
Let (G, X) be a finitely generated contracting group. The *Schreier graphs* $\Gamma_n = \Gamma_n(G, S)$ are the graphs with the set of vertices X^n where v is connected to $s(v)$ for all $v \in \mathsf{X}^n$ and $s \in S$.

The Schreier graphs of contracting groups seem to converge as $n \to \infty$ to fractals.

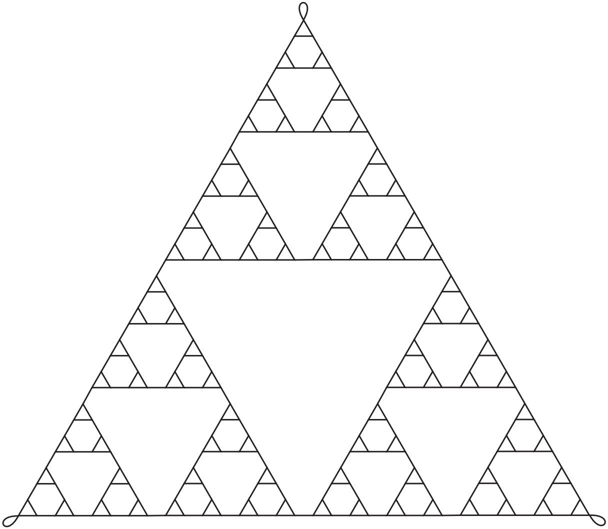
Graphs for $\mathrm{IMG}\left(z^2 - 1\right)$.



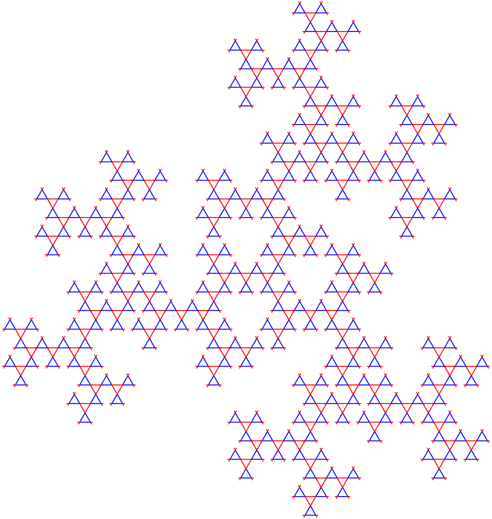
Julia set of $z^2 - 1$.



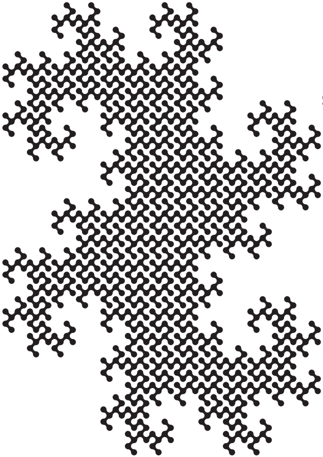
"Hanoi tower" group

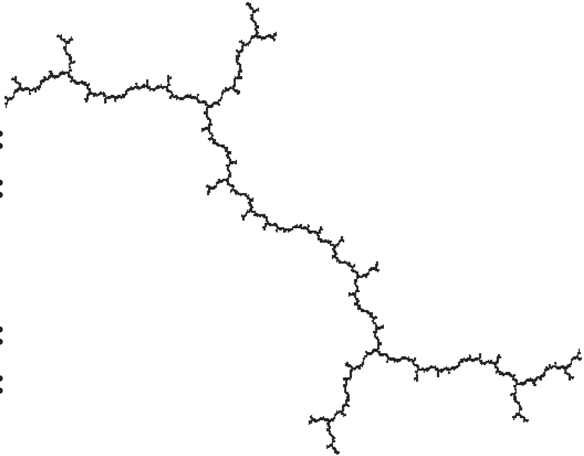


“Gupta-Fabrikowski group”



$\mathrm{IMG}\left(z^2 - 0.2282\ldots + 1.1151\ldots i\right)$





Real numeration systems

If $A \in M_{n\times n}(\mathbb{Z})$ is an expanding matrix and $r_1, r_2, \ldots, r_d$ is a coset transversal of $\mathbb{Z}^n/A(\mathbb{Z}^n)$, then every series

$$r + A^{-1}(r_{i_1}) + A^{-2}(r_{i_2}) + \cdots$$

converges in $\mathbb{R}^n$, and we get an "A-adic" numeration system on $\mathbb{R}^n$.

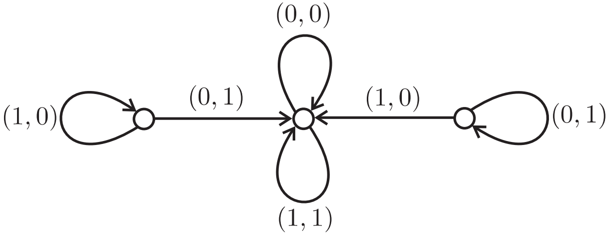
Limit space $\mathcal{J}_G$

Consider the space $\mathsf{X}^{-\omega}$ of the left-infinite words $\ldots x_2 x_1$.
Fix a self-similar group G. Two sequences $\ldots x_2 x_1, \ldots y_2 y_1$ are equivalent if there exists a finite set $A \subset G$ and a sequences $g_k \in A$ such that

$$g_k(x_k \ldots x_1) = y_k \ldots y_1.$$

for all k.
The quotient of $\mathsf{X}^{-\omega}$ by this equivalence relation is the *limit space* $\mathcal{J}_G$.
The equivalence relation is invariant under the shift $\ldots x_2 x_1 \mapsto \ldots x_3 x_2$, hence the shift induces a continuous map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$, called the *limit dynamical system*.

The *Moore diagram* of the nucleus $\mathcal{N}$ of a contracting group is the labeled graph with the set of vertices $\mathcal{N}$, where g is connected to $g|_x$ by an arrow labeled by $(x, g(x))$.



is the Moore diagram of the adding machine action.

Proposition

Sequences $\ldots x_2 x_1, \ldots y_2 y_1 \in \mathsf{X}^{-\omega}$ *are equivalent if and only if there exists a path* $\ldots e_2 e_1$ *in the Moore diagram of the nucleus* $\mathcal{N}$ *such that the arrow* e_n *is labeled by* (x_n, y_n)*.*

Elementary properties

The limit space $\mathcal{J}_G$ is metrizable, finite-dimensional, compact.

It is connected if the group G is level-transitive.

It is locally connected if the group G is self-replicating.

Julia sets and limit spaces

If $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is an *expanding* partial self-covering, then $\mathrm{IMG}(f)$ is contracting and $(\mathcal{J}_{\mathrm{IMG}(f)}, \mathsf{s})$ is topologically conjugate to $(\mathcal{J}_f, f)$,where $\mathcal{J}_f$ is the set of accumulation points of the inverse orbit

$$\bigcup_{n\geq 0} f^{-n}(t),$$

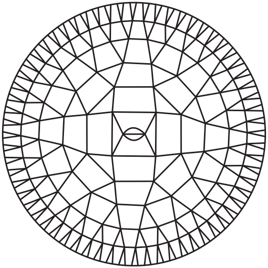
i.e., the support of the measure of maximal entropy.

A choice of connecting paths provides a symbolic presentation of the limit dynamical system.

In particular, the limit space of the A-adic adding machine action of $\mathbb{Z}^n$ (if A is expanding) is $\mathbb{R}^n/\mathbb{Z}^n$, since this action is the iterated monodromy group of the self-covering of $\mathbb{R}^n/\mathbb{Z}^n$ induced by A.

Limit spaces as Gromov boundaries

Let a contracting group G be generated by a finite set S. Consider the graph with the set of vertices X^* where a vertex v is connected to $s(v)$ for $s \in S$ and to xv for $x \in \mathsf{X}$.



This graph is Gromov hyperbolic and its boundary is homeomorphic to $\mathcal{J}_G$.

Limit G-space

Let (G, X) be a contracting group. Consider the topological space $\mathsf{X}^{-\omega} \times G$ and take its quotient by the equivalence relation

$$\ldots x_2 x_1 \cdot g \sim \ldots y_2 y_1 \cdot h \quad \text{iff}$$

$$g_k(x_k \ldots x_2 x_1) = y_k \ldots y_2 y_1, \quad g_k|_{x_k \ldots x_2 x_1} g = h$$

for some bounded sequence $\{g_k\}$.
The condition can be written as

$$g_k(x_k \ldots x_2 x_1 g(w)) = y_k \ldots y_2 y_1 h(w)$$

for all $w \in \mathsf{X}^*$.

The quotient $\mathcal{X}_G$ of $\mathsf{X}^{-\omega} \times G$ is called the *limit G-space*.

The group G acts on $\mathcal{X}_G$ by a natural right action. This action is proper and co-compact.

The space $\mathcal{X}_G$ is locally compact, metrizable and finite-dimensional. It is connected and locally connected if (G, X) is self-replicating.

In the case of the A-adic adding machine action, the limit G-space of $\mathbb{Z}^n$ is $\mathbb{R}^n$ with the natural action.

The symbolic representation of the points of $\mathcal{X}_{\mathbb{Z}^n}$ as $\ldots x_2 x_1 \cdot g$ corresponds to the representation of the points of $\mathbb{R}^n$ as series

$$g + A^{-1}(r_{i_1}) + A^{-2}(r_{i_2}) + \cdots .$$

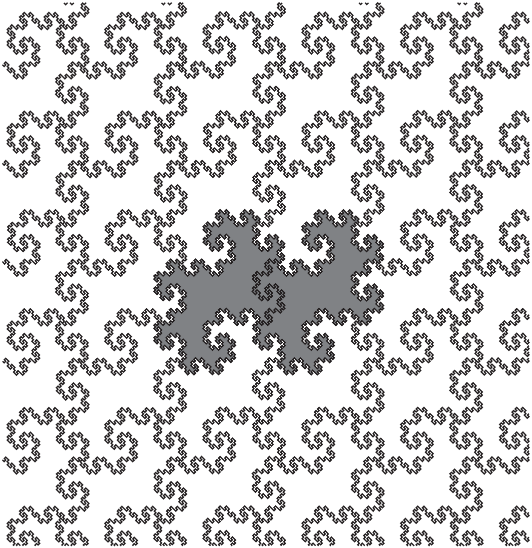
Tiles

The image of the set $\mathsf{X}^{-\omega} \cdot 1$ in $\mathcal{X}_G$ is called the *tile* $\mathcal{T}$ of the group (G, X). We have

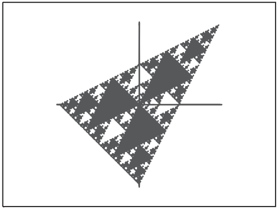
$$\mathcal{X}_G = \bigcup_{g \in G} \mathcal{T} \cdot g.$$

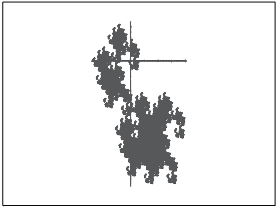
If for every $g \in G$ there exists $v \in \mathsf{X}^*$ such that $g|_v = 1$, then tiles $\mathcal{T} \cdot g$ have disjoint interiors and are closures of their interiors.

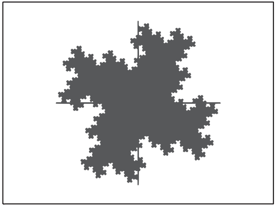
A tile of a $\mathbb{Z}^2$ action

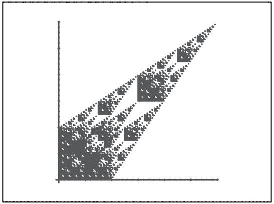


Some other tiles of $\mathbb{Z}^2$ actions









Proposition

The tile $\mathcal{T}$ is the quotient of the space $\mathsf{X}^{-\omega}$ by the equivalence relation identifying $\ldots x_2x_1$ and $\ldots y_2y_1$ if there is a path $\ldots e_2e_1$ in the Moore diagram of $\mathcal{N}$ ending in identity and such that e_n is labeled by (x_n, y_n).

Proposition

Two tiles $\mathcal{T} \cdot g$ and $\mathcal{T} \cdot h$ intersect if and only if $gh^{-1} \in \mathcal{N}$.

Proposition

If for every $g \in G$ there exists $v \in \mathsf{X}^*$ such that $g|_h = 1$, then $\partial\mathcal{T}$ is the intersection of $\mathcal{T}$ with $\bigcup_{g\neq 1} \mathcal{T} \cdot g$. It consists of the images of sequences $\ldots x_2x_1 \cdot 1$ such that there exists a path $\ldots e_2e_1$ in the Moore diagram of the nucleus ending in a non-trivial element of $\mathcal{N}$ and such that for all k the arrow e_k is labeled by (x_k, y_k) for some y_k.

For given $v \in \mathsf{X}^*$ and $g \in G$ the corresponding tile $\mathcal{T} \cdot v \cdot g$ is the image of the sequences $\ldots x_{n+1} v \cdot g$.
The tile $\mathcal{T} \cdot v \cdot g$ is homeomorphic to the tile $\mathcal{T}$ and

$$\mathcal{T} = \bigcup_{v \in \mathsf{X}^n} \mathcal{T} \cdot v.$$

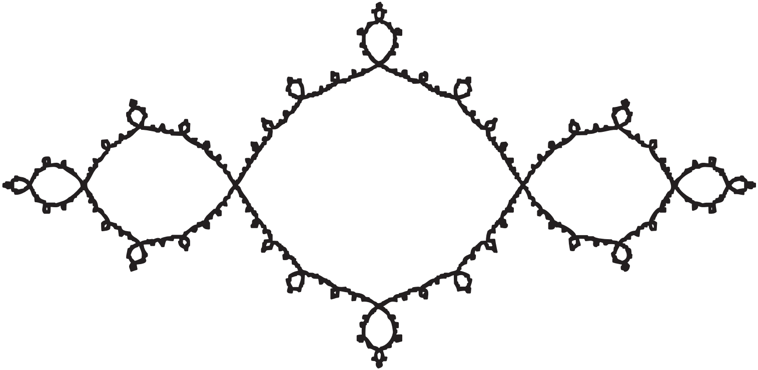
For $\xi \in \mathcal{X}_G$ the set

$$U_n(\xi) = \bigcup_{v \cdot g \in \mathsf{X}^n \times G, \xi \in \mathcal{T} \cdot v \cdot g} \mathcal{T} \cdot v \cdot g$$

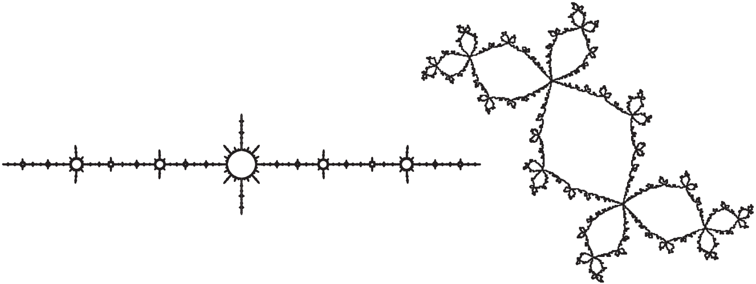
is a neighborhood of ξ and $\{U_n(\xi) \ : \ n \geq 0\}$ is a basis of neighborhoods of ξ.

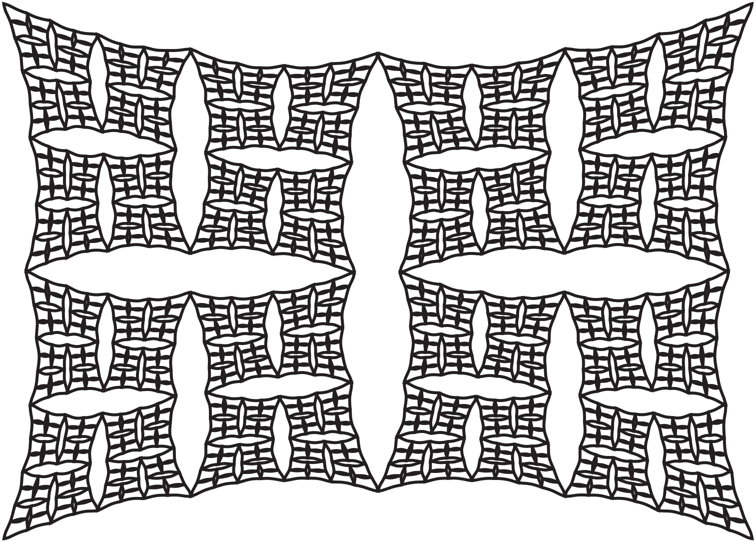
If $\partial\mathcal{T}$ is finite, then $\mathcal{J}_G$ and $\mathcal{X}_G$ are topologically one dimensional.

“Basilica”.



“Airplane and Rabbit”



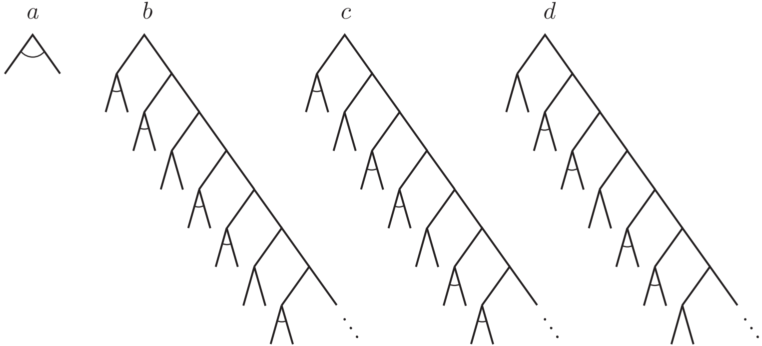


An automorphism g of the tree X^* is *finitary* if there exists n such that $g|_v = 1$ for all $v \in X^n$. The smallest n is called *finitary depth* of g.

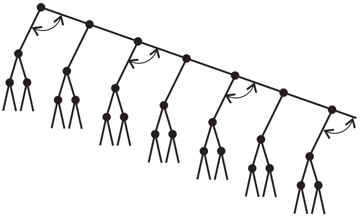
An automorphism g is called *directed* if there exists n and a path $w \in X^\omega$ such that $g|_v$ is finitary of depth $\leq n$ if v is not a beginning of w.

An automorphism g of X^* is *bounded* if there exists n such that $g|_v$ is either finitary or directed for $v \in X^n$. The set of bounded automorphisms of X^* is a group.

Grigorchuk group



IMG $\left(z^2 - 1\right)$



An automorphism g of X^* is *automatic* if the set $\{g|_v \ : \ v \in \mathsf{X}^*\}$ is finite.

Theorem (E. Bondarenko, V.N.)

A self-similar group of bounded automatic automorphisms of X^* *is contracting and has finite boundary of the tile* $\mathcal{T} \subset \mathcal{X}_G$.
If a contracting group (G, X) *satisfies the "open set condition" and has finite boundary of the tile, then it consists of bounded automatic automorphisms of* X^*.

The group of bounded automorphisms of X^* has no free subgroups.

Is it amenable?

Theorem (L. Bartholdi, B. Virag, V. Kaimanovich, V.N.)

The group of bounded automatic automorphisms of X^* is amenable.

Corollary

Iterated monodromy groups of post-critically finite polynomials are amenable.

Sketch of the proof

Notation:

$$a = \pi(a_0, a_1, \dots, a_{d-1})$$

for $\pi \in Symm(\mathsf{X})$ and $\mathsf{X} = \{0, 1, 2, \dots, d-1\}$ means

$$a(iw) = \pi(i)a_i(w).$$

It is sufficient to prove the theorem for finitely generated subgroups G of the group of bounded automatic automorphisms. Passing to sections and replacing X by X^n, we may assume that the generators of G are either elements of $Symm(\mathsf{X})$ (acting on the first letter) or of the form

$$a = \pi(a_0, \dots, a_{d-1}),$$

where $a_x = a$ for some x and $a_i \in Symm(\mathsf{X})$ for $i \neq x$.

Conjugating by

$$\delta = (\delta, \varsigma^{-1}\delta, \varsigma^{-2}\delta, \ldots, \varsigma^{-(d-1)}\delta),$$

where $\varsigma = (0, 1, \ldots, d-1)$, we get

$$\delta a \delta^{-1} = (\delta, \varsigma^{-1}\delta, \ldots, \varsigma^{-(d-1)}\delta)\pi(a_0, \ldots, a_{d-1})(\delta^{-1}, \delta^{-1}\varsigma, \ldots, \delta^{-1}\varsigma^{d-1})$$

$$= \pi(\varsigma^{-\pi(0)}\delta a_0 \delta^{-1}, \varsigma^{-\pi(1)}\delta a_1 \delta^{-1}\varsigma, \ldots, \varsigma^{-\pi(d-1)}\delta a_{d-1}\delta^{-1}\varsigma^{d-1}).$$

Note that the coordinate number x is $\varsigma^{-\pi(x)}\delta a \delta^{-1}\varsigma^{x}$, which is

$$\varsigma^{-\pi(x)}\pi\varsigma^{x}(\varsigma^{-\pi(x)}\delta a \delta^{-1}\varsigma^{x}, \ldots, \varsigma^{-\pi(x+d-1)}\delta a_{x+d-1}\delta^{-1}\varsigma^{x+d-1}),$$

and that $\varsigma^{-\pi(x)}\pi\varsigma^{x}(0) = 0$.

If $a \in Symm(\mathsf{X})$, then $\delta a \delta^{-1}$ is finitary of depth ≤ 2. Hence, passing again to X^2, we may assume that the generators of G either belong to $Symm(\mathsf{X})$, or are of the form

$$a = \pi(a, a_1, \ldots, a_{d-1})$$

for $a_i \in Symm(\mathsf{X})$ and $\pi \in Symm(\mathsf{X})$ such that $\pi(0) = 0$.

The set of automorphisms of X^* of the second type form a group B isomorphic to $Symm(\mathsf{X}) \wr Symm(\mathsf{X} \setminus 0)$. We denote $A = Symm(\mathsf{X})$.

Let $M(\mathsf{X})$ be the group generated by A and B.

We have proved

Proposition

Any finitely generated group of bounded automatic automorphisms of X^* *can be embedded as a subgroup into* $M(\mathsf{X}^n) \wr Symm(\mathsf{X}^n)$.

Hence it is sufficient to show that $M(\mathsf{X})$ is amenable for every X.

Let m_A and m_B be the uniform probability measures on the finite groups $A = \mathit{Symm}(\mathsf{X})$ and $B = \mathit{Symm}(\mathsf{X}) \wr \mathit{Symm}(\mathsf{X} \setminus 0)$. Consider their convolution $\mu = m_B * m_A$ and consider the corresponding random walk on $M(\mathsf{X})$.

By self-similarity of $M(\mathsf{X})$ we get a natural Markov chain on $\mathsf{X} \cdot M(\mathsf{X})$:

$$(i, g) \mapsto (h(i), h|_i \cdot g) \quad \text{with probability } \mu(h).$$

Projection of this chain onto X is a sequence of independent X-valued random variables. Projection onto G is the random walk determined by the measure

$$\widetilde{\mu} = \frac{d-1}{d} m_A + \frac{1}{d} m_B.$$

Considering the projections we conclude that

$$H(\mu^{*n}) \leq dH(\widetilde{\mu}^{*n}) + d\log d,$$

hence $h(M(\mathsf{X}), \mu) \leq dh(M(\mathsf{X}), \widetilde{\mu})$.

On the other side, using that m_A and m_B are idempotents, one can estimate

$$h(M(\mathsf{X}), \widetilde{\mu}) \leq \frac{d-1}{d^2} h(M(\mathsf{X}), \mu),$$

hence $h(M(\mathsf{X}), \mu) \leq \frac{d-1}{d} h(M(\mathsf{X}), \mu)$, i.e., $h(\mu) = 0$, which implies amenability of $M(\mathsf{X})$.

The Markov chain on $\mathsf{X} \times G$ is described by the transition matrix

$$M = (\mu_{xy})_{x,y \in \mathsf{X}},$$

where $\mu_{xy}(h)$ is the probability of the transition from $y \cdot g$ to $x \cdot hg$. In general, if $a = \sum_{g \in G} \alpha_g g \in \mathbb{C}[G]$, then the corresponding matrix $\phi(a) = (a_{xy})_{x,y \in \mathsf{X}}$ is given by

$$a_{xy} = \sum_{g(y)=x} \alpha_g g|_y.$$

The map $\phi : \mathbb{C}[G] \longrightarrow M_{d \times d}(\mathbb{C}[G])$ is a homomorphism of algebras called the *matrix recursion*.

For example, in the case of the binary adding machine the recursion is

$$a \mapsto \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}.$$

For $\mathrm{IMG}\left(z^2 - 1\right)$ we have

$$a \mapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

since

$$a(0w) = 1w, \quad a(1w) = 0b(w), \qquad b(0w) = 0w, \quad b(1w) = 1a(w).$$

The *Cuntz-Pimsner algebra* $\mathcal{O}_G = \mathcal{O}_{(G,\mathsf{X})}$ of a self-similar group (G, X) is the universal C^*-algebra generated by G and S_x (for $x \in \mathsf{X}$) satisfying

1. relations of G;
2. *Cuntz algebra relations*

$$S_x^* S_x = 1, \qquad \sum_{x \in \mathsf{X}} S_x S_x^* = 1;$$

3. for all $g \in G, x \in \mathsf{X}$:

$$g \cdot S_x = S_y \cdot h$$

whenever $g(xw) = yh(w)$ for all w.

The action of a self-similar group G on X^* extends naturally onto the boundary X^ω. We get in this way a natural unitary representation of G on $L^2(\mathsf{X}^\omega)$.
There is also a natural representation of the Cuntz algebra on $L^2(\mathsf{X}^\omega)$ induced by the transformations

$$T_x : w \mapsto xw.$$

The condition $g(xw) = yh(w)$ is equivalent to

$$g \cdot T_x = T_y \cdot h.$$

Hence we get a representation of $\mathcal{O}_G$.

Let $f : \mathcal{X}_1 \longrightarrow \mathcal{X}$ be an expanding covering of a space by a subset. Choose $t \in \mathcal{C}$ and consider

$$T_f = \bigsqcup_{n \geq 0} f^{-n}(t).$$

$\mathcal{O}_{\mathrm{IMG}(f)}$ is the universal C^*-algebra generated by S_γ, where γ are homotopy classes of paths connecting the points of T_f, satisfying

1. $S_{1_t} = 1$.
2. $S_{\gamma_1\gamma_2} = S_{\gamma_1} S_{\gamma_2}$.
3. $S_{\gamma^{-1}} = S_\gamma^*$.
4. $S_\gamma = \sum_{\delta \in f^{-1}(\gamma)} S_\delta$.

Define for $z \in \mathbb{C}, |z| = 1$

$$\Gamma_z(g) = g$$

for all $g \in G$ and

$$\Gamma_z(S_x) = zS_x$$

for all $x \in \mathsf{X}$.
We get a *gauge* action of the circle on $\mathcal{O}_G$.
The algebra of fixed points of the gauge action is the closed linear span of

$$S_v g S_u^*, \qquad \text{for } |v| = |u|.$$

We denote it $\mathcal{M}_G$.

Proposition

The gauge invariant sub-algebra $\mathcal{M}_G$ is isomorphic to the inductive limit of

$$C^*(G) \longrightarrow M_d(C^*(G)) \longrightarrow M_{d^2}(C^*(G)) \longrightarrow \cdots ,$$

where the homomorphism are induced by the matrix recursions.

Theorem

If the self-similar group (G, X) *is contracting (e.g., it is* $\mathrm{IMG}(f)$ *for an expanding* f*), then* $\mathcal{O}_G$ *is defined by a finite number of relations.*

For example, $\mathcal{O}_{\mathrm{IMG}(f)}$, where f is a *hyperbolic quadratic polynomial* is generated by the Cuntz algebra $\mathcal{O}_2 = \langle S_0, S_1 \rangle$ and one unitary a such that

$$a = S_1 S_0^* + S_0(1 - S_v S_v^* + S_v a S_v^*) S_1^*.$$

Theorem

If (G, X) is a regular (i.e., the groups of germs are trivial) contracting self-similar group, then $\mathcal{O}_G$ is simple, purely infinite, nuclear.

Theorem

Let f be a hyperbolic rational function of degree d. Denote by c the number of attracting cycles of f, by k the sum and by l the greatest common divisor of their lengths.
Then

$$K_0(\mathcal{M}_{\mathrm{IMG}(f)}) = \mathbb{Z}[1/d], \qquad K_1(\mathcal{M}_{\mathrm{IMG}(f)}) = \mathbb{Z}^{k-1}$$

and

$$K_0(\mathcal{O}_{\mathrm{IMG}(f)}) = \mathbb{Z}/(d-1)\mathbb{Z} \oplus \mathbb{Z}^{c-1}, \qquad K_1(\mathcal{O}_{\mathrm{IMG}(f)}) = \mathbb{Z}/l\mathbb{Z} \oplus \mathbb{Z}^{c-1}.$$

Theorem

Let f_1, f_2 be hyperbolic rational functions. Then the following two conditions are equivalent.

1. *The C^*-dynamical systems $(\mathcal{O}_{\mathrm{IMG}(f_1)}, \Gamma_z)$ and $(\mathcal{O}_{\mathrm{IMG}(f_2)}, \Gamma_z)$ are conjugate.*
2. *The topological dynamical systems (J_{f_1}, f_1) and (J_{f_2}, f_2) are conjugate, where J_{f_i} are the Julia sets of f_i, i.e., the closure of the set of repelling cycles of f_i.*